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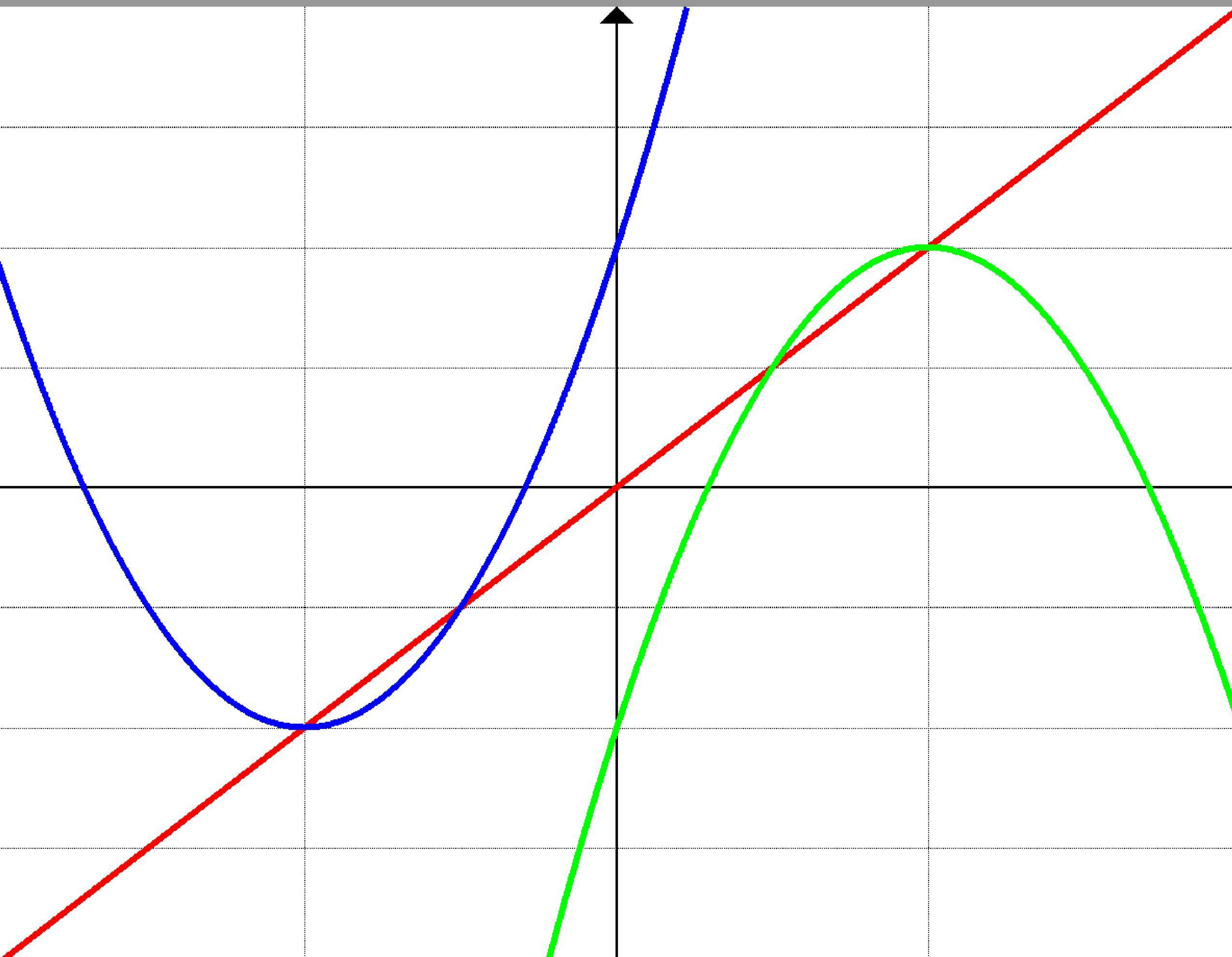
MAT1033 Intermediate Algebra at Santa Fe College



MAT1033 Intermediate Algebra at Santa Fe College

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Contents

Chapter 1 Unit 1 - Review - Real Numbers, Order of Operations, Graphs, and Exponent Rules

Chapter Outline

Introduction

This chapter contains of review of several topics that you have probably seen before but probably have not practiced recently. These include the classification of numbers, operations and properties of real numbers, graphing, and rules and properties of exponents.

1.1 Types of Numbers and the Real Number Line

Learning Objectives

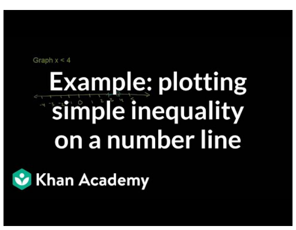
Here you will review the number sets that make up the real number system. In addition, you will graph inequalities on a real number line.

Concept Problem

Can you describe the number 13? Can you say what number sets the number 13 belongs to?

Watch This

Khan Academy Inequalities on a Number Line



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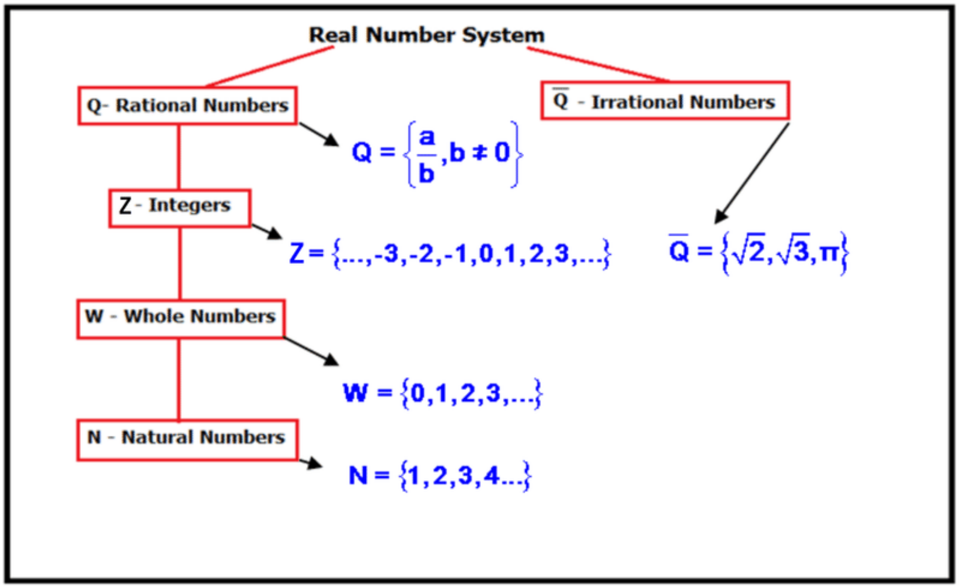
Guidance

All of the numbers you have learned about so far in math belong to the real number system. Positives, negatives, fractions, and decimals are all part of the real number system. The diagram below shows how all of the numbers in the real number system are grouped.



Natural Numbers

The **natural numbers** (denoted with the symbol $\mathbb{N}$) are also called the 'counting numbers', because they are the numbers we use to count objects:

$$1,2,3,4...$$

In many ways they are the building blocks upon which the rest of the real number system rests.

Whole Numbers

The **whole numbers** (denoted with the symbol $\mathbb{W}$) are identical to the natural numbers except that they also include the number 0.

$$0,1,2,3,4...$$

This distinction may seem small but is important historically. The concept of using a symbol to represent the absence of something was a difficult one for people to grasp. Ancient Egyptian and Mayan mathematicians had symbols for zero, but Europeans did not until their adoption of the Hindu-Arabic numeral system.

Integers

The **integers** (denoted with the symbol $\mathbb{Z}$) build on the whole numbers by incorporating both positive and negative values:

$$...-3,-2,-1,0,1,2,3...$$

Negative numbers have many practical applications, such as situations involving debt and deficit. They are referenced in early Chinese record-keepers (200 BCE) who used red ink for negative values and blank ink for positive values. The Indian mathematician Brahmagupta (620 CE) used a special symbol for negative values the same way we use the $-$ sign today.

Rational Numbers

The **rational numbers** (denoted with the symbol $\mathbb{Q}$) are numbers that may be written as a *ratio* of two integers i.e. $\frac{p}{q}$ where p and q are integers. It is not easy to list all of the rational numbers, but here are some examples:

$$\frac{2}{3}, -\frac{5}{7}, \frac{21}{11}, 8, 0$$

Note that an integer like 8 qualifies as a rational number, because it may be written as $\frac{8}{1}$, even though it is not usually written that way.

Decimal numbers whose representations terminate (like 3.75) or repeat (like 5.31313131...) also are rational numbers.

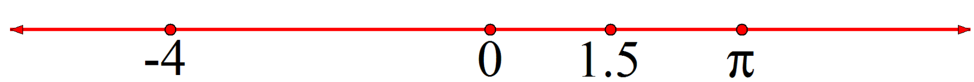
Irrational Numbers

The **irrational numbers** (denoted with the symbol $\mathbb{I}$) are literally numbers that are 'not rational' i.e. they cannot be written as a ratio of two integers. Their decimal representations do not terminate and do not repeat. Examples of irrational numbers include:

$$\begin{aligned}
\pi &= 3.141592653...\\
e &= 2.7182818284590...\\
\sqrt{2} &= 1.414213562...\\
\phi &= \frac{1+\sqrt{5}}{2} = 1.618033989...
\end{aligned}$$

Real Numbers

The **real numbers** (denoted with the symbol $\mathbb{R}$) are the union (combination) of the sets of rational and irrational numbers. They are the overarching category that includes all of the types of numbers discussed in this section. The set of real numbers is often represented as a number line.



The numbers $-4, 0, 1, 5$ and π are marked, but *any* spot on the line represents a real number.

In practice, there may be more than one symbol or combination of symbols used to describe a particular set of numbers. For example, the set of all irrational numbers may be written as $\overline{\mathbb{Q}}$, or $\mathbb{I}$, or even $\mathbb{R} - \mathbb{Q}$ (Real Numbers with Rational Numbers removed). The most important thing is to be consistent, and to state the definition of the symbol(s) you use.

Any number in the real number system can be plotted on a real number line. You can also graph inequalities on a real number line. In order to graph inequalities, make sure you know the following symbols:

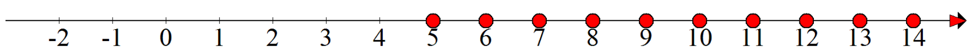
- The symbol >means "is greater than."
- The symbol <means "is less than."
- The symbol $\geq$ means "is greater than or equal to."
- The symbol $\leq$ means "is less than or equal to."

The inequality symbol indicates the type of dot that is placed on the beginning point and the number set indicates whether an arrow is drawn on the number line or if points are used.

Example A

Represent $x > 4$ where x is an integer, on a number line.

Solution:

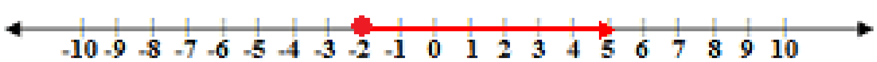


Since x must be an integer, we will only include points that have exactly integer values and not the points in between them. The closed dots on 5, 6, 7, 8 means that these numbers are included. Notice that 4 is not included on the graph, because x must be greater than 4. $x = 4$ is not a valid possibility in this case. The red arrow pointing to the right means that all integers to the right of 8 are also included in the graph.

Example B

Represent this inequality statement on a number line $\{x|x \geq -2, x \in R\}$.

Solution: The statement can be read as "the set of all x such that x is greater than or equal to -2 and x is a member of the real numbers." In other words, represent all real numbers greater than or equal to -2.



The inequality symbol says that x is greater than or equal to -2. This means that -2 is included in the graph. A solid dot is placed on -2 and on all numbers to the right of -2. The arrow indicates that the graph extends indefinitely in that direction.

Example C

Represent this inequality statement, also known as set-builder notation, on a number line $\{x|2 < x \leq 7, x \in \mathbb{R}\}$.

Solution: This inequality statement can be read as the set of all x such that x is greater than 2 and less than or equal to 7 and x is a member of the real numbers. In other words, all real numbers between 2 and 7, including 7 but not including 2.

Notice that there is no arrow on this graph, because it does not extend indefinitely; it stops at 2 and at 7.

Concept Problem Revisited

To what number set(s) does the number 13 belong?

The number 13 is a natural number because it is in the set $N = \{1,2,3,4\ldots\}$.

The number 13 is a whole number because it is in the set $W = \{0,1,2,3\ldots\}$.

The number 13 is an integer because it is in the set $Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.

The number 13 is a rational number because it is in the set $Q = \{\frac{a}{b}, b \neq 0\}$.

The number 13 belongs to the real number system.

Vocabulary

Inequality

An ***inequality*** is a mathematical statement relating expressions by using one or more inequality symbols. The inequality symbols are $>,<,\geq,\leq$.

Number Line

A ***number line*** is a set of real numbers represented by a continuous line with the numbers in order from smallest to largest.

Set-Builder Notation

Set-builder notation is a mathematical statement that shows an inequality and the set of numbers to which the variable belongs. $\{x|x \geq -3, x \in Z\}$ is an example of set-builder notation.

Guided Practice

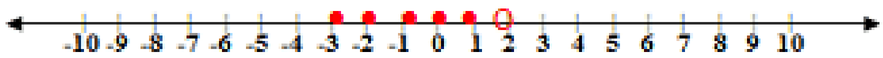
1. Check the set(s) to which each number belongs. The number may belong to more than one set.

TABLE 1.1:

Number	$\mathbb{N}$	$\mathbb{W}$	$\mathbb{Z}$	$\mathbb{Q}$	$\overline{\mathbb{Q}}$
5					
$-\frac{47}{3}$					
1.48					
$\sqrt{7}$					
0					
π					

2. Graph $\{x|-3 \leq x \leq 8, x \in R\}$ on a number line.

3. Use set notation to describe the set shown on the number line.



Answers:

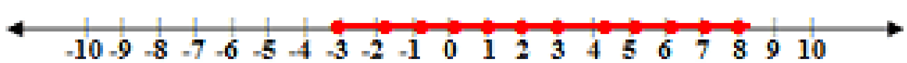
1. Review the definitions for each set of numbers.

TABLE 1.2:

Number	$\mathbb{N}$	$\mathbb{W}$	$\mathbb{Z}$	$\mathbb{Q}$	$\overline{\mathbb{Q}}$
5	X	X	X	X	
$-\frac{47}{3}$				X	
1.48				X	
$\sqrt{7}$					X
0		X	X	X	
π					X

2. $\{x|-3 \le x \le 8, x \in R\}$

The set notation means to graph all real numbers between -3 and +8. The line joining the solid dots represents the fact that the set belongs to the real number system.



3. The closed dot means that -3 is included in the answer. The remaining dots are to the right of -3. The open dot means that 2 is not included in the answer. This means that the numbers are all less than 2. Graphing on a number line is done from smallest to greatest or from left to right. There is no line joining the dots so the variable does not belong to the set of real numbers. However, negative whole numbers, zero and positive whole numbers make up the integers. The set notation that is represented on the number line is $\{x|-3 \le x < 2, x \in Z\}$.

Practice

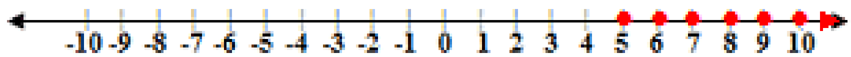
Describe each set notation in words.

1. $\{x|x > 8, x \in R\}$
2. $\{x|x \le -3, x \in Z\}$
3. $\{x|-4 \le x \le 6, x \in R\}$
4. $\{x|5 \le x \le 11, x \in W\}$
5. $\{x|x \ge 6, x \in N\}$

Represent each graph using set notation

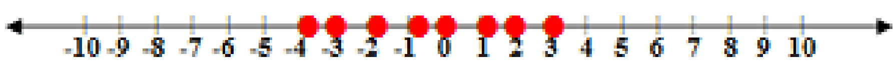
6.

7.



8.

9.



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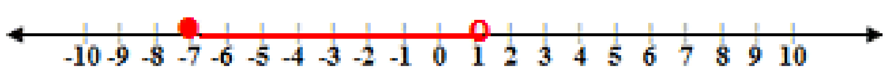


For each of the following situations, use set notations to represent the limits.

11. To ride the new tilt-a whirl at the fairgrounds, a child can be no taller than 4.5 feet.
12. A dance is being held at the community hall to raise money for breast cancer. The dance is only for those people 19 years of age or older.
13. A sled driver in the Alaska Speed Quest must start the race with no less than 10 dogs and no more than 16 dogs.
14. The residents of a small community are planning a skating party at the local lake. In order for the event to take place, the outdoor temperature needs to be above $-6^{\circ}C$ and not above $-1^{\circ}C$.
15. Juanita and Hans are planning their wedding supper at a local venue. To book the facility, they must guarantee that at least 100 people will have supper but no more than 225 people will eat.

Represent the following set notations on a number line.

16. $\{x|x > 6, x \in N\}$
17. $\{x|x \leq 8, x \in R\}$
18. $\{x|-3 \leq x < 6, x \in Z\}$

1.2 Addition of Integers

Learning Objectives

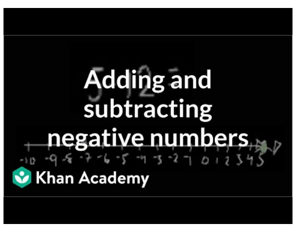
Here you'll learn to add integers using different representations including a number line. These methods will lead to the formation of two rules for adding integers.

Concept Problem

On Monday, Marty borrows \$50.00 from his father. On Thursday, he gives his father \$28.00. Can you write an addition statement to describe Marty's financial transactions?

Watch This

Khan Academy Adding/Subtracting Negative Numbers



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Guidance

When adding integers, you need to make sure you follow two rules:

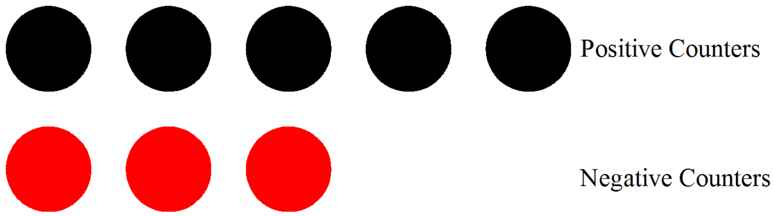
1. Numbers with unlike signs must be subtracted. The answer will have the same sign as that of the higher number.
2. Numbers with the same sign must be added. The answer will have the same sign as that of the numbers being added.

In order to understand why these rules work, you can represent the addition of integers with manipulatives such as color counters or algebra tiles. A number line can also be used to show the addition of integers. The following examples show how to use these manipulatives to understand the rules for adding integers.

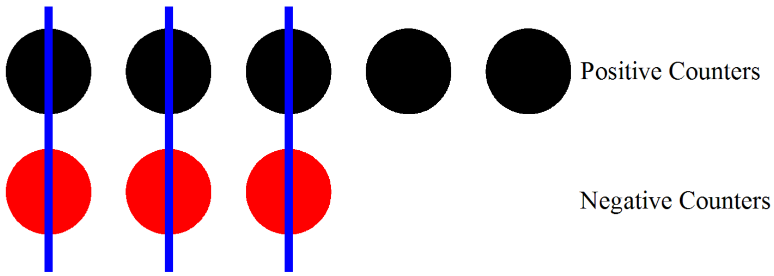
Example A

$5+(-3)=?$

Solution: This problem can be represented with color counters. In this case, the black counters represent positive numbers and the red ones represent the negative numbers.



One positive counter and one negative counter equals zero because $1+(-1)=0$. Draw a line through the counters that equal zero.

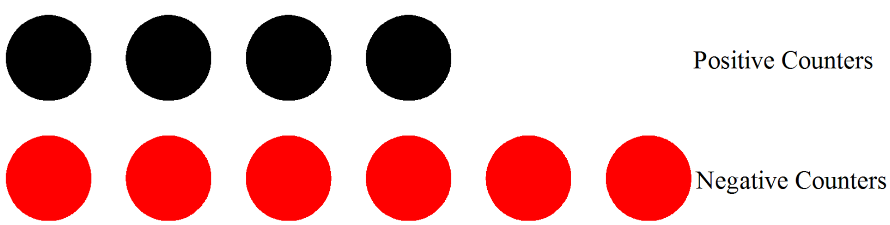


The remaining counters represent the answer. Therefore, $5+(-3)=2$. The answer is the difference between 5 and 3. The answer takes on the sign of the larger number. In this case, the five has a positive value and it is greater than 3.

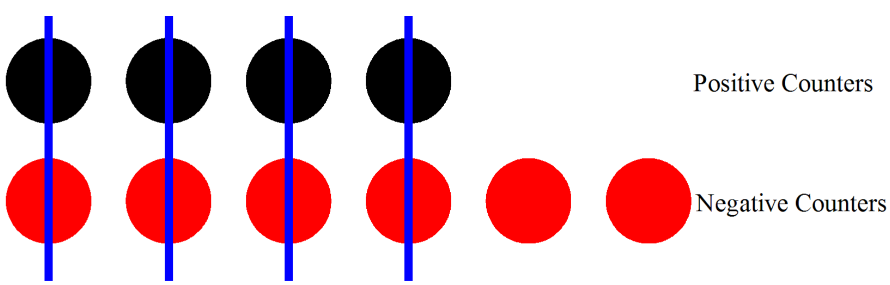
Example B

$4+(-6)=?$

Solution:



Draw a line through the counters that equal zero.

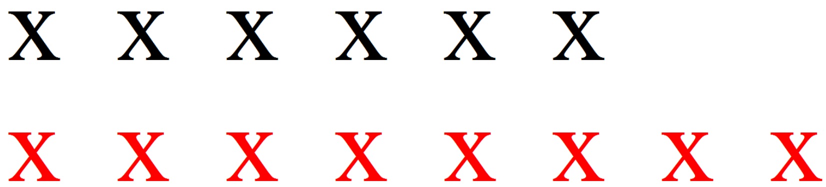


The remaining counters represent the answer. Therefore, $4+(-6)=-2$. The answer is the difference between 6 and 4. The answer takes on the sign of the larger number, which is 6 in this case.

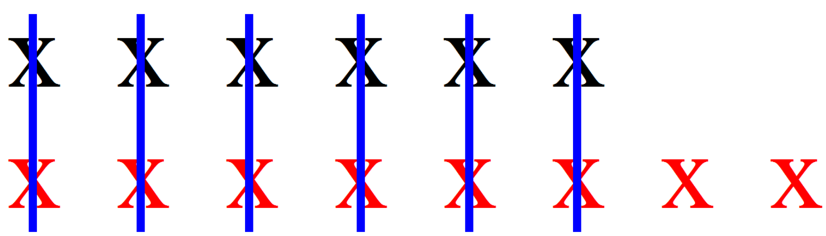
Example C

$6x+(-8x)=?$

Solution: This same method can be extended to adding variables. We can use black to represent positive $x's$ and red to represent negative $x's$.



Cross off pairs of $x's$ as we did in the last example:

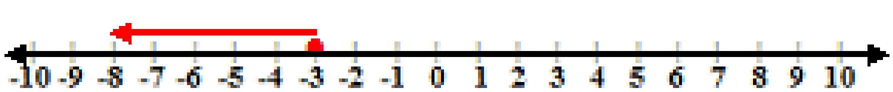


The remaining $x's$ represent the answer. There are two negative $x's$ remaining. Therefore, $(6x)+(-8x)=-2x$. The answer is the difference between $8x$ and $6x$. The answer takes on the sign of the larger coefficient, which in this case is 8.

Example D

$(-3)+(-5)=?$

Solution: You can solve this problem with a number line. Indicate the starting point of -3 by using a dot. From this point, add a -5 by moving five places to the left. You will stop at -8.



The point where you stopped is the answer to the problem. Therefore, $(-3)+(-5)=-8$

Concept Problem Revisited

On Monday, Marty borrows \$50.00 from his father. On Thursday, he gives his father \$28.00.

Marty borrowed \$50.00 which he must repay to his father. Therefore Marty has $-\$50.00$.

He returns \$28.00 to his father. Now Marty has $-\$50.00+(\$28.00)=-\$22.00$. He still owes his father \$22.00.

Guided Practice

1. $(-7)+(+5)=?$

2. $8+(-2)=?$

3. Determine the answer to $(-6)+(-3)=?$ and $(2)+(-5)=?$ by using the rules for adding integers.

Answers:

1. $(-7)+(+5)=5-7=-2$

2. $8+(-2)=8-2=6$

3. $(-6)+(-3)=-9$.

$(2)+(-5)=2-5=-3$.

Practice

Complete the following addition problems using any method.

1. $(-7)+(-2)$
2. $(6)+(-8)$
3. $(5)+(4)$
4. $(-7)+(9)$
5. $(-1)+(5)$
6. $(8)+(-12)$
7. $(-2)+(-5)$
8. $(3)+(4)$
9. $(-6)+(10)$
10. $(-1)+(-7)$
11. $(-13)+(9)$
12. $(-3)+(-8)+(12)$
13. $(14)+(-6)+(5)$
14. $(15)+(-8)+(-9)$
15. $(7)+(6)+(-9)+(-8)$

For each of the following models, write an addition problem using variables and use it to simplify the expression.

16. $6x+(-4x)$

17. $-7x+2x$

18. $7x+(-6x)$

19. $6x+(-6x)$

20. $(-7x)+12x$

Answers for Practice Problems

To view the practice problem answers, open this PDF file and look for section 1.1.

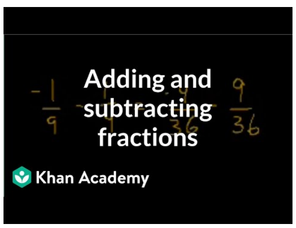
1.3 Addition and Subtraction of Fractions

Concept Problem

Lily and Howard ordered a pizza that was cut into 8 slices. Lily ate 3 slices and Howard ate 4 slices. What portion of the pizza did each person eat? What portion of the pizza did they eat all together? How much of the pizza remains?

Watch This

Khan Academy Adding and Subtracting Fractions



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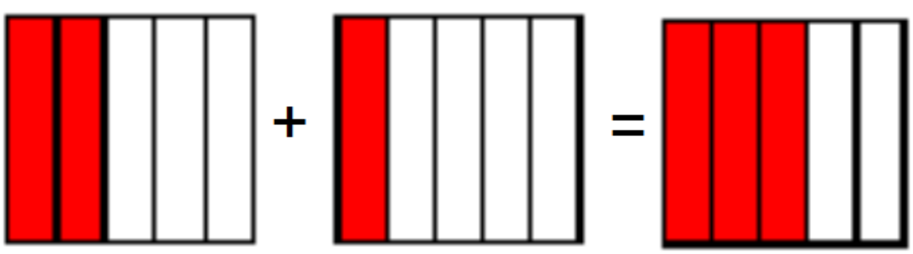
Guidance

$\frac{2}{5}+\frac{1}{5}=?$

The problem above can be represented using fraction strips.



$$\frac{2}{5}+\frac{1}{5}=\frac{2+1}{5}=\frac{3}{5}$$

To add or subtract fractions, the fractions must have the same denominators (bottom numbers). In this case, both fractions have a denominator of 5, so we can proceed to the next step: combining the numerators (top numbers). When adding fractions, the numerators are added, but the denominator does **not** change. So we have $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.

A number line can also be used to show the addition of fractions, as you will explore in Example C.

The sum of two fractions will sometimes result in an answer that is an **improper fraction**, which is a fraction in which the numerator is greater than or equal to the denominator. Numbers like $\frac{5}{5}, -\frac{19}{7}$, and $\frac{100}{3}$ are improper fractions.

An improper fraction can also be written as a **mixed number**, which is a number made up of a whole number and a fraction. The mixed number forms of $\frac{7}{5}, \frac{-19}{7}$, and $\frac{100}{3}$ are $2\frac{2}{5}, -2\frac{5}{7}$, and $33\frac{1}{3}$. Mixed numbers are useful for showing the magnitude of a number, but usually improper fractions are easier to work with arithmetically.

Adding Fractions

Suppose we want to add $\frac{7}{12}+\frac{11}{15}$. In order to add fractions that have different denominators, the fractions must be expressed as equivalent fractions with a common denominator. Any common denominator will work in theory, but it is usually easiest to use the smallest one, which is called the *least common denominator (LCD)*. The LCD is the smallest number that is a multiple of both denominators. The LCD of $\frac{7}{12}$and $\frac{11}{15}$ is 60, because 60 is the smallest number that is a multiple of both 12 and 15.

Once the LCD is determined, the fractions must be manipulated to get the LCD in each denominator. The tool to do this is multiplication, because we can multiply the top and bottom of a fraction by a number without changing the value of the fraction. Looking at $\frac{7}{12}$ we see that we can change the 12 to 60 by multiplying by 5, so we do this to the numerator and denominator: $\frac{7}{12}\cdot\frac{5}{5}=\frac{35}{60}$. Similarly, we multiply $\frac{11}{15}$ by $\frac{4}{4}$ to obtain $\frac{11}{15}\cdot\frac{4}{4}=\frac{44}{60}$.

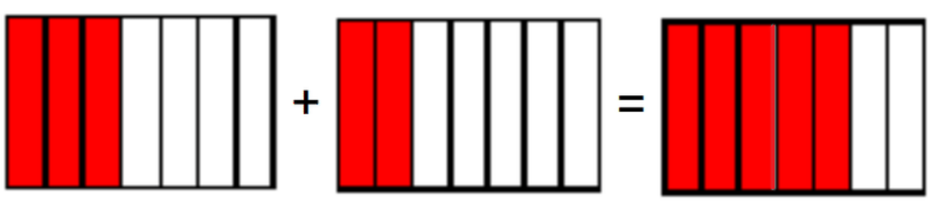
Once we have a common denominator, we can add like we did in the example above:

$$\frac{7}{12}+\frac{11}{15}=\frac{35}{60}+\frac{44}{60}=\frac{79}{60}$$

Example A

$\frac{3}{7}+\frac{2}{7}=?$

Solution:

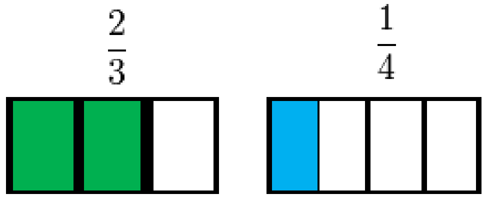


$$\boxed{\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}}$$

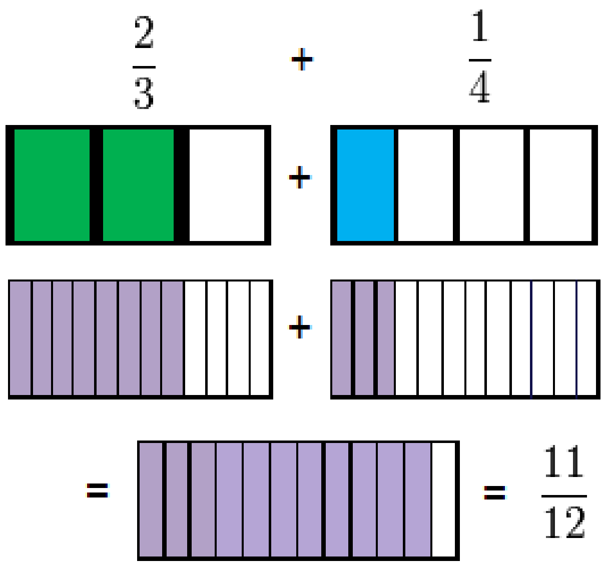
Example B

Louise is taking inventory of the amount of water in the water coolers located in the school. She estimates that one cooler is $\frac{2}{3}$ full and the other is $\frac{1}{4}$ full. What single fraction could Louise use to represent the amount of water of the two coolers together?

Solution: Use fraction strips to represent each fraction.



$\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions. $\frac{2}{3}\left(\frac{4}{4}\right)=\frac{8}{12}$.
$\frac{1}{4}$ and $\frac{3}{12}$ are equivalent fractions. $\frac{1}{4}\left(\frac{3}{3}\right)=\frac{3}{12}$.
The two green pieces will be replaced with eight purple pieces and the one blue piece will be replaced with three purple pieces.



The amount of water in the two coolers can be represented by the single fraction $\frac{11}{12}$.

$$\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}=\frac{11}{12}$$

The denominator of 12 is the LCD (least common denominator) of $\frac{2}{3}$ and $\frac{1}{4}$ because it is the LCM (least common multiple) of the numbers 3 and 4.

Example C

Subtract $\frac{2}{3}-\frac{7}{5}$.

Solution: To subtract, we follow the same steps as addition, but we subtract the numerators instead of adding them. In this case our LCD is 15, so we have:

$$\frac{2}{3}-\frac{7}{5}=\frac{2}{3}\cdot\frac{5}{5}-\frac{7}{5}\cdot\frac{3}{3}=\frac{10}{15}-\frac{21}{15}=-\frac{11}{15}$$

Example D

Subtract $5-\frac{7}{4}$.

Solution: When combining an integer and a fraction, rewrite the integer as a fraction with denominator 1, and then follow the same process we have been using.

$$5-\frac{7}{4} \rightarrow \frac{5}{1}-\frac{7}{4}$$

What is the LCD of 1 and 4? It's 4!

$$\frac{5}{1}-\frac{7}{4}=\frac{5}{1}\cdot\frac{4}{4}-\frac{7}{4}=\frac{20}{4}-\frac{7}{4}=\frac{13}{4}$$

Concept Problem Revisited

Lily ate $\frac{3}{8}$ of the pizza because she ate 3 out of the 8 slices. Howard ate $\frac{1}{2}$ of the pizza. So together they ate

$$\frac{3}{8}+\frac{1}{2}=\frac{3}{8}+\frac{4}{8}=\frac{7}{8}$$

which is $\frac{7}{8}$ of the pizza.

How much pizza remains? We started with 1 pizza, and now $\frac{7}{8}$ of it is gone, so we can subract $1-\frac{7}{8}$. First, we need to rewrite 1 as $\frac{1}{1}$.

$$\frac{1}{1}-\frac{7}{8}=\frac{1}{1}\cdot\frac{8}{8}-\frac{7}{8}=\frac{8}{8}-\frac{7}{8}=\frac{1}{8}$$

So $\frac{1}{8}$ of the pizza remains.

Vocabulary

Denominator

The ***denominator*** of a fraction is the number on the bottom that indicates the total number of equal parts in the whole or the group. $\frac{5}{8}$ has ***denominator*** 8.

Fraction

A ***fraction*** is any rational number that is not an integer.

Improper Fraction

An ***improper fraction*** is a fraction in which the numerator is larger than the denominator.
$\frac{8}{3}$ is an ***improper fraction.***

LCD

The ***least common denominator*** is the lowest common multiple of the denominators of two or more fractions. The ***least common denominator*** of $\frac{3}{4}$ and $\frac{1}{5}$ is 20.

LCM

The ***least common multiple*** is the lowest common multiple that two or more numbers share. The ***least common multiple*** of 6 and 4 is 12.

Mixed Number

A ***mixed number*** is a number made up of a whole number and a fraction such as $4\frac{3}{5}$.

Numerator

The ***numerator*** of a fraction is the number on top that is the number of equal parts being considered in the whole or the group. $\frac{5}{8}$ has ***numerator*** 5.

Guided Practice

1. $\frac{1}{2}+\frac{1}{6}=?$
2. $\frac{1}{6}+\frac{3}{4}=?$
3. $\frac{2}{5}-\frac{2}{3}=?$

Answers:

1.

$$\frac{1}{2}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

2.

$$\frac{1}{6}+\frac{3}{4}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$$

3.

$$\frac{2}{5}-\frac{2}{3}=\frac{6}{15}-\frac{10}{15}=-\frac{4}{15}$$

Practice

Complete the following addition problems using any method.

1. $\frac{1}{4}+\frac{5}{8}$
2. $\frac{2}{5}+\frac{1}{3}$
3. $\frac{2}{9}+\frac{2}{3}$
4. $\frac{3}{7}+\frac{2}{3}$
5. $\frac{7}{10}+\frac{1}{5}$
6. $\frac{2}{3}+\frac{1}{2}$
7. $\frac{2}{5}+\frac{3}{10}$
8. $\frac{5}{9}+\frac{2}{3}$
9. $\frac{3}{8}+\frac{3}{4}$
10. $\frac{3}{5}+\frac{3}{10}$
11. $\frac{7}{11}+\frac{1}{2}$
12. $\frac{7}{8}+\frac{5}{12}$
13. $\frac{3}{4}+\frac{5}{6}$
14. $\frac{5}{6}+\frac{2}{5}$

15. $\frac{4}{5}+\frac{3}{4}$

For each of the following questions, write an addition statement and find the result. Express all answers as either proper fraction or mixed numbers.

16. Karen used $\frac{5}{8}$ *cups* of flour to make cookies. Jenny used $\frac{15}{16}$ *cups* of flour to make a cake. How much flour did they use altogether?
17. Lauren used $\frac{3}{4}$ *cups* of milk, $1\frac{1}{3}$ *cups* of flour and $\frac{3}{8}$ *cups* of oil to make pancakes. How many cups of ingredients did she use in total?
18. Write two fractions with different denominators whose sum is $\frac{5}{6}$.
19. Allan's cat ate $2\frac{2}{3}$ *cans* of food in one week and $3\frac{1}{4}$ *cans* the next week. How many cans of food did the cat eat in two weeks?
20. Amanda and Justin each solved the same problem.

Amanda's Solution:

$$\frac{1}{6}+\frac{3}{4}$$
$$\frac{2}{12}+\frac{9}{12}$$
$$=\frac{11}{24}$$

Justin's Solution:

$$\frac{1}{6}+\frac{3}{4}$$
$$\frac{2}{12}+\frac{9}{12}$$
$$=\frac{11}{12}$$

Who is correct? What would you tell the person who has the wrong answer?

1.4 Decimal Notation

Learning Objectives

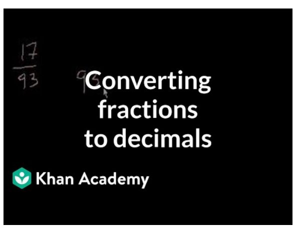
Here you will learn the difference between a terminating decimal number and a periodic decimal number. You will also learn how to express a given fraction in decimal form and how to express a number in decimal form as a fraction.

Concept Problem

Which is greater, $\frac{18}{99}$ or $\frac{15}{80}$? How can you use decimals to help you with this problem?

Watch This

Khan Academy Converting Fractions to Decimals



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Guidance

To change a fraction to a number in decimal form, the numerator must be divided by the denominator. You can use long division or a calculator for this calculation. If, after the division, the numbers after the decimal point end, it is a terminating decimal. If, after the division, the numbers after the decimal point repeat in a pattern forever, it is a periodic decimal (also known as a repeating decimal).

- $\frac{3}{4} = 3 \div 4 = 0.75$. This is a terminating decimal.
- $\frac{3}{13} = 3 \div 13 = 0.230769230769\ldots$. This is a periodic or repeating decimal. The period is 230769 because this is the set of numbers that repeats.

Keep in mind that a rational number is any number that can be written in the form $\frac{a}{b}$ where $b \neq 0$. Therefore, periodic decimals and terminating decimals are both rational numbers.

Decimal numbers that don't do these things (i.e. non-terminating and non-repeating) are irrational numbers. Examples of these include:

$\pi = 3.14159265358979323846264338\ldots$
$e = 2.71828182845904523536028747713526\ldots$
$\sqrt{2} = 1.41421356237309504880168872420 96\ldots$

Example A

What fraction is equal to 0.45454545... ?

Solution: This is a periodic or repeating decimal. The period has a length of two because the pattern that is repeating consists of 2 digits. To express the number as a fraction, follow these steps:

Step 1: Let $x = 0.45454545...$

Step 2: The repeating digit is 45. We would like to move 45 to the left of the decimal place. We can do this by multiplying by 100: $100x = 45.454545$

Step 3: Subtract the two equations and solve for x.

$$\begin{array}{rl} 100x = & 45.454545... \\ - \quad x = & 0.45454545... \\ \hline 99x = & 45 \end{array}$$

Now, we solve for x:

$$\begin{aligned} 99x &= 45 \\ \frac{99x}{99} &= \frac{45}{99} \\ x &= \frac{45}{99} = \frac{5}{11} \end{aligned}$$

Example B

What fraction is equal to 0.125?

Solution: This number is a terminating decimal number. The steps to follow to express 0.125 as a fraction are:

Step 1: Let $x = .125$. Express the number as a whole number by moving the decimal point to the right. In this case, the decimal must be moved three places to the right. To do this, we multiply both sides of the equation by 1000:

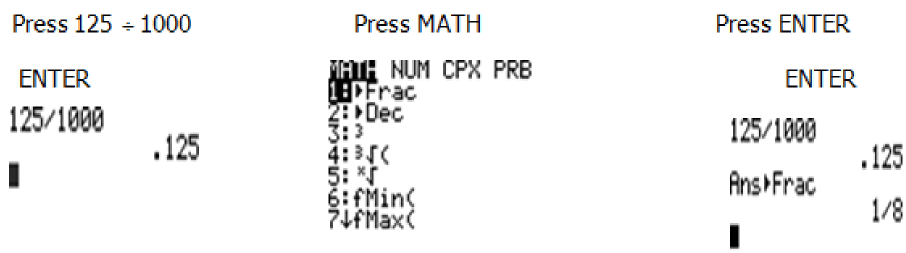
$$\begin{aligned} (1000) \cdot x &= .125 \cdot (1000) \\ 1000x &= 125 \end{aligned}$$

Step 2: $1000x = 125$

Solve for x by dividing both sides of the equation by 1000

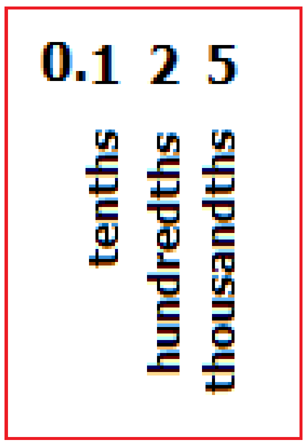
$$x = \frac{125}{1000}$$

Step 3: If possible, simplify the fraction. If you are not sure of the simplified form, your graphing calculator can do the calculations.



Therefore, the decimal number of 0.125 is equivalent to the fraction $\frac{1}{8}$.

The method shown above is one that can be used if you can't remember the place value associated with the decimal numbers. If you remember the place values, you can simply write the decimal as a fraction and simplify that fraction.



$$\frac{125}{1000} = \frac{1}{8}$$

Example C

Are the following decimal numbers terminating or periodic? If they are periodic, what is the period and what is its length?

i) 0.318181818...

ii) 0.375

iii) 0.3125

iv) 0.1211221112...

Solution:

i) A periodic decimal with a period of 18. The length of the period is 2.

ii) 0.375 A terminating decimal.

iii) 0.3125 A terminating decimal.

iv) 0.1211221112 This decimal is not a terminating decimal nor is it a periodic decimal. Therefore, the decimal is not a rational number. Decimals that are non-periodic belong to the set of irrational numbers.

Concept Problem Revisited

You can convert both fractions to decimals in order to figure out which is greater.

$\frac{18}{99} = .1818...$

$\frac{15}{80} = .1875$

You can see that $\frac{15}{80}$ is greater.

Vocabulary

Irrational Numbers

An ***irrational number*** is the set of non-periodic decimal numbers. Some examples of ***irrational numbers*** are $\sqrt{3}$, $\sqrt{2}$ and π.

Periodic Decimal

A ***periodic decimal*** is a decimal number that has a pattern of digits that repeat. The decimal number 0.1465325325..., is a ***periodic decimal.***

Rational Numbers

A rational number is any number that be written in the form $\frac{a}{b}$ where $b \neq 0$. Therefore, periodic decimal numbers and terminating decimal numbers are ***rational numbers.***

Terminating Decimal

A ***terminating decimal*** is a decimal number that ends. The process of dividing the fraction ends when the remainder is zero. The decimal number 0.25 is a ***terminating decimal.***

Guided Practice

1. Express 2.018181818 in the form $\frac{a}{b}$.

2. Express $\frac{15}{11}$ in decimal form.

3. If one tablet of micro K contains 0.5 grams of potassium, how much is contained in $2\frac{3}{4}$ tablets?

Answers:

1. Let $x = 2.018181818$ The period is 18.

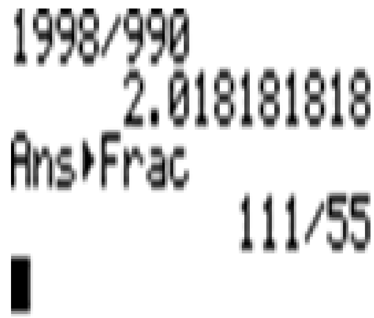
$1000x = 2018.181818$

$10x = 20.18181818$

$$\begin{aligned} 1000x &= 2018.181818 && \text{These are the two equations that must be subtracted.} \\ \underline{-10x} &\underline{= 20.18181818} && \\ \frac{990x}{990} &= \frac{1998}{990} && \text{Solve for } x. \\ x &= \frac{1998}{990} && \end{aligned}$$

Use your calculator to simplify the fraction.

$x = \frac{1998}{990}$



$x = \frac{111}{55}$

2. $\frac{15}{11} = 1.3636...$

3. The number of tablets is given as a mixed number. $2\frac{3}{4} = 2.75$. $2.75 \times 0.5 = 1.375$ *grams*.

Practice

Express the following fractions in decimal form.

1. $\frac{1}{12}$
2. $\frac{6}{11}$
3. $\frac{3}{20}$
4. $\frac{1}{13}$
5. $\frac{3}{8}$

Express the following numbers in the form $\frac{a}{b}$.

6. 0.325
7. 3.72727272...
8. 0.245454545...
9. 0.618
10. 0.36363636...

Complete the following table.

TABLE 1.3:

	Fraction	Decimal
11.	$\frac{5}{64}$	
12.	$\frac{11}{32}$	
13.	$\frac{1}{20}$	
14.		0.0703125
15.		0.1875

1.5 Multiplication of Real Numbers

Learning Objectives

Here you will learn to multiply integers, fractions and decimals.

Concept Problem

Jacob received tips of $4.50 each from three of his paper route customers. How much did he receive in total?

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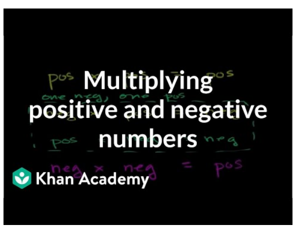
Khan Academy Multiplying Positive and Negative Numbers



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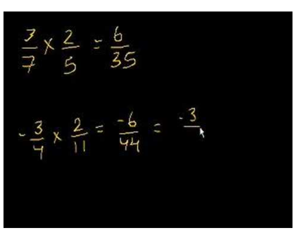
Khan Academy Multiplying Fractions



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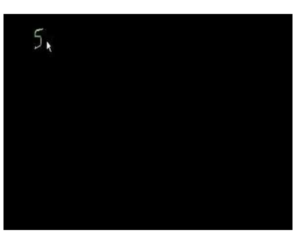
Khan Academy Multiplication 8: Multiplying Decimals



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Guidance

Multiplication of two integers with the same signs produces a positive result and multiplication of two integers with unlike signs results in a negative answer. These same rules apply to the multiplication of all real numbers.

To multiply fractions, you multiply the numerators and then you multiply the denominators, then simplify the result.

The rules for multiplying integers also apply to multiplying decimals. The sum of the number of digits after the decimal points determines the placement of the decimal point in the answer.

Example A

Sam spent \$2.00 for a bottle of chocolate milk at the school cafeteria every school day. At the end of the week, how does this affect his net worth?

Solution: The result of $(5) \cdot (2) = 10$ so Sam spent \$10 that week.

Example B

What is $(-2) \cdot (-3)$?

Solution: $(-2) \cdot (-3) = 6$, since the product of two negative integers is always positive.

Example C

i) $\left(\frac{2}{3}\right) \cdot \left(\frac{5}{7}\right)$

ii) $\left(\frac{7}{8}\right) \cdot \left(\frac{15}{4}\right)$

Solution: Remember, there are three simple steps to follow to multiply fractions:

1. Multiply the numerators of the fractions
2. Multiply the denominators of the fractions.
3. Simplify the fraction if necessary.

i) $\left(\frac{2}{3}\right) \cdot \left(\frac{5}{7}\right)$

$$
\begin{aligned}
&= \frac{2 \cdot 5}{3 \cdot 7} \\
&= \frac{10}{21}
\end{aligned}
$$

ii) $\left(\frac{7}{8}\right) \cdot \left(\frac{15}{4}\right)$

$$
\begin{aligned}
&= \left(\frac{7}{8}\right) \cdot \left(\frac{15}{4}\right) \\
&= \frac{7 \cdot 15}{8 \cdot 4} \\
&= \frac{105}{32}
\end{aligned}
$$

Concept Problem Revisited

Jacob received tips of $4.50 each from three of his paper route customers. How much did he receive in total?

$(3) \cdot (4.5) = 13.5$, so Jacob received $13.50 in tips.

Guided Practice

Multiply the following fractions:

1. $\left(\frac{5}{9}\right) \cdot \left(\frac{-4}{7}\right)$
2. $\left(\frac{11}{3}\right) \cdot \left(\frac{21}{5}\right)$

Answers:

1. Multiply the numerators. Multiply the denominators. Simplify the fraction.

$$\left(\frac{5}{9}\right) \cdot \left(\frac{-4}{7}\right) = \frac{5 \cdot (-4)}{9 \cdot 7} = -\frac{20}{63}$$

The answer can be written as $\frac{-20}{63}$ or $-\frac{20}{63}$.

2. Follow the steps for multiplying fractions. Multiply the numerators with each other and the denominators with each other, then simplify the fraction if necessary.

$$\left(\frac{11}{3}\right) \cdot \left(\frac{21}{5}\right)$$
$$\left(\frac{11}{3}\right) \cdot \left(\frac{21}{5}\right) = \frac{231}{15}$$

Practice

Multiply.

1. $(-7) \cdot (-2)$
2. $(+3) \cdot (+4)$
3. $(-5) \cdot (+3)$
4. $(+2) \cdot (-4)$
5. $(+4) \cdot (-1)$

Match each given phrase with the correct multiplication statement. Then, determine each product.

6. take away six groups of 3 balls
7. net worth after losing seven $5 bills
8. take away nine sets of 8 forks
9. take away four sets of four plates

10. receive eight groups of 4 glasses
11. buy seven sets of 12 placemats

a) $(8) \cdot (4)$
b) $(+7) \cdot (-5)$
c) $(-4) \cdot (+4)$
d) $(-9) \cdot (+8)$
e) $(+7) \cdot (+12)$
f) $(-6) \cdot (+3)$

Use the rules that you have learned for multiplying real numbers to answer the following problems.

12. $(-13) \cdot (-9)$
13. $(-3.68) \cdot (82.4)$
14. $\left(\frac{4}{9}\right) \cdot \left(\frac{5}{7}\right)$
15. $\left(\frac{23}{3}\right) \cdot \left(\frac{13}{2}\right)$
16. $(15.734) \cdot (-8.1)$

1.6 Division of Real Numbers

Learning Objectives

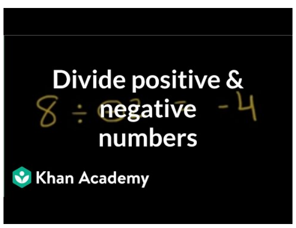
Here you will review how to divide real numbers including fractions and decimals.

Concept Problem

The meteorologist on the local radio station just announced that a cold front caused the temperature to drop $12^\circ C$ in just four hours. What was the mean temperature change per hour over these four hours?

Watch This

Khan Academy Dividing Positive and Negative Numbers



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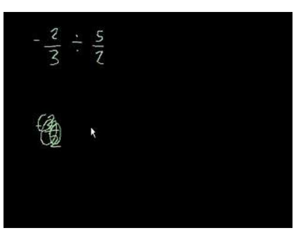
Khan Academy Dividing Fractions



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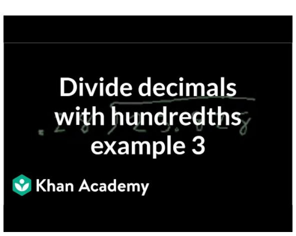
Khan Academy Dividing Decimals



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Guidance

Real numbers include rational numbers (which include integers), and irrational numbers. The process of dividing differs depending on the type of real number.

The rules for dividing integers are the same as the rules for multiplying integers:

1. When you divide two integers that have the same signs, the answer is always positive.
2. When you divide two integers that have opposite signs, the answer is always negative.

Dividing fractions is like multiplying fractions with one additional step. To divide fractions, multiply the first fraction by the reciprocal of the second fraction. For example, $\frac{3}{5} \div \frac{2}{9} = \frac{3}{5} \cdot \frac{9}{2}$.

Example A

Miguel was doing a science project on weather and he reported a total temperature change of $-15^\circ F$ and a mean hourly change of $-3^\circ C$. How many hourly temperature changes did Miguel record?

Solution: The result of $(-15) \div (-3)$ is 5.

Example B

i) $\left(\frac{6}{11}\right) \div \left(\frac{5}{7}\right)$

ii) $\left(\frac{13}{3}\right) \div \left(\frac{19}{7}\right)$

Solution:

i)

$$\left(\frac{6}{11}\right) \div \left(\frac{5}{7}\right)$$
$$\frac{6}{11} \times \frac{7}{5}$$
$$\frac{6 \times 7}{11 \times 5}$$
$$= \frac{42}{55}$$

ii)

$$\left(\frac{13}{3}\right) \div \left(\frac{19}{7}\right) \text{ Multiply by the reciprocal of } \frac{19}{7}.$$
$$\frac{13}{3} \times \frac{7}{19}$$
$$= \frac{91}{57}$$

Example C

i) $(0.68) \div (1.7)$

ii) $0.365 \div -18.25$

Solution:

i) $(0.68) \div (1.7)$

$$\begin{array}{r} 0.4 \\ 1.\underrightarrow{7}\overline{)0.\underrightarrow{6}\ 8} \\ \underline{-6\ 8} \\ 0 \end{array}$$

The decimal point of the divisor was moved one place to the right. The decimal point of the dividend was moved one place to the right. The decimal point was placed in the quotient directly above the new decimal point of the dividend.

ii) $0.365 \div -18.25$

You have learned that when you divide a positive number by a negative number, the answer will always be negative.

$$\begin{array}{r} -.02 \\ -18.\underrightarrow{25}\overline{)0.\underrightarrow{36}\ 50} \\ \underline{-36\ \ 50} \\ 0 \end{array}$$

The decimal point of the divisor was moved two places to the right. The decimal point of the dividend was moved two place to the right. The decimal point was placed in the quotient directly above the new decimal point of the dividend.

Concept Problem Revisited

The meteorologist on the local radio station just announced that a cold front caused the temperature to drop $12^\circ C$ in just four hours.

The mean temperature change per hour is the result of $(-12) \div (+4)$, which is -3.

Vocabulary

Reciprocal

The ***reciprocal*** of a number is the multiplicative inverse of that number. If $\frac{a}{b}$ is a nonzero number, then $\frac{b}{a}$ is its reciprocal. The product of a number and its reciprocal is one.

Quotient

The ***quotient*** is the answer of a division problem.

Guided Practice

1. $\left(\frac{7}{10}\right) \div \left(\frac{5}{6}\right) = ?$

2. $\left(\frac{32}{5}\right) \div \left(\frac{5}{3}\right) = ?$

3. How many pieces of plywood 0.375 in. thick are in a stack of 30 in. high?

Answers:

1. $\left(\frac{7}{10}\right) \div \left(\frac{5}{6}\right)$

$= \frac{7}{10} \times \frac{6}{5}$

$= \frac{7 \times 6}{10 \times 5}$

$= \frac{42}{50} = \frac{21}{25}$

2. $\left(\frac{32}{5}\right) \div \left(\frac{5}{3}\right)$

$= \left(\frac{32}{5}\right) \times \left(\frac{3}{5}\right)$

$= \frac{32 \times 3}{5 \times 5}$

$$= \frac{96}{25}$$

3. To determine the number of pieces of plywood in the stack, divide the thickness of one piece into the height of the pile.

$$\begin{array}{r} 80 \\ 0.\underrightarrow{375})\overline{30.\underrightarrow{000}} \\ \underline{-3000} \\ 0 \\ \underline{-\quad 0} \\ 0 \end{array}$$

There are 80 pieces of plywood in the pile.

Practice

Find each quotient or product.

1. $(+14) \div (+2)$
2. $(-14) \div (+2)$
3. $(-9) \div (-3)$
4. $(+16) \div (+4)$
5. $(+25) \div (-5)$
6. $(-9) \times (7)$
7. $(-8) \times (-8)$
8. $(+4) \times (-7)$
9. $(-10) \times (-3)$

10. $(+5)\times(+2)$
11. $\left(\frac{5}{16}\right)\div\left(\frac{3}{7}\right)$
12. $(-8.8)\div(-3.2)$
13. $(7.23)\div(0.6)$
14. $\left(\frac{11}{4}\right)\div\left(\frac{9}{8}\right)$
15. $(-30.24)\div(-0.42)$

For each of the following questions, write a division statement and find the result.

16. A truck is delivering fruit baskets to the local food banks for the patrons. Each fruit basket weighs 3.68 lb. How many baskets are in a load weighing 5888 lb?
17. A wedding invitation must be printed on card stock measuring 4.25 in. wide. If the area of the invitation is 23.375 in^2, what is its length? (Hint: The area of a rectangle is found by multiplying the length times the width.)
18. A seamstress needs to divide 32.675 ft. of piping into 3 equal pieces. Calculate the length of each piece.
19. The floor area of a room on a house plan measures 3.5 in. by 4.625 in. If the house plan is drawn to the scale 0.25 in. represents 1 ft, what is the actual size of the room?
20. How many hair bows of 3.5 in. can be cut from 24.75 in. of ribbon?

1.7 Properties of Real Number Addition

Concept Problem

On the first day of school, you are all dressed in your new clothes. When you got dressed, you put one sock on your left foot and one sock on your right foot. Would it have made a difference if you had put one sock on your right foot first and then one sock on your left foot?

Guidance

There are five properties of addition that are important for you to know.

Commutative Property

In algebra, the operation of addition is **commutative**. This means that the order in which you add two real numbers does not change the result, as shown below:

$$(+7)+(+20)=? \qquad (+20)+(+7)=?$$
$$(+7)+(+20)=+27 \qquad (+20)+(+7)=+27$$

The order in which you added the numbers did not affect the answer. If a and b are real numbers, then $a+b=b+a$.

Closure Property

The real numbers are **closed** under the operation of addition. This means that the sum of any two real numbers will result in a real number. If a and b are real numbers and $a+b=c$, then c is a real number.

Associative Property

The associative property of addition says that the order in which three or more real numbers are added not affect the result. The result will always be the same real number. If a,b and c are real numbers, then $(a+b)+c=a+(b+c)$.

Additive Identity

Zero is known as the **additive identity** or the identity element of addition. If zero is added to any real number the answer is always the real number. If a is a real number, then $a+0=a$.

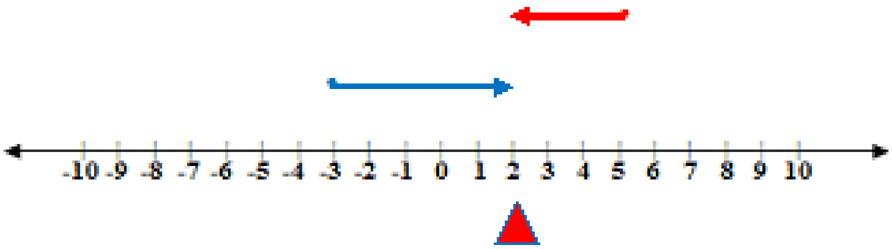
Additive Inverse

Each real number a has an **additive inverse** called $-a$.The sum of any real number and its additive inverse is zero. If a is a real number, then $a+(-a)=0$.

Example A

Use a number line to show that $(5)+(-3)=(-3)+(5)$.

Solution: On a number line, you add a positive number by moving to the right on the number line and you add a negative number by moving to the left on the number line.



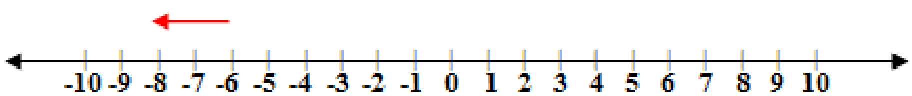
$(5)+(-3)=+2$ *The red dot is placed at +5. Then the (-3) is added by moving three places to the left. The result is +2.*

$(-3)+(5)=+2$ *The blue dot is placed at -3. Then the (+5) is added by moving five places to the right. The result is +2.*

Example B

Is $(-6)+(-2)$ a real number?

Solution:



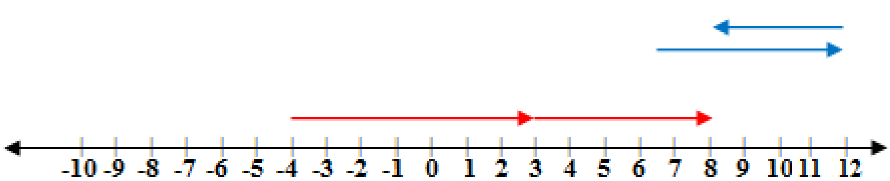
-6 and -2 are integers, so they are also real numbers. Even without calculating the result, the closure property tells us that their sum must also be a real number. In this case, the result is -8.

Example C

Does $(-4+7)+5=-4+(7+5)$?

Solution:

This is an example of the associative property. Without calculating, the property tells us that the result should be the same on both sides. But let's check to make sure:



$(-4+7)+5=$ *The red dot is placed at -4. Then the (+7) is added by moving seven places to the left. Then (+5) is added by moving five places to the right. The result is +8.*

$-4+(7+5)=$ *The blue dot is placed at +7. Then the (+5) is added by moving five places to the left. Then (-4) is added by moving four places to the left. The result is +8.*

$(-4+7)+5=-4+(7+5)$

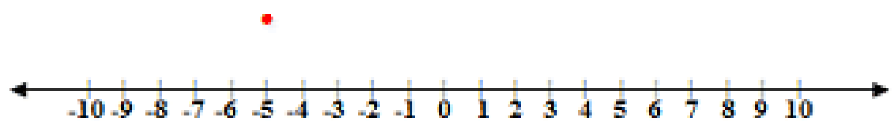
The numbers in the problem were the same, but were grouped differently. The answer was the same in both cases.

Example D

Does $(-5)+0=-5$?

Solution:

Since 0 is the additive identity, it does not change any number it gets added to. Hence $(-5)+0$ should be -5. Let's check:



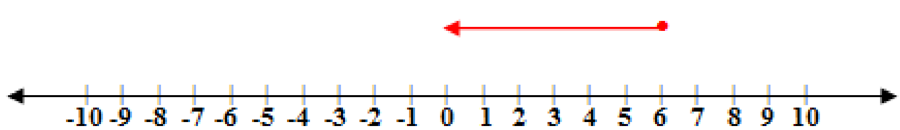
$(-5)+0=-5$ *The red dot is placed at -5. If zero is being added to the number, there is no movement to the right and no movement to the left. Therefore the result is -5.*

Example E

Does $(+6)+(-6)=0$?

Solution:

The additive inverse of 6 is -6, so when we add them the result should be 0. Let's check:



$(+6)+(-6)=0$ *The red dot is placed at +6. Then the (-6) is added by moving six places to the left. The result is 0.*

Concept Problem Revisited

Think back to the question about putting on socks. The order in which you put on the socks does not affect the outcome - you have one sock on each foot.

This is like the commutative property in algebra. The order in which you add two real numbers does not change the result.

Guided Practice

1. Add using the properties of addition: $-1.6+4.2+1.6$

2. What property justifies the statement? $(-21+6)+8=-21+(6+8)$

3. Apply the commutative property of addition to the following problem. $17x-15y$

Answers:

1. $-1.6+4.2+1.6=-1.6+1.6+4.2=0+4.2=4.2$

2. associative property

3. This is the same as $-15y+17x$.

Practice

Match the following addition statements with the correct property of addition.

1. $(-5)+5=0$

2. $(-16+4)+5=-16+(4+5)$

3. $-9+(-7)=-16$

4. $45+0=45$

5. $9+(-6)=(-6)+9$

a) Commutative Property

b) Closure Property

c) Inverse Property

d) Identity Property

e) Associative Property

Add the following using the properties of addition:

6. $24+(-18)+12$
7. $-21+34+21$
8. $5+\left(-\frac{2}{5}\right)+\left(-\frac{3}{5}\right)$
9. $19+(-7)+9$
10. $8+\frac{3}{7}+\left(-\frac{3}{7}\right)$

Name the property of addition that is being shown in each of the following addition statements:

11. $(-12+7)+10=-12+(7+10)$
12. $-18+0=-18$
13. $16.5+18.4=18.4+16.5$
14. $52+(-75)=-23$
15. $(-26)+(26)=0$

1.8 Properties of Real Number Multiplication

Concept Problem

Does $(-2)\cdot(-3)$ give the same result as $(-3)\cdot(-2)$?

Guidance

There are five properties of multiplication that are important for you to know. These properties are analogous to the properties of addition discussed in the last section.

Commutative Property

The commutative property of multiplication states that the order in which two numbers are multiplied does not affect the sum. If a and b are real numbers, then $a\cdot b = b\cdot a$.

Closure Property

The real numbers are closed under the operation of multiplication. This means that the product of any two real numbers will result in a real number. If a and b are real numbers and $a\cdot b = c$ then c is a real number.

Associative Property

The associative property of multiplication says that the order in which three or more real numbers are grouped for multiplication will not affect the product. The result will always be the same real number. If a,b and c are real numbers, then $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.

Multiplicative Identity

The number 1 is called the **multiplicative identity** or the identity element of multiplication. When any real number is multiplied by the number one, the real number does not change. If a is a real number, then $a\cdot 1 = a$.

Multiplicative Inverse

Each nonzero real number a has a **multiplicative inverse** (also called its **reciprocal**) called $\frac{1}{a}$. The product of any nonzero real number and its reciprocal is the number one. If a is a nonzero real number, then $a\cdot\frac{1}{a} = 1$.

Example A

Does $(-3)\cdot(+2) = (+2)\cdot(-3)$?

Solution: $(-3)\cdot(+2) = (+2)\cdot(-3) = -6$.

Yes. This is an example of the commutative property of multiplication.

Example B

Is $(-6)\cdot(+3)$ a real number?

Solution: Yes. Since -6 and 3 are real numbers, $(-6)\cdot(+3)$ must be a real number. This is an example of the closure property of multiplication.

Example C

Does $(-3\cdot 2)\cdot 2=-3\cdot(2\cdot 2)$?

Solution: $(-3\cdot 2)\cdot 2=-3\cdot(2\cdot 2)=-12$. Even though the numbers are grouped differently, the result is the same. This is an example of the associative property of multiplication.

Example D

Does $8\cdot 1=8$?

Solution: Yes. This is an example of the identity property of multiplication.

Example E

Does $7\cdot\frac{1}{7}=1$?

Solution: Yes. This is an example of the inverse property of multiplication.

Concept Problem Revisited

$(-2)\cdot(-3)=6$ and $(-3)\cdot(-2)=6$.

The order in which you multiplied the numbers did not affect the answer. This is an example of the commutative property of multiplication.

Guided Practice

1. Multiply using the properties of multiplication: $\left(6\cdot\frac{1}{6}\right)\cdot(3\cdot-1)$
2. What property of multiplication justifies the statement $(-9\cdot 5)\cdot 2=-9\cdot(5\cdot 2)$?
3. What property of multiplication justifies the statement $-176\cdot 1=-176$?

Answers:

1. $\left(6\cdot\frac{1}{6}\right)\cdot(3\cdot-1)=\frac{6}{6}\cdot-3=1\cdot-3=-3$
2. associative property of multiplication
3. identity property of multiplication

Practice

Match the following multiplication statements with the correct property of multiplication.

1. $9 \cdot \frac{1}{9} = 1$

2. $(-7 \cdot 4) \cdot 2 = -7 \cdot (4 \cdot 2)$

3. $-8 \cdot (4) = -32$

4. $6 \cdot (-3) = (-3) \cdot 6$

5. $-7 \cdot 1 = -7$

a) Commutative Property

b) Closure Property

c) Inverse Property

d) Identity Property

e) Associative Property

In each of the following, circle the correct answer.

6. What does $-5(4)\left(-\frac{1}{5}\right)$equal?
   a. -20
   b. -4
   c. +20
   d. +4
7. What is another name for the reciprocal of any real number?
   a. the additive identity
   b. the multiplicative identity
   c. the multiplicative inverse
   d. the additive inverse
8. What is the multiplicative identity?
   a. -1
   b. 1
   c. 0
   d. $\frac{1}{2}$
9. What is the product of a nonzero real number and its multiplicative inverse?
   a. 1
   b. -1
   c. 0
   d. there is no product
10. Which of the following statements is NOT true?
    a. The product of any real number and negative one is the opposite of the real number.
    b. The product of any real number and zero is always zero.
    c. The order in which two real numbers are multiplied does not affect the product.
    d. The product of any real number and negative one is always a negative number.

Name the property of multiplication that is being shown in each of the following multiplication statements:

11. $(-6 \cdot 7) \cdot 2 = -6 \cdot (7 \cdot 2)$

12. $-12 \cdot 1 = -12$
13. $25 \cdot 3 = 3 \cdot 25$
14. $10 \cdot \frac{1}{10} = 1$
15. $-12 \cdot 3 = -36$

Answers for Practice Problems

To view the practice problem answers, open this PDF file and look for section 1.10.

1.9 Order of Operations with Real Numbers

Concept Problem

Rosa walked into Math class and saw the following question on the board.

$$\boxed{6+12 \div 2 \times 3+1}$$

Her teacher, Ms. Black, directed the class to evaluate the mathematical expression. When the students had completed the task, Ms. Black then asked four students to put their work on the board. Here are the results:

$$\begin{aligned}6+12 \div 2 \times 3+1 &= 18+2 \cdot 3+1\\ &= 9 \cdot 3+1\\ &= 27+1\\ &= 28\end{aligned}$$

$$\begin{aligned}6+12 \div 2 \times 3+1 &= 18+2 \cdot 4\\ &= 6+6 \cdot 4\\ &= 6+24\\ &= 30\end{aligned}$$

$$\begin{aligned}6+12 \div 2 \times 3+1 &= 6+6 \cdot 3+1\\ &= 12 \cdot 3+1\\ &= 36+1\\ &= 37\end{aligned}$$

$$\begin{aligned}6+12 \div 2 \times 3+1 &= 6+6 \cdot 3+1\\ &= 6+18+1\\ &= 24+1\\ &= 25\end{aligned}$$

Which answer is correct?

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Khan Academy Introduction to Order of Operations

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/5257

Guidance

$$\boxed{6+12\div 2\times 3+1}$$

To avoid confusion in evaluating mathematical expressions like the one shown above, mathematicians have adopted a standard order of operations for arithmetic calculations. This adopted standard of operations consists of the following rules:

1. Evaluate any expressions shown inside **parentheses** first.
2. Evaluate **exponents**.
3. Perform all **multiplication** and **division**, in the order they occur, working from left to right.
4. Perform all **addition** and **subtraction**, in the order they occur, working from left to right.

If you look at the first letters of the bolded words, you will see that they form the acronym **PEMDAS** - parentheses, exponents, multiplication, division, addition, subtraction. PEMDAS serves as a method for you to remember the order in which to perform the arithmetic calculations.

Example A

Perform the following calculations, using PEMDAS.

$360\div(18+6\cdot 2)-2$

Solution: When performing the calculations in parentheses, follow the rules for order of operations.

$360\div(18+12)-2$

When the calculations in parentheses have been completed, the parentheses are no longer necessary.

$360\div 30-2$

There are no exponents in this problem. The next step is to perform the division.

$12-2$

The final step is to subtract 2 from 12. The final answer is 10.

$=10$

Example B

$\left(\frac{3}{4}+\frac{1}{6}\right)\cdot(5\cdot 3^2-5)$

Solution: There are two sets of parentheses. Work from left to right in the first set of parentheses.

$\frac{11}{12}\cdot(5\cdot 3^2-5)$

$\frac{11}{12}\cdot(5\cdot 9-5)$

$\frac{11}{12} \cdot (45-5)$

$\frac{11}{12} \cdot 40$

$= \frac{110}{3}$

Example C

$(1+6)^2 - \frac{2+4 \cdot 12}{18-4 \cdot 2} + (72 \div 8)$

Solution: To start, add the numbers in the first parentheses.

$(7)^2 - \frac{2+4 \cdot 12}{18-4 \cdot 2} + (72 \div 8)$

$49 - \frac{2+4 \cdot 12}{18-4 \cdot 2} + (72 \div 8)$

$49 - \frac{2+4 \cdot 12}{18-4 \cdot 2} + 9$

Remember that the line of a fraction means divide. Before the division can be completed, you must obtain an answer for the calculations in the numerator and in the denominator. PEMDAS must be applied when doing the calculations.

$49 - \frac{2+48}{18-8} + 9$

$49 - \frac{50}{10} + 9$

$49 - 5 + 9$

$= 53$

Example D

$6.12 + 8.6 \cdot 0.9 - (10.26 \div 3.8)$

Solution:

$$\begin{aligned}
&= 6.12 + 8.6 \cdot 0.9 - 2.7 \\
&= 6.12 + 7.74 - 2.7 \\
&= 13.86 - 2.7 \\
&= = 11.16
\end{aligned}$$

Example E

If $m = 2$ and $n = 3$, evaluate $m^2 + 3n - 7$.

Solution: The first step is to substitute the values into the given statement.

$m^2 + 3n - 7$

$= (2)^2 + 3 \cdot 3 - 7$

$$\begin{aligned}
&= 4 + 3 \cdot 3 - 7 \\
&= 4 + 9 - 7 \\
&= 13 - 7 \\
&= 6
\end{aligned}$$

Concept Problem Revisited

The last solution is correct.

$$\begin{aligned}6+12\div 2\cdot 3+1 &= 6+6\cdot 3+1\\ &= 6+18+1\\ &= 24+1\\ &= 25\end{aligned}$$

Ms. Black could have minimized the confusion by writing the statement with parentheses.

$$\boxed{6+((12\div 2)\times 3)+1}$$

Guided Practice

1. Perform the following operations using PEMDAS: $8\times 9+19\div(30-11)-6$

2. A remodeling job requires 132 square feet of countertops. Two options are being considered. The more expensive option is to use all Corian at \$66 per sq ft. The less expensive option is to use 78 sq ft of granite at \$56 per sq ft and 54 sq ft of laminate at \$23 per sq ft. Write a mathematical statement to calculate the difference in cost between the more expensive option and the less expensive option. What is the cost difference?

3. Determine the answer to $\frac{12+6}{6+3}+\frac{36}{4}-(12\div 12)$ by using the rules for the standard order of operations.

Answers:

1.

$$\begin{aligned}&8\cdot 9+19\div(30-11)-6\\ &8\cdot 9+19\div 19-6\\ &72+19\div 19-6\\ &72+1-6\\ &73-6\\ &=67\end{aligned}$$

2. The first option is \$3102 more than the second option.

$$\begin{aligned}&(132\cdot \$66)-(78\cdot \$56+54\cdot \$23)\\ &\$8712-(78\cdot \$56+54\cdot \$23)\\ &\$8712-(\$4368+\$1242)\\ &\$8712-\$5610\\ &=\$3102\end{aligned}$$

3.

$$\frac{12+6}{6+3}+\frac{36}{4}-(12\div 12)$$
$$\frac{12+6}{6+3}+\frac{36}{4}-1$$
$$\frac{12+6}{6+3}+9-1$$
$$\frac{18}{9}+9-1$$
$$2+9-1$$
$$11-1$$
$$=10$$

Problem Set

Perform the indicated calculations, using PEMDAS to determine the answer.

1. $\frac{4^2(8+7)}{6}$
2. $\frac{2\cdot 6}{4}(5-2)$
3. $\frac{15\cdot 3}{5}+4(7\cdot 1)-2\cdot 3$
4. $4+27\div 3\cdot 2-6$
5. $7^2-3\cdot 2^3-5$

For each of the following problems write a single mathematical statement to represent the problem. Then use the statement to determine the answer.

6. At the beginning of the day on Monday, the cafeteria has 520 tortilla wraps. The supervisor estimates that she will need 68 wraps each day. A new shipment of 300 wraps will arrive on Thursday. Calculate the number of wraps she will have at the end of the day on Friday.
7. The students enrolled in the masonry course are estimating the cost for building a stone wall and gate. They estimate that the job will require 40 hours to complete. They will need the services of two laborers and they will be paid $12 per hour. They will also need three masons who will be paid $16 per hour. The cost of the materials is $2140. What is the estimated cost of the job?
8. Mrs. Forsythe purchased 15 scientific calculators at $19 each and received $8 credit for each of the seven regular calculators that she returned. How much money did she spend to buy the scientific calculators?
9. A landscaper charged a customer $472 for labor and $85 each for eight flats of Bedding plants. What was the total cost of the job?
10. A painter had a one hundred dollar bill when he went to the hardware store to purchase supplies for a job. He bought 2 quarts of white latex paint for $8 a quart and 4 gallons of white enamel paint for $19 a gallon. How much change did he receive?

If $a=2, b=3$ and $c=5$, evaluate, using PEMDAS to determine the answer.

11. $6a-3b+4c$
12. $2a^2-3a+b^2$
13. $3ac-2ab+bc$
14. $a^2+b^2+c^2$
15. $3a^2(4c-3b)$

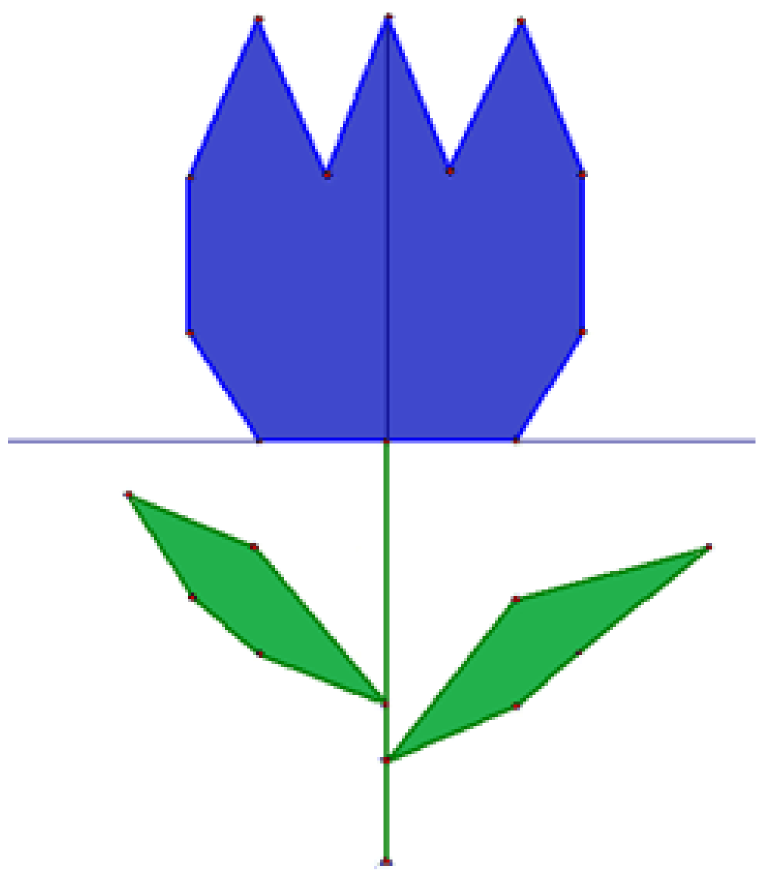
1.10 The Cartesian Plane

Learning Objectives

Here you will learn about the Cartesian plane and review how to plot points on the Cartesian plane.

Concept Problem

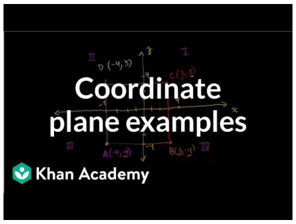
Kaitlyn walked into Math class and saw the following image displayed on the board. Her teacher asked everyone in the class to duplicate the picture on a blank sheet of paper.



When the teacher felt that the students had completed the drawing, she asked them to share their results with the class. Most of the students had difficulty reproducing the picture. Kaitlyn told the class that she could not make the picture the same size as the one shown. She also said that she had a problem locating the leaves in the same places on the stem. Her teacher said that she could offer a solution to these problems. What was the solution?

Watch This

Khan Academy The Coordinate Plane



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/74

Guidance

The ***Cartesian plane*** is a system of four areas or quadrants produced by the perpendicular intersection of two number lines. The two number lines intersect at right angles. The point of intersection is known as the ***origin***. One number line is a horizontal line and this is called the ***x-axis***. The other number line is a vertical line and it is called the ***y-axis***. The two number lines are referred to as the ***axes*** of the Cartesian plane. The Cartesian plane, also known as the ***coordinate plane***, has four quadrants that are labeled counterclockwise.

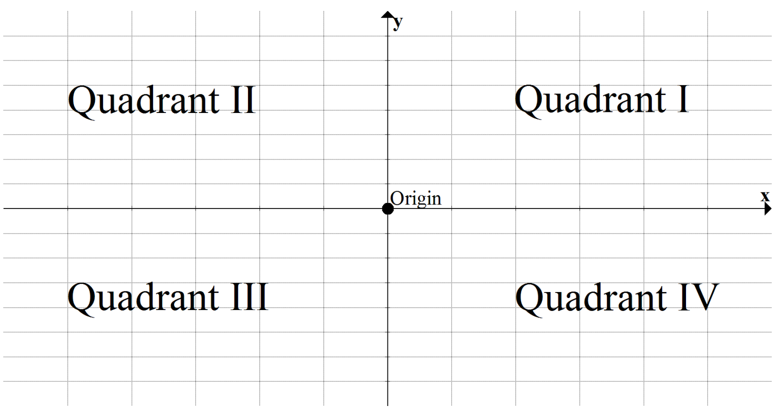


FIGURE 1.1

The value of the origin on the x-axis is zero. If you think of the x-axis as a number line, the numbers to the right of zero are positive values, and those to the left of zero are negative values. The same can be applied to the y-axis. The value of the origin on the y-axis is zero. The numbers above zero are positive values and those below zero are negative values.

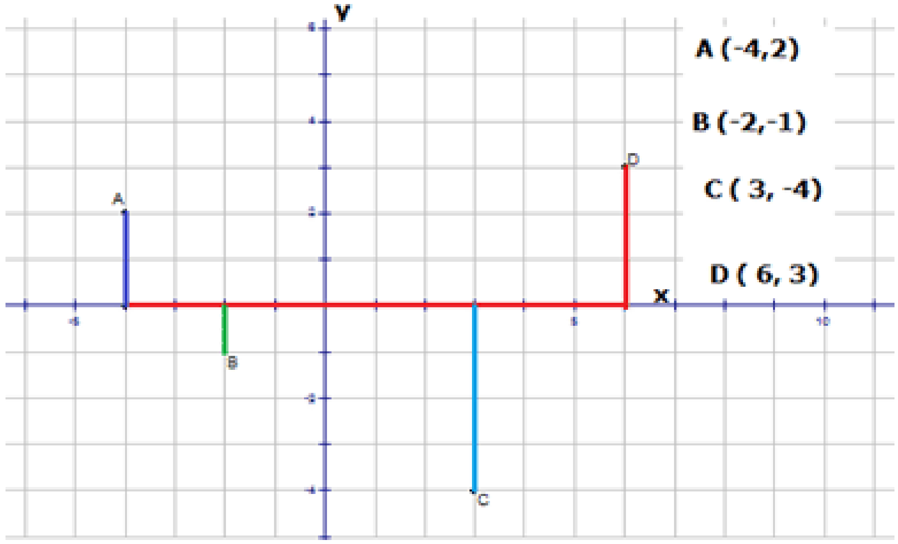
Every point that is plotted on a Cartesian plane has two values associated with it. The first value represents the x-value and the second value represents the y-value. These two values are called the ***coordinates*** of the point and are written as the ordered pair (x, y).

To plot a point on the Cartesian plane:

- Start at zero (the origin) and locate the $x-$coordinate on the x-axis.
- If the $x-$coordinate is positive, move to the right of the origin the number of units displayed by the number. If the $x-$coordinate is negative, move to the left of the origin the number of units displayed by the number.
- Once the $x-$coordinate (also called the ***abscissa***) has been located, move vertically the number of units displayed by the $y-$coordinate (also called the ***ordinate***). If the $y-$coordinate is positive, move vertically upward from the $x-$coordinate, the number of units displayed by the $y-$coordinate. If the $y-$coordinate is negative, move vertically downward from the $x-$coordinate, the number of units displayed by the $y-$coordinate.

- The point can now be plotted.

Examine the points A, B, C and D that have been plotted on the graph below.



- $A(-4,2)$ - From the origin, move to the left four units (along the red line on the x-axis). Now, move vertically upward two units. Plot the point A.
- $B(-2,-1)$ - From the origin, move to the left two units (along the red line on the x-axis). Now, move vertically downward one unit. Plot the point B.
- $C(3,-4)$ - From the origin, move to the right three units (along the red line on the x-axis). Now, move vertically downward four units. Plot the point C.
- $D(6,3)$ - From the origin, move to the right six units (along the red line on the x-axis). Now, move vertically upward three units. Plot the point D.

Example A

For each quadrant, say whether the values of x and y are positive or negative.

Solution: The graph below shows where x and y values are positive and negative.

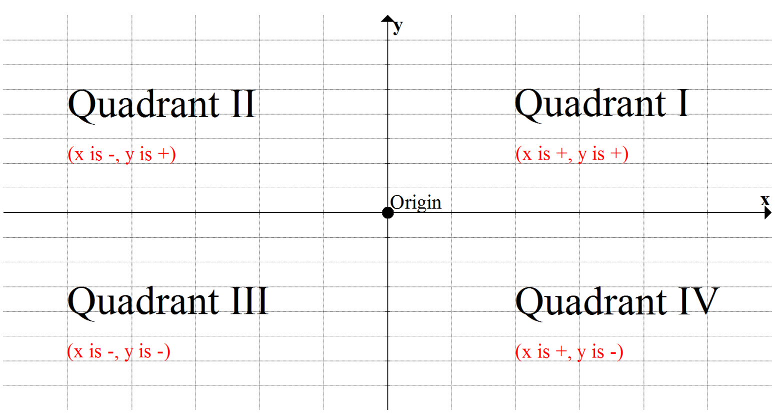


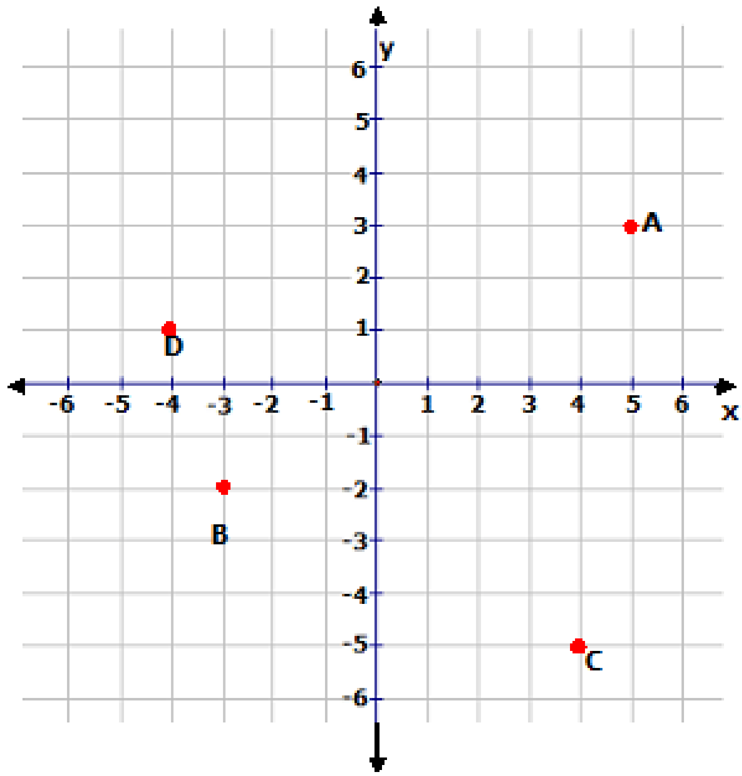
FIGURE 1.2

Example B

On a Cartesian plane, plot the following points:

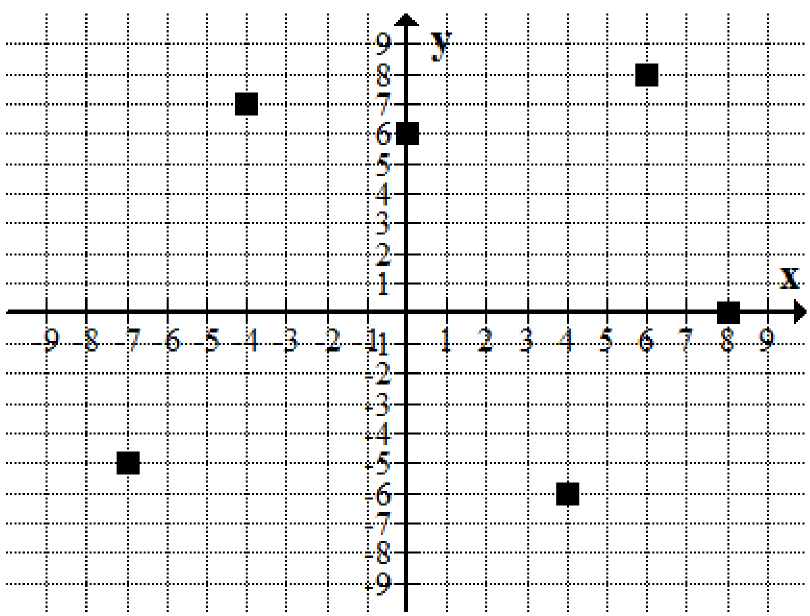
$A(5,3) \quad B(-3,-2) \quad C(4,-5) \quad D(-4,1)$

Solution:

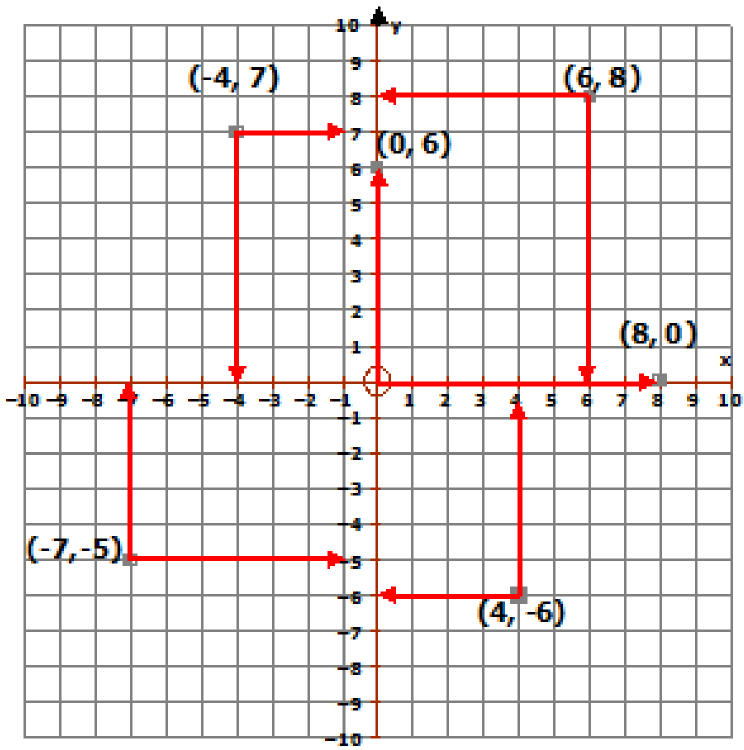


Example C

Determine the coordinates of each of the plotted points on the following graph.



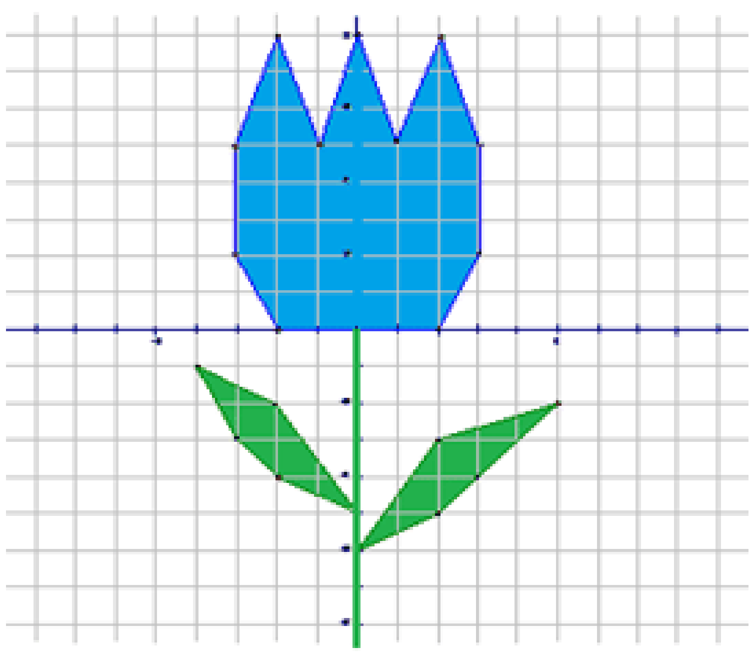
Solution:



Concept Problem Revisited

Now, let us return to the beginning of the lesson to find out the solution that the teacher had for the students.

If the teacher lets the students see the picture on a Cartesian plane, the reproduction process should be much easier.



Guided Practice

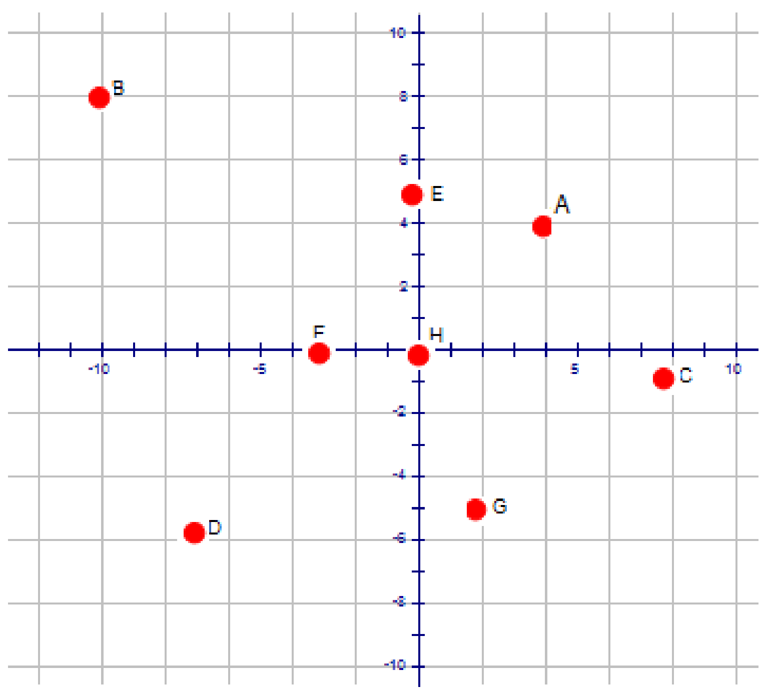
1. Draw a Cartesian plane that displays only positive values. Number the x and y axes to twelve. Plot the following coordinates and connect them in order. Use a straight edge to connect the points. When the word “STOP” appears, begin the next line. Plot the points in the order they appear in each Line row.

LINE 1 (6, 0) (8, 0) (9, 1) (10, 3) (10, 6) (9, 8) (7, 9) (5, 9) **STOP**
LINE 2 (6, 0) (4, 0) (3, 1) (2, 3) (2, 6) (3, 8) (5, 9) **STOP**
LINE 3 (7, 9) (6, 12) (4, 11) (5, 9) **STOP**
LINE 4 (4, 8) (3, 6) (5, 6) (4, 8) **STOP**
LINE 5 (8, 8) (7, 6) (9, 6) (8, 8) **STOP**
LINE 6 (5, 5) (7, 5) (6, 3) (5, 5) **STOP**
LINE 7 (3, 2) (4, 1) (5, 2) (6, 1) (7, 2) (8, 1) (9, 2) **STOP**
LINE 8 (4, 1) (6, 1) (8, 1) **STOP**

2. In which quadrant would the following points be located?

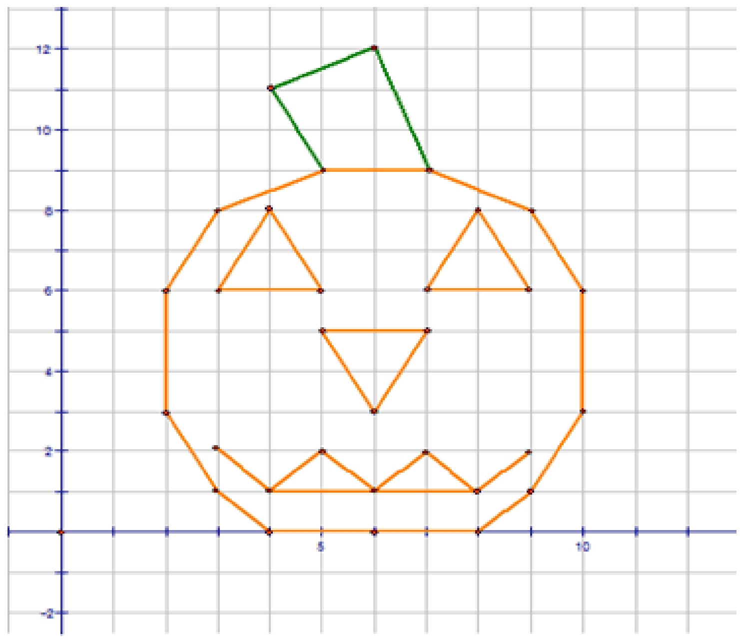
i) (3, -8)
ii) (-5, 4)
iii) (7, 2)
iv) (-6, -9)
v) (-3, 3)
vi) (9, -7)

3. State the coordinates of the points plotted on the following Cartesian plane.



Answers:

1. The following picture is the result of plotting the coordinates and joining them in the order in which they were plotted. Your pumpkin can be any color you like.



2. i) (3, -8) - the x coordinate is positive and the $y-$coordinate is negative. This point will be located in Quadrant IV.

ii) (-5, 4) - the x coordinate is negative and the $y-$coordinate is positive. This point will be located in Quadrant II.
iii) (7, 2) - the x coordinate is positive and the $y-$coordinate is positive. This point will be located in Quadrant I.
iv) (-6, -9) - the x coordinate is negative and the $y-$coordinate is negative. This point will be located in Quadrant III.
v) (-3, 3) - the x coordinate is negative and the $y-$coordinate is positive. This point will be located in Quadrant II.
vi) (9, -7) - the x coordinate is positive and the $y-$coordinate is negative. This point will be located in Quadrant IV.

3. $A(4,4)$ $B(-10,8)$ $C(8,-1)$ $D(-6,-6)$ $E(0,5)$ $F(-3,0)$ $G(2,-5)$ $H(0,0)$

Practice

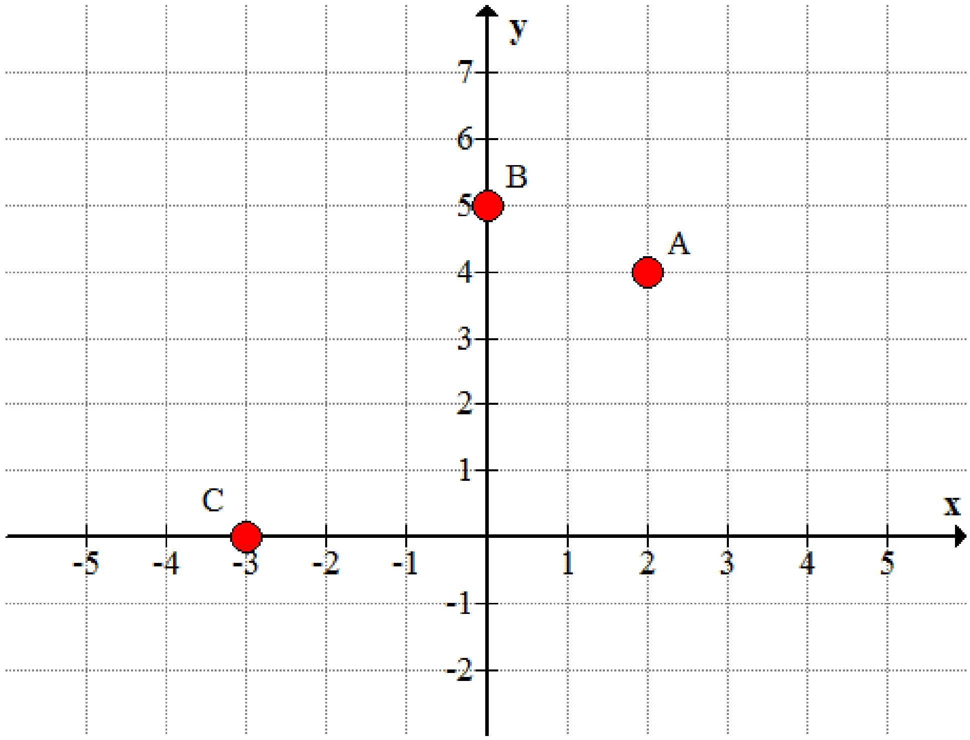
Answer the following questions with respect to the Cartesian plane:

1. What name is given to the horizontal number line on the Cartesian plane?
2. What name is given to the four areas of the Cartesian plane?
3. What are the coordinates of the origin?
4. What name is given to the vertical number line on the Cartesian plane?
5. What other name is often used to refer to the $x-$coordinate of a point on the Cartesian plane?

On a Cartesian plane, plot each of the following points. For each point, name the quadrant it is in or axis it is on.

6. $(2,0)$
7. $(-3,1)$
8. $(0,4)$
9. $(1,-2)$
10. $(5,5)$

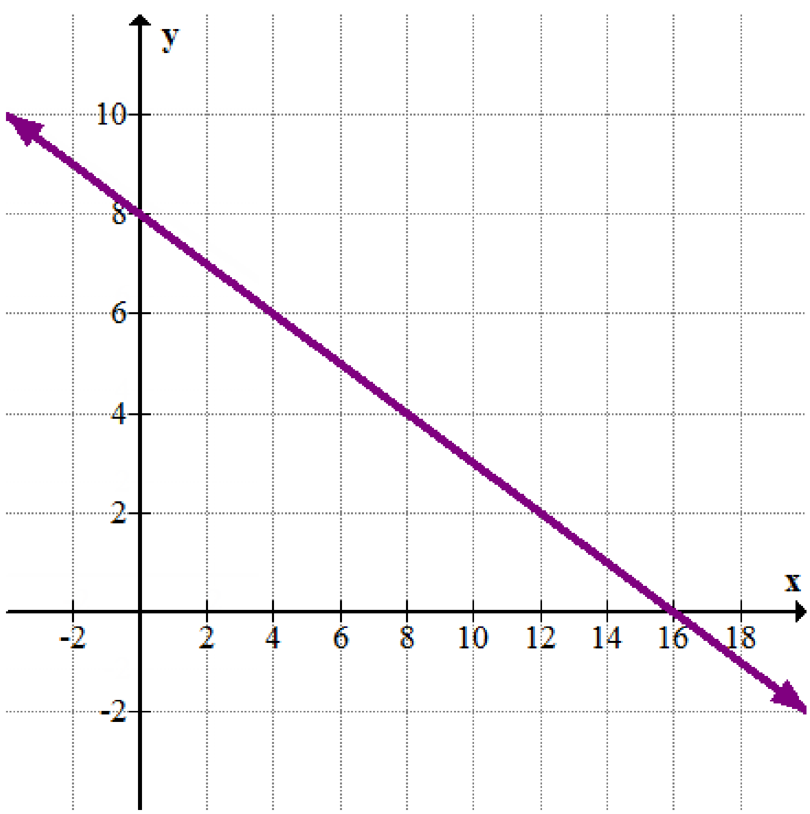
Use the graph below for #11-#13.



11. The coordinates of point A are ______.
12. The coordinates of point B are ______.
13. The coordinates of point C are ______.

For each of the following graphs, select three points on the graph and state the coordinates of these points.

14.



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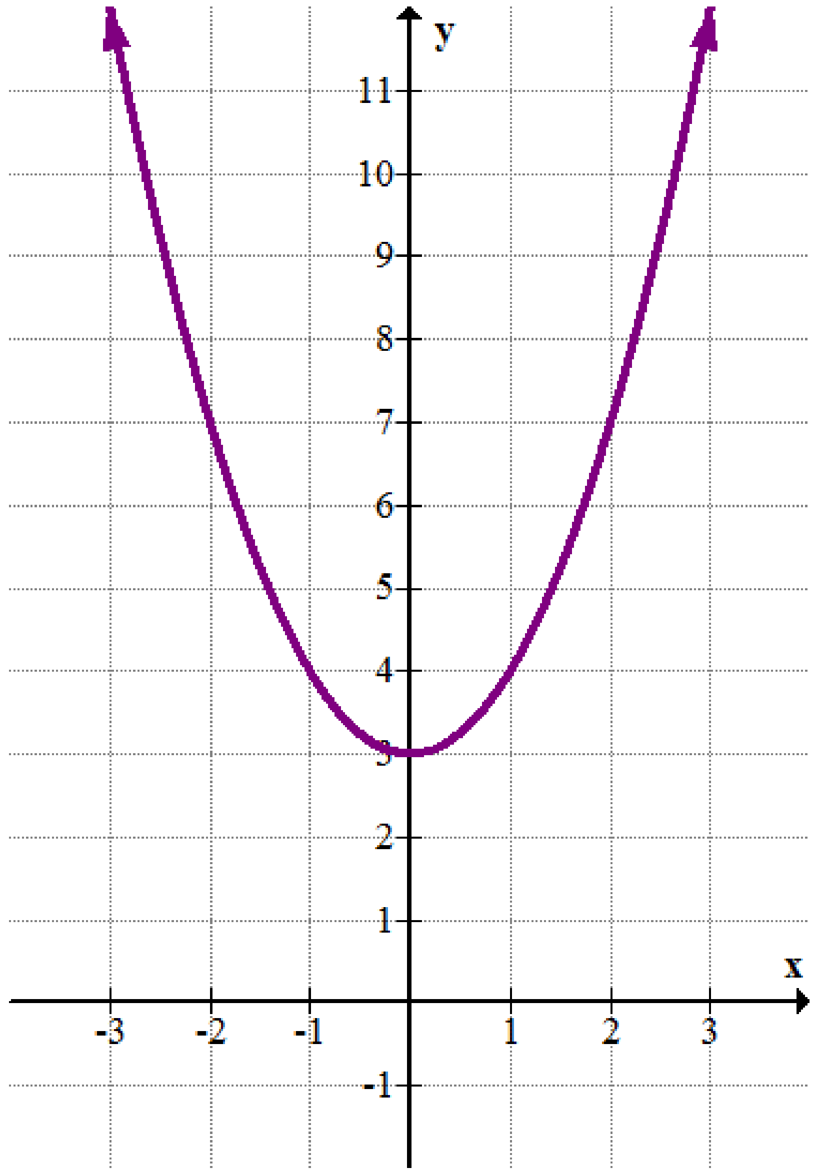
15.



1.11 Equations with Variables on Both Sides

Learning Objectives

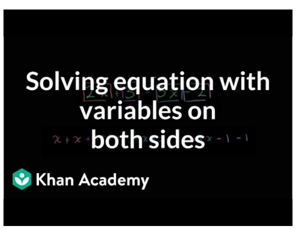
Here you will learn to solve equations where there are variables on both sides of the equals sign.

Concept Problem

Thomas has \$50 and Jack has \$100. Thomas is saving \$10 per week for his new bike. Jack is saving \$5 a week for his new bike. Can you represent their funds with equations? How long will it be before the two boys have the same amount of money?

Watch This

Khan Academy Equations with Variables on Both Sides



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Guidance

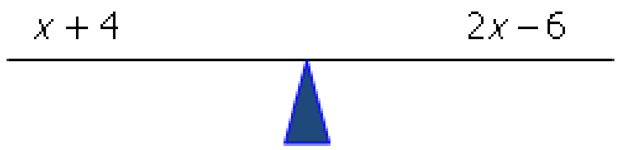
The methods used for solving equations with variables on both sides of the equation are the same as the methods used to solve equations with variables on one side of the equation. What differs is that first you must add or subtract a term from both sides in order to have the variable on only one side of the equals sign.

Remember that your goal for solving any equation is to get the variables on one side and the constants on the other side. You do this by adding and subtracting terms from both sides of the equals sign. Then you isolate the variables by multiplying or dividing. You must remember in these problems, as with any equation, whatever operation (addition, subtraction, multiplication, or division) you do to one side of the equals sign, you must do to the other side. This is a big rule to remember in order for equations to remain equal or to remain in balance.

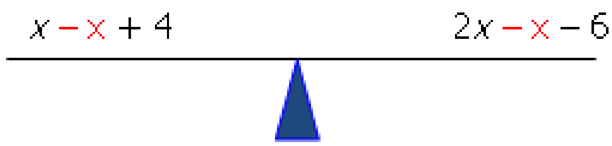
Example A

$x+4=2x-6$

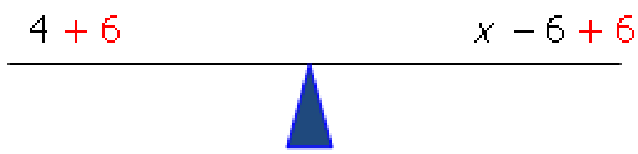
Solution: You can solve this problem using the balance method.

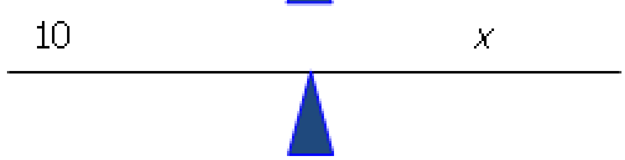


You could first try to get the variables all on one side of the equation. You do this by subtracting x from both sides of the equation.



Next, isolate the x variable by adding 6 to both sides.





Therefore $x = 10$.

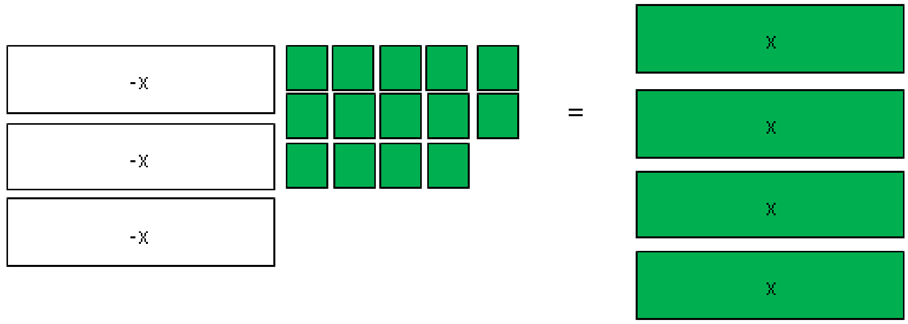
Check:

$$\begin{aligned} x+4 &= 2x-6 \\ (10)+4 &= 2(10)-6 \\ 14 &= 20-6 \\ 14 &= 14 \end{aligned}$$

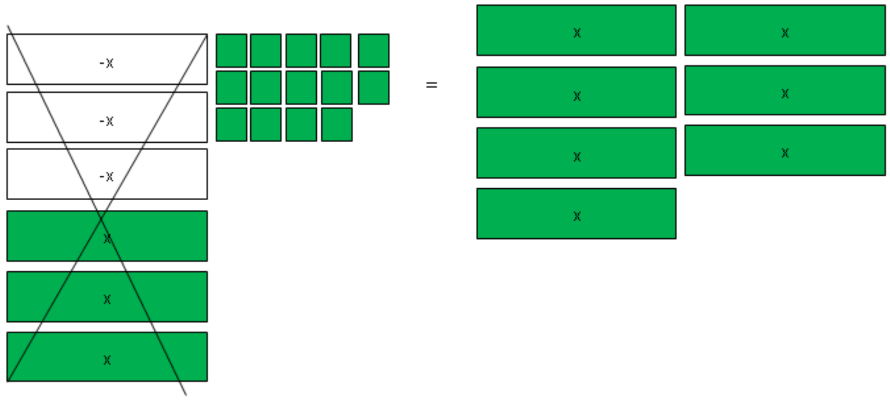
Example B

$14 - 3x = 4x$

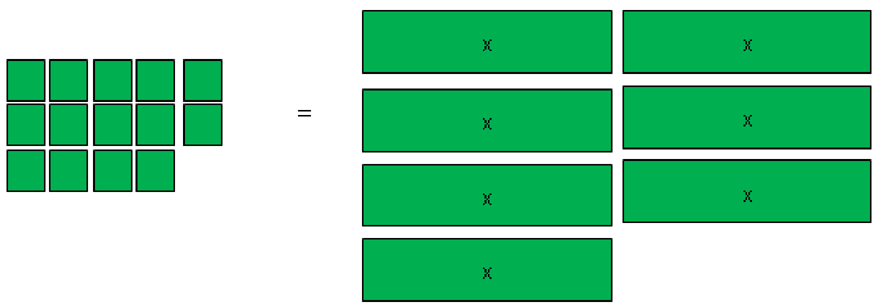
Solution: You can solve this equation using algebra tiles.



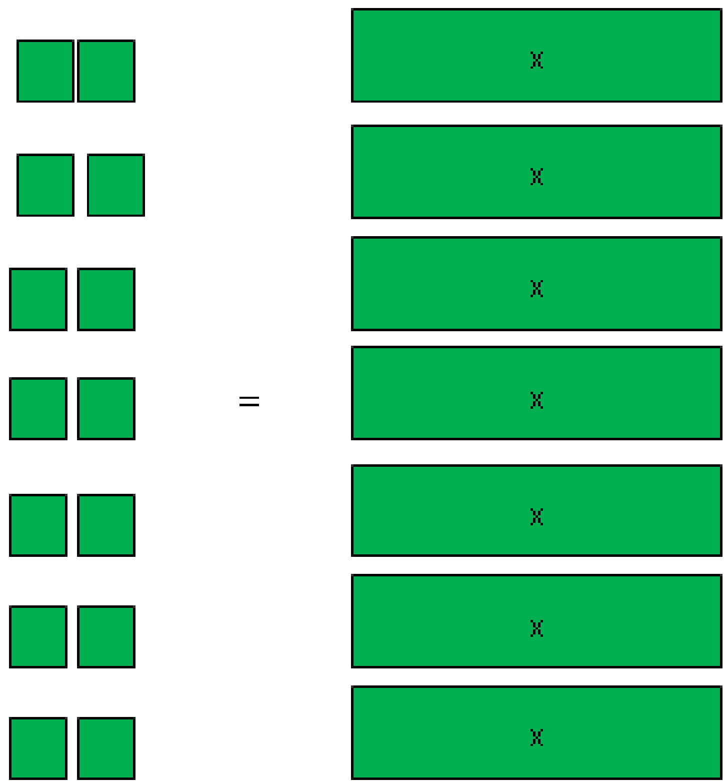
You first have to combine our variables (x) tiles onto the same side of the equation. You do this by adding $3x$ tiles to both sides of the equals sign. In this way the $-3x$ will be eliminated from the left hand side of the equation.



By isolating the variable (x) you are left with these algebra tiles.



Rearranging you will get the following.



Check:

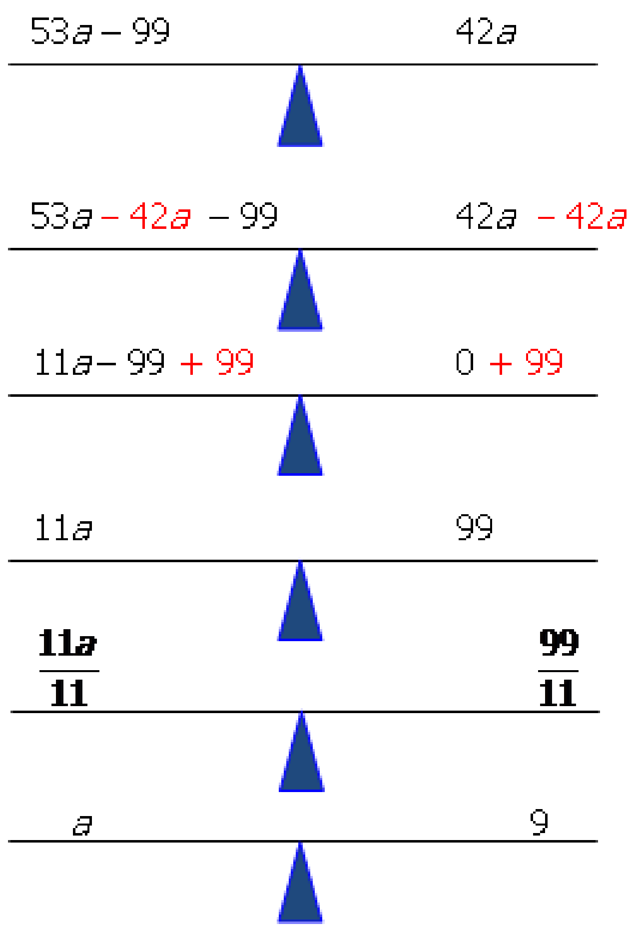
$$
\begin{aligned}
14 - 3x &= 4x \\
14 - 3(2) &= 4(2) \\
14 - 6 &= 8 \\
8 &= 8
\end{aligned}
$$

Therefore $y = 2$.

Example C

$53a - 99 = 42a$

Solution: To solve this problem, you would need to have a large number of algebra tiles! It might be more efficient to use the balance method to solve this problem.



$$\text{Check:}$$
$$53a - 99 = 42a$$
$$53(9) - 99 = 42(9)$$
$$477 - 99 = 378$$
$$378 = 378$$

Therefore, $a = 9$.

Concept Problem Revisited

Thomas has \$50 and Jack has \$100. Thomas is saving \$10 per week for his new bike. Jack is saving \$5 a week for his new bike.

If you let x be the number of weeks, you can write the following equation.

$$\underbrace{10x + 50}_{\text{Thomas's money: \$10 per week+\$50}} = \underbrace{5x + 100}_{\text{Jack's money: \$5 per week+\$100}}$$

You can solve the equation now by first combining like terms.

$$\begin{aligned}
10x+50 &= 5x+100 \\
10x-5x+50 &= 5x-5x+100 && \text{-moving the } x \text{ variables to left side of the equation} \\
5x+50-50 &= 100-50 && \text{-moving the constants to right side of the equation} \\
5x &= 50
\end{aligned}$$

You can now solve for x to find the number of weeks until the boys have the same amount of money.

$$\begin{aligned}
5x &= 50 \\
\frac{5x}{5} &= \frac{50}{5} \\
x &= 10
\end{aligned}$$

Therefore, in 10 weeks Jack and Thomas will each have the same amount of money.

Guided Practice

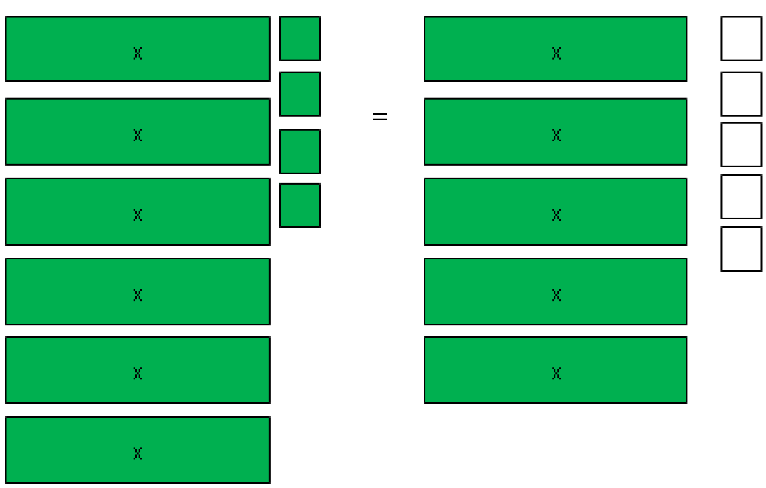
1. Solve for the variable in the equation $6x+4=5x-5$.

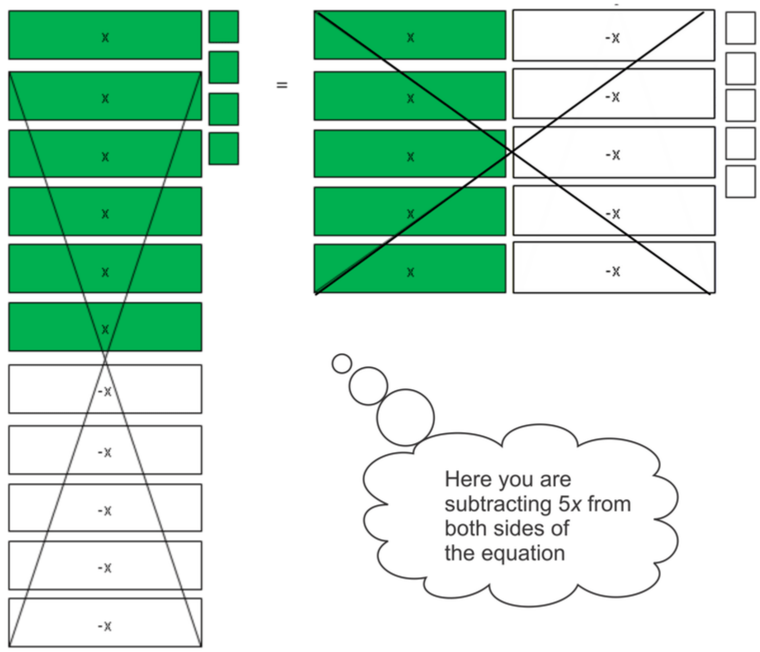
2. Solve for the variable in the equation $7r-4=3+8r$.

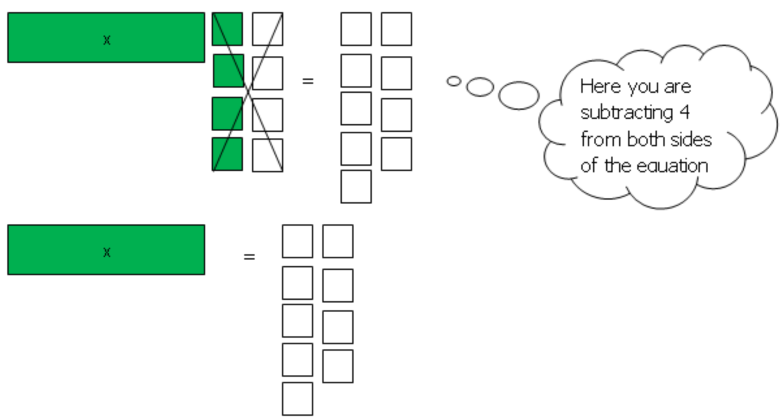
3. Determine the most efficient method to solve for the variable in the problem $10b-22=29-7b$. Explain your choice of method for solving this problem.

Answers:

1. $6x+4=5x-5$

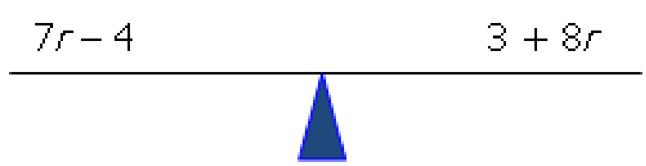




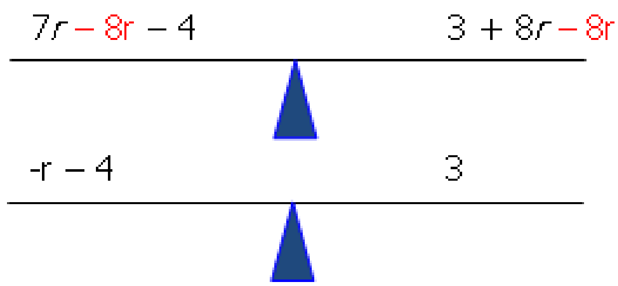


Therefore $x = -9$.

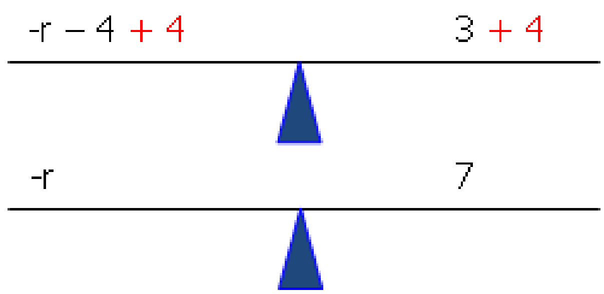
2. $7r - 4 = 3 + 8r$



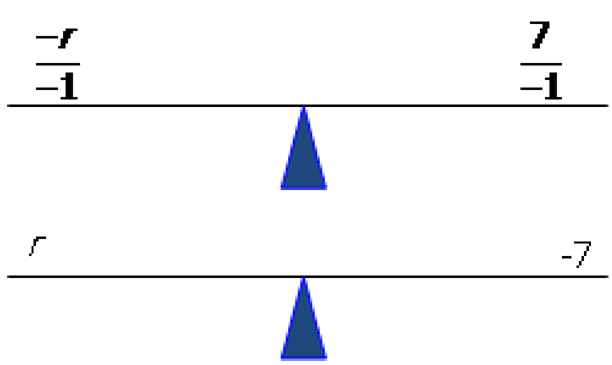
You can begin by combining the r terms. Subtract $8r$ from both sides of the equation.



You next have to isolate the variable. To do this, add 4 to both sides of the equation.

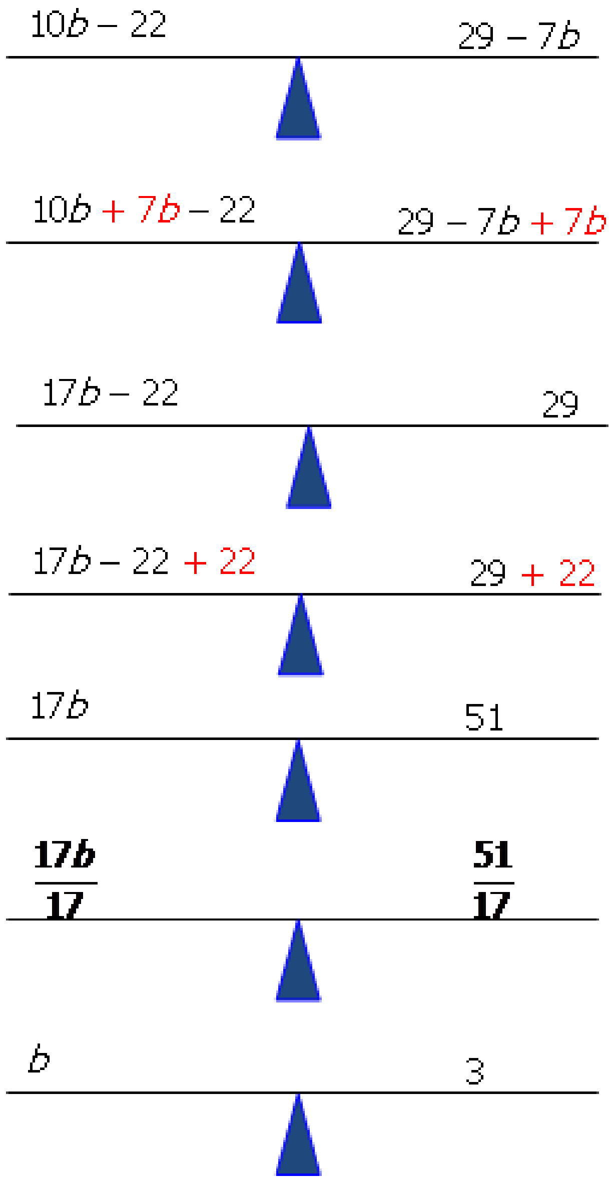


But there is still a negative sign with the r term. You now have to divide both sides by -1 to finally isolate the variable.



Therefore $r = -7$.

3. You could choose either method but there are larger numbers in this equation. With larger numbers, the use of algebra tiles is not an efficient manipulative. You should solve the problem using the balance method. Work through the steps to see if you can follow them.



Therefore $b = 3$.

Practice Problems

Use the balance method to find the solution for the variable in each of the following problems.

1. $5p+3 = -3p-5$
2. $6b-13 = 2b+3$
3. $2x-5 = x+6$
4. $3x-2x = -4x+4$
5. $4t-5t+9 = 5t-9$

Use algebra tiles to find the solution for the variable in each of the following problems.

6. $6-2d = 15-d$

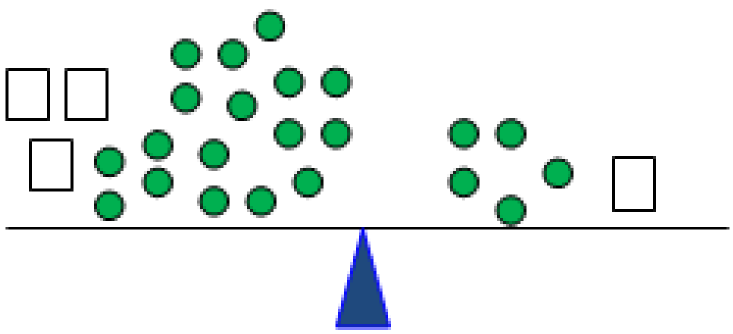
7. $8 - s = s - 6$
8. $5x + 5 = 2x - 7$
9. $3x - 2x = -4x + 4$
10. $8 + t = 2t + 2$

Use the methods that you have learned for solving equations with variables on both sides to solve for the variables in each of the following problems. Remember to choose an efficient method to solve for the variable.

11. $4p - 7 = 21 - 3p$
12. $75 - 6x = 4x - 15$
13. $3t + 7 = 15 - t$
14. $5 + h = 11 - 2h$
15. $9 - 2e = 3 - e$

For each of the following models, write a problem to represent the model and solve for the variable for the problem.

16.



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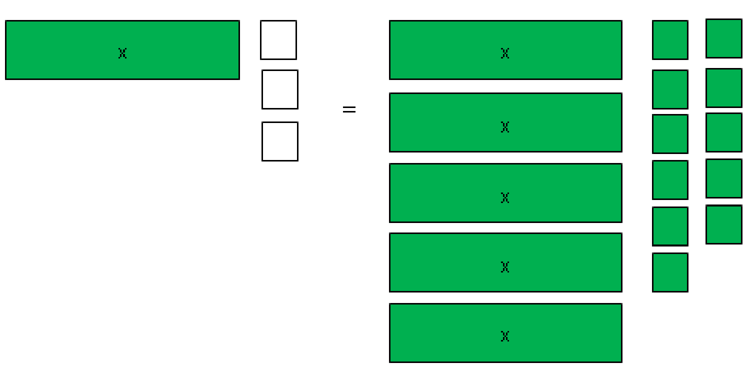

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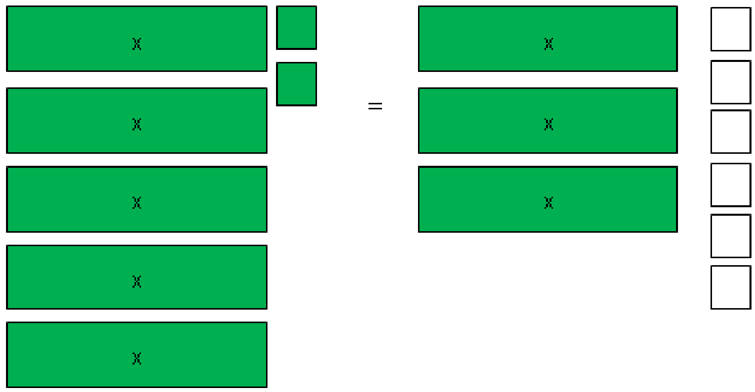

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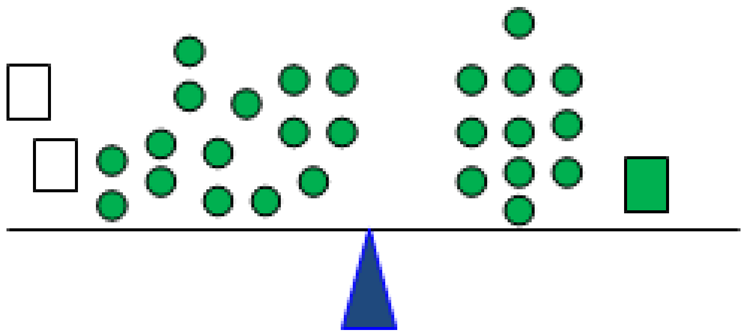

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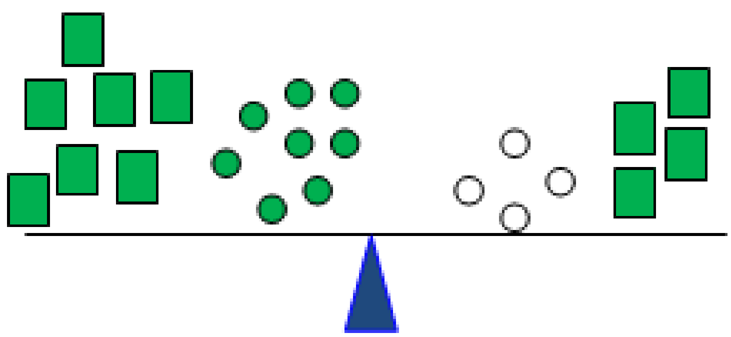


1.12 Review of Exponential Expressions

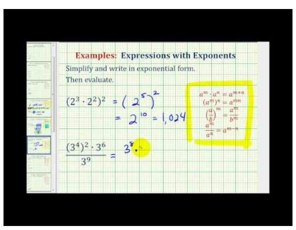
Concept Problem

Can you simplify the following expression so that it has only positive exponents?

$$\frac{8x^3y^{-2}}{(-4a^2b^4)^{-2}}$$

Watch This

James Sousa: Simplify Exponential Expressions



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Guidance

In the expression a^n, a is called the **base** and n is called the **exponent**. We think of an expression like this as telling us how many times the number a is multiplied by itself i.e.

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \ldots}_{\text{n times}}$$

You have probably dealt with exponents at different points in your mathematical coursework. We will be examining them in more detail later, but here is a quick review of their properties.

Laws of Exponents

If a and b are real numbers and m and n are integers, then...

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$ (if $m > n, a \neq 0$)
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$
6. $a^0 = 1$ $(a \neq 0)$
7. $a^{-m} = \frac{1}{a^m}$

Example A

Evaluate 3^{-4}.

Solution: First, rewrite with a positive exponent:

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

.

Example B

Simplify $(4x^3y)(3x^5y^2)^4$.

Solution:

$$\begin{aligned}(4x^3y)(3x^5y^2)^4 &= (4x^3y)(81x^{20}y^8)\\ &= 324x^{23}y^9\end{aligned}$$

Example C

Simplify $\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2}$.

Solution:

$$\begin{aligned}\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2} &= \left(\frac{x^4y^3}{x^{-2}y}\right)^2\\ &= (x^6y^2)^2\\ &= x^{12}y^4\end{aligned}$$

Concept Problem Revisited

$$\begin{aligned}\frac{8x^3y^{-2}}{(-4x^2y^4)^{-2}} &= (8x^3y^{-2})(-4x^2y^4)^2\\ &= (8x^3y^{-2})(16x^4y^8)\\ &= 8 \cdot 16 \cdot x^3 \cdot x^4 \cdot y^{-2} \cdot y^8\\ &= 128x^7y^6\end{aligned}$$

Guided Practice

Use the laws of exponents to simplify each of the following:

1. $(-2x)^5(2x^2)$
2. $(16x^{10})\left(\frac{3}{4}x^5\right)$

3. $\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8}$

Answers:

1. $(-2x)^5(2x^2) = (-32x^5)(2x^2) = -64x^7$

2. $(16x^{10})\left(\frac{3}{4}x^5\right) = 12x^{15}$

3. $\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8} = \frac{x^{64}}{x^{56}} = x^8$

Practice Problems

Simplify each expression.

1. $(x^{10})(x^{10})$
2. $(7x^3)(3x^7)$
3. $(x^3y^2)(xy^3)(x^5y)$
4. $\frac{(x^3)(x^2)}{(x^4)}$
5. $\frac{x^2}{x^{-3}}$
6. $\frac{x^6y^8}{x^4y^{-2}}$
7. $(2x^{12})^3$
8. $(x^5y^{10})^7$
9. $\left(\frac{2x^{10}}{3y^{20}}\right)^3$

Express each of the following as a power of 3. Do not evaluate.

10. $(3^3)^5$
11. $(3^9)(3^3)$
12. $(9)(3^7)$
13. 9^4
14. $(9)(27^2)$

Apply the laws of exponents to evaluate each of the following without using a calculator.

15. $(2^3)(2^2)$
16. $6^6 \div 6^5$
17. $-(3^2)^3$
18. $(1^2)^3 + (1^3)^2$
19. $\left(\frac{1}{3}\right)^6 \div \left(\frac{1}{3}\right)^8$

Use the laws of exponents to simplify each of the following.

20. $(4x)^2$
21. $(-3x)^3$
22. $(x^3)^4$
23. $(3x)(x^7)$
24. $(5x)(4x^4)$
25. $(-3x^2)(-6x^3)$
26. $(10x^8) \div (2x^4)$

1.13 Scientific Notation

Learning Objectives

Here you'll learn about scientific notation.

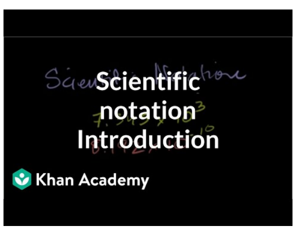
Very large and very small quantities and measures are often used to provide information in magazines, textbooks, television, newspapers and on the Internet. Some examples are:

- The distance between the sun and Neptune is 4,500,000,000 km.
- The diameter of an electron is approximately 0.00000000000022 inches.

Scientific notation is a convenient way to represent such numbers. How could you write the numbers above using scientific notation?

Watch This

Khan Academy Scientific Notation



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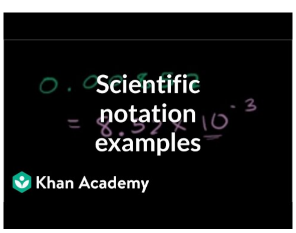
Khan Academy Scientific Notation Examples



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Guidance

To represent a number in scientific notation means to express the number as a product of two factors: a number between 1 and 10 (including 1) and a power of 10. A positive real number 'x' is said to be written in **scientific notation** if it is expressed as

$$x = a \times 10^n$$

where

$$1 \leq a < 10 \text{ and } n \in Z.$$

In other words, a number in scientific notation is a single nonzero digit followed by a decimal point and other digits, all multiplied by a power of 10.

When working with numbers written in scientific notation, you can use the following rules. These rules are proved by example in Example B and Example C.

$$\boxed{(A\times 10^n)+(B\times 10^n)=(A+B)\times 10^n}$$

$$\boxed{(A\times 10^n)-(B\times 10^n)=(A-B)\times 10^n}$$

$$\boxed{(A\times 10^m)\times(B\times 10^n)=(A\times B)\times(10^{m+n})}$$

$$\boxed{(A\times 10^m)\div(B\times 10^n)=(A\div B)\times(10^{m-n})}$$

Example A

Write the following numbers using scientific notation:

i) 2,679,000

ii) 0.00005728

Solutions:

i)

$$\begin{aligned}2,679,000 &= 2.679\times 1,000,000\\ 2.679\times 1,000,000 &= 2.679\times 10^{6}\end{aligned}$$

The exponent, $n=6$, represents the decimal point that is 6 places to the right of the **standard position of the decimal point.**

ii)

$$\begin{aligned}0.00005728 &= 5.728\times 0.00001\\ 5.728\times 0.00001 &= 5.728\times\frac{1}{100,000}\\ 5.728\times\frac{1}{100,000} &= 5.728\times\frac{1}{10^{5}}\\ 5.728\times\frac{1}{100,000} &= 5.728\times 10^{-5}\end{aligned}$$

The exponent, $n=-5$, represents the decimal point that is 5 places to the left of the **standard position of the decimal point.**

One advantage of scientific notation is that calculations with large or small numbers can be done by **applying the laws of exponents.**

Example B

Complete the following table.

TABLE 1.4:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$			
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$			
$4.6 \times 10^4 - 2.2 \times 10^4$			
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$			

Solution:

TABLE 1.5:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$	$130,000 + 250,000$	380,000	3.8×10^5
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$	$0.037 + 0.051$	0.088	8.8×10^{-2}
$4.6 \times 10^4 - 2.2 \times 10^4$	$46,000 - 22,000$	24,000	2.4×10^4
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$	$0.079 - 0.054$	0.025	2.5×10^{-2}

Note that the numbers in the last column have the same power of 10 as those in the first column.

Example C

Complete the following table.

TABLE 1.6:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$			
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$			
$(4.4 \times 10^4) \div (2.2 \times 10^2)$			
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$			

Solution:

TABLE 1.7:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$	360×1400	504,000	5.04×10^5
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$	2500×0.0000011	0.00275	2.75×10^{-3}
$(4.4 \times 10^4) \div (2.2 \times 10^2)$	$44,000 \div 220$	200	2.0×10^2
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$	$0.00068 \div 0.032$	0.02125	2.125×10^{-2}

Note that for multiplication, the power of 10 is the result of adding the exponents of the powers in the first column. For division, the power of 10 is the result of subtracting the exponents of the powers in the first column.

Example D

Calculate each of the following:

i) $4.6 \times 10^4 + 5.3 \times 10^5$

ii) $4.7 \times 10^{-3} - 2.4 \times 10^{-4}$

iii) $(7.3 \times 10^5) \times (6.8 \times 10^4)$

iv) $(4.8 \times 10^9) \div (5.79 \times 10^7)$

Solution:

i) Before the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.6×10^4

$4.6 \times 10^4 = (0.46 \times 10^1) \times 10^4$ The power 10^1 indicates the number of places to the right that the decimal point must be moved to return 0.46 to the original number of 4.6.

$(0.46 \times 10^1) \times 10^4 = 0.46 \times 10^5$ Add the exponents of the power.

Rewrite the question and substitute 4.6×10^4 with 0.46×10^5.

$0.46 \times 10^5 + 5.3 \times 10^5$

Apply the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

.

$$(0.46 \times 10^5) + (5.3 \times 10^5) = (0.46 + 5.3) \times 10^5$$
$$(0.46 + 5.3) \times 10^5 = 5.76 \times 10^5$$
$$\boxed{4.6 \times 10^4 + 5.3 \times 10^5 = 5.76 \times 10^5}$$

ii) Before the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.7×10^{-3}

$4.7 \times 10^{-3} = (47 \times 10^{-1}) \times 10^{-3}$ The power 10^{-1} indicates the number of places to the left that the decimal point must be moved to return 47 to the original number of 4.7.

$(47 \times 10^{-1}) \times 10^{-3} = 47 \times 10^{-4}$ Add the exponents of the power.

Rewrite the question and substitute 4.7×10^{-3} with 47×10^{-4}.

$47 \times 10^{-4} - 2.4 \times 10^{-4}$

Apply the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

.

$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = (47 - 2.4) \times 10^{-4}$$
$$(47 \times 10^{-4}) - (2.4 \times 10^{-4}) = 44.6 \times 10^{-4}$$

The answer must be written in scientific notation.

$44.6 \times 10^{-4} = (4.46 \times 10^{1}) \times 10^{-4}$ Apply the law of exponents – add the exponents of the power.

$4.46 \times 10 \times 10^{-4} = 4.46 \times 10^{-3}$

$\boxed{4.7 \times 10^{-3} - 2.4 \times 10^{-4} = 4.46 \times 10^{-3}}$

iii) $(7.3 \times 10^5) \times (6.8 \times 10^4)$

$7.3 \times 10^5 \times 6.8 \times 10^4$ Apply the rule $\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$.

$(7.3 \times 10^5) \times (6.8 \times 10^4) = (7.3 \times 6.8) \times (10^{5+4})$

$(7.3 \times 6.8) \times (10^{5+4}) = (49.64) \times (10^9)$

$(49.64) \times (10^9) = 49.64 \times 10^9$ Write the answer in scientific notation.

$49.64 \times 10^9 = (4.964 \times 10^1) \times 10^9$ Apply the law of exponents – add the exponents of the power.

$49.64 \times 10^9 = 4.964 \times 10^{10}$

$\boxed{(7.3 \times 10^5) \times (6.8 \times 10^4) = 4.964 \times 10^{10}}$

iv) $(4.8 \times 10^9) \div (5.79 \times 10^7)$

$(4.8 \times 10^9) \div (5.79 \times 10^7)$ Apply the rule $\boxed{(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})}$.

$(4.8 \times 10^9) \div (5.79 \times 10^7) = (4.8 \div 5.79) \times 10^{9-7}$ Apply the law of exponents – subtract the exponents of the power.

$(4.8 \div 5.79) \times 10^{9-7} = (0.829) \times 10^2$ Write the answer in scientific notation.

$(0.829) \times 10^2 = (8.29 \times 10^{-1}) \times 10^2$ Apply the law of exponents – add the exponents of the power.

$\boxed{(8.29 \times 10^{-1}) \times 10^2 = 8.29 \times 10^1}$

Concept Problem Revisited

The distance between the sun and Neptune would be written as 4.5×10^9 *km* and the diameter of an electron would be written as 2.2×10^{-13} *in*.

Guided Practice

1. Express the following product in scientific notation: $(4 \times 10^{12})(9.2 \times 10^{7})$

2. Express the following quotient in scientific notation: $\frac{6,400,000}{0.008}$

3. If $a = 0.000415, b = 521$, and $c = 71,640$, find an approximate value for $\frac{ab}{c}$. Express the answer in scientific notation.

Answers:

1. Apply the rule

$$\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$$

$$(4 \times 10^{12}) \times (9.2 \times 10^7) = (4 \times 9.2) \times (10^{12+7})$$
$$(4 \times 9.2) \times (10^{12+7}) = 36.8 \times 10^{19}$$

Express the answer in scientific notation.

$$36.8 \times 10^{19} = (3.68 \times 10^1) \times 10^{19}$$
$$(3.68 \times 10^1) \times 10^{19} = 3.68 \times 10^{20}$$
$$\boxed{(4 \times 10^{12})(9.2 \times 10^7) = 3.68 \times 10^{20}}$$

2. Begin by expressing the numerator and the denominator in scientific notation.

$\frac{6.4 \times 10^6}{8.0 \times 10^{-3}}$

Apply the rule

$$\boxed{(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m+n})}$$

.

$(6.4 \times 10^6) \div (8.0 \times 10^{-3}) = (6.4 \div 8.0) \times (10^{6--3})$ Apply the law of exponents – subtract the exponents of the powers.

$(6.4 \div 8.0) \times (10^{6--3}) = (0.8) \times (10^9)$

$(0.8) \times (10^9) = 0.8 \times 10^9$ Express the answer in scientific notation.

$0.8 \times 10^9 = (8.0 \times 10^{-1}) \times 10^9$

$0.8 \times 10^9 = 8.0 \times 10^{-1} \times 10^9$ Apply the law of exponents – add the exponents of the powers.

$8.0 \times 10^{-1} \times 10^9 = 8.0 \times 10^8$

$\boxed{\frac{6,400,000}{0.008} = 8.0 \times 10^8}$ Express the answer in scientific notation.

3. Express all values in scientific notation.

$$0.000415 = 4.15 \times 10^{-4}$$
$$521 = 5.21 \times 10^2$$
$$71,640 = 7.1640 \times 10^4$$

Use the values in scientific notation to determine an approximate value for $\frac{ab}{c}$.

$\frac{ab}{c} = \frac{(4.15\times10^{-4})(5.21\times10^{2})}{7.1640\times10^{4}}$

In the numerator, apply the rule

$$\boxed{(A\times10^{m})\times(B\times10^{n}) = (A\times B)\times(10^{m+n})}$$

$$\frac{(4.15\times10^{-4})(5.21\times10^{2})}{7.1640\times10^{4}} = \frac{(4.15\times5.21)\times(10^{-4}\times10^{2})}{7.1640\times10^{4}}$$
$$\frac{(4.15\times5.21)\times(10^{-4}\times10^{2})}{7.1640\times10^{4}} = \frac{21.6215\times10^{-2}}{7.1640\times10^{4}}$$

Apply the rule $\boxed{(A\times10^{m})\div(B\times10^{n}) = (A\div B)\times(10^{m-n})}$.

$$\frac{21.6215\times10^{-2}}{7.1640\times10^{4}} = (21.6215\div7.1640)\times(10^{-2-4})$$
$$\boxed{(21.6215\div7.1640)\times(10^{-2}\times10^{4}) = 3.018\times10^{-6}}$$

Practice

Express each of the following in scientific notation:

1. 42,000
2. 0.00087
3. 150.64
4. 56,789
5. 0.00947

Express each of the following in standard form:

6. 4.26×10^{5}
7. 8×10^{4}
8. 5.967×10^{10}
9. 1.482×10^{-6}
10. 7.64×10^{-3}

Perform the indicated operations and express the answer in scientific notation

11. $8.9\times10^{4}+4.3\times10^{5}$
12. $8.7\times10^{-4}-6.5\times10^{-5}$
13. $(5.3\times10^{6})\times(7.9\times10^{5})$
14. $(3.9\times10^{8})\div(2.8\times10^{6})$

For the given values, perform the indicated operations for $\frac{ab}{c}$ and express the answer in scientific notation and standard form.

15.

$$a = 76.1$$
$$b = 818,000,000$$
$$c = 0.000016$$

16.

$$a = 9.13 \times 10^9$$
$$b = 5.45 \times 10^{-23}$$
$$c = 1.62$$

Summary

Now that you have warmed up with a little review, it's time to go on to new material. Onward, to Chapter 2!

1.14 References

1. . The Real Number Line.

CHAPTER 2

Unit 2 - Functions and Graphs

Chapter Outline

Introduction

In this chapter, we introduce the idea of a function, which is a core notion in mathematics that underlies most of the rest of the course. Functions provide an efficient way to describe how two quantities are related and to state particular values that are related to each other.

In essence, a function is a mechanism for converting values of one type to values of another. For instance, you can create a function that converts a temperature measured in Fahrenheit to the equivalent Celsius temperature. A computer program or table that tells you how many people live in each ZIP code could also be considered a function. In that case, values of ’ZIP code’ are being converted into ’number of people’.

We will make heavy use of functions and their notation throughout this course, so it is important that you have a firm grasp of the material in this chapter before going on.

2.1 Relations and Functions

Learning Objectives

Here you will learn about relations, and what makes a relation a function.

Concept Problem

Consider the two scenarios laid out below:

Scenario 1

Workers at Milliways Diner earn $12 per hour of work. Let x represent the number of hours a person works and M represent the number of dollars they have earned.

Scenario 2

Consider this year's graduating class at San Andreas University. Let t represent the total number of semesters that a graduate took classes, and let S represent their annual salary after graduation.

There are some similarities between these two scenarios: Both involve measuring money and time, for instance. But there is a fundamental difference between them that we will examine in this section.

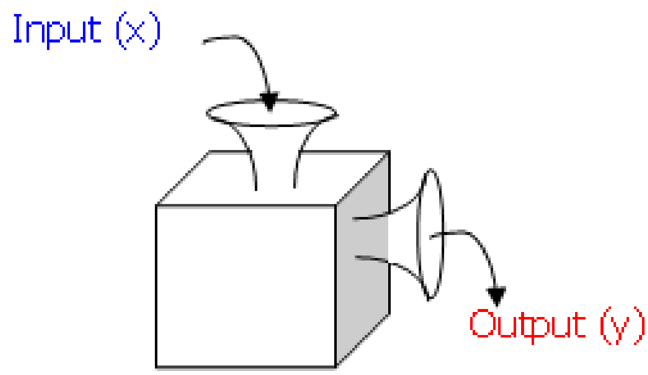
For each scenario, ask these questions: Are the two quantities related to each other? If so, how closely are they related? If we know the value of one quantity, can we determine the value of the other? Do either of these scenarios describe a function?

Watch This

Khan Academy - Relations and Function

Guidance

Consider a relationship between two quantities (or variables), so values of one can be put into some kind of correspondence with values of the other. You can think of this relationship in terms of an input/output machine.



If two quantities can be put into correspondence with each other then they are related in some way. A **relation** is any set of ordered pairs.

For instance, the following set of order pairs defines a relation: $\{(2,1),(5,-2),(8,0),(3,1),(5,4)\}$. The point $(8,0)$ indicates that the x value (input) 8 corresponds to the y value (output) 0. We could also arrange this data into a table, which would be easier to read:

x	2	5	8	3	5	7
y	1	-2	0	1	4	7

So when $x = 3$ the corresponding value of y is 1. Notice that when $x = 5$ we have both $y = -2$ and $y = 4$.

In the table above we know for sure that the value of y is 1 when $x = 3$; it is the only value that corresponds with it. When $x = 5$, however, we can't say for sure whether the value of y is -2 or 4 since both are associated with it. This kind of ambiguity is difficult to work with.

Mathematicians like to work with relations where there are no ambiguous values i.e. where each input has exactly one associated output. They like them so much, in fact, that they have a special name. A **function** is a relation where each input (x value) has exactly one output (y value). Thus, every function is a relation, but not every relation is a function (as we will see below). The table we examined above is a relation, but it is not a function.

There are several different ways to express relations and functions. In this course we focus on three: numerically (with a table), graphically (with a picture), symbolically (with an equation)

Look at the two tables below. **Table A** shows a relation that is a function because every x value has only one y value. **Table B** shows a relation that is not a function because there are two different y values for the x value of 0.

TABLE 2.1: Table A

x	y
0	4
1	7
2	7
3	6

TABLE 2.2: Table B

x	y
0	4
0	2

TABLE 2.2: (continued)

x	y
2	6
2	7

Example A

Determine if the following relation is a function.

TABLE 2.3:

x	y
-3.5	-3.6
-1	-1
4	3.6
7.8	7.2

Solution:

The relation is a function because there is only one value of y for every value of x.

Graphs

We can get a visual representation of a relation or function by using a graph.

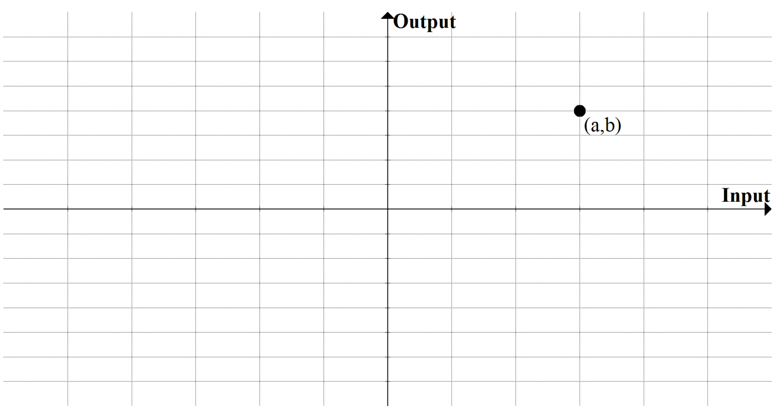


FIGURE 2.1

The graph above consists of a single point (a,b), which indicates that the input value a corresponds to the output value b. Most graphs that you will see, though, contain many points in the form of a solid curve.

How many points are on a curve like this? *An infinite number!* There are some that are easy to identify, like (0,-2), (-2,2), and (3,7), but between those are innumerable other points that don't have integer coordinates, such as (2.5,4.25) and (1.414213...,0).

An arbitrary graph may or may not represent a function. When looking at the graph of a relation, you can determine whether or not it is a function using the **vertical line test**. If a vertical line can be drawn anywhere through the graph such that it intersects the graph more than once, it corresponds to an input value that produces more than one output, and so the graph is not function.

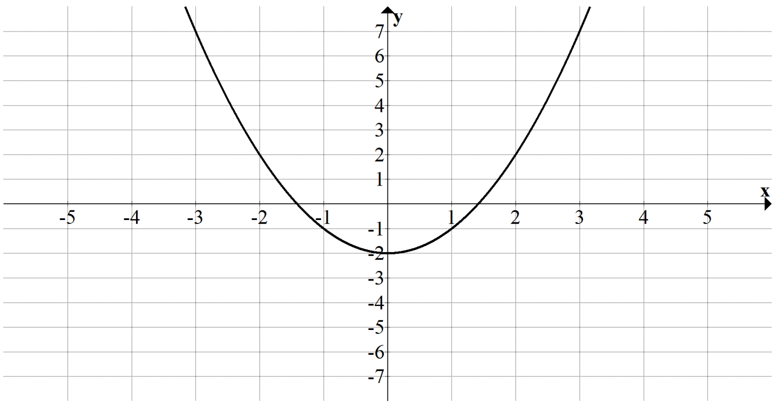
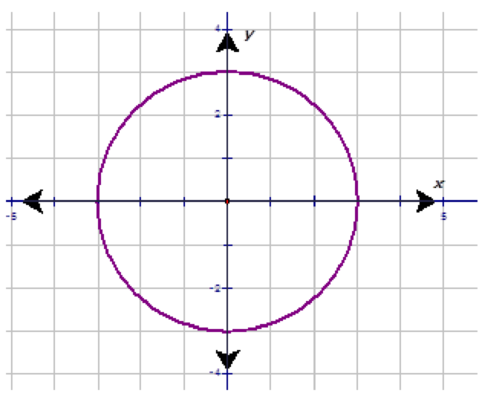
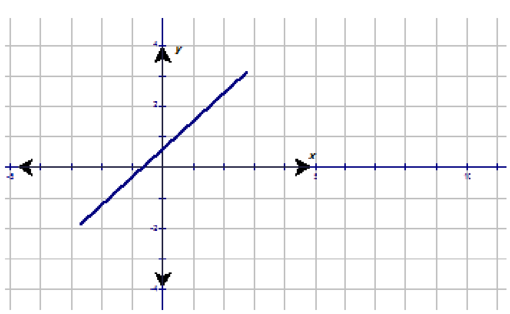


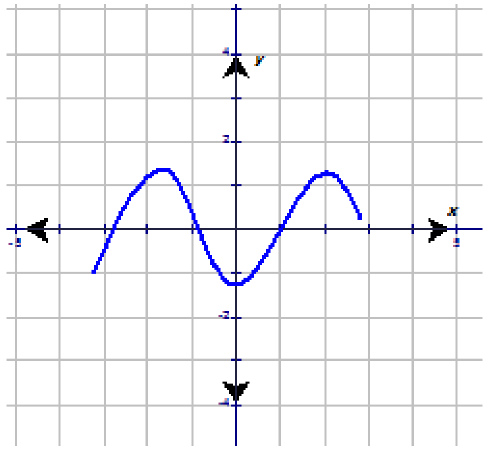
FIGURE 2.2

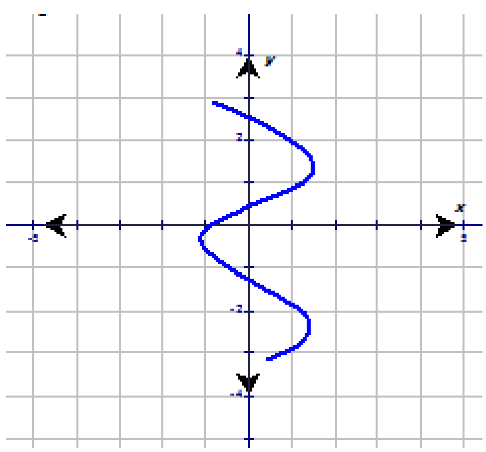
Example B

Which of the following graphs represent a function?



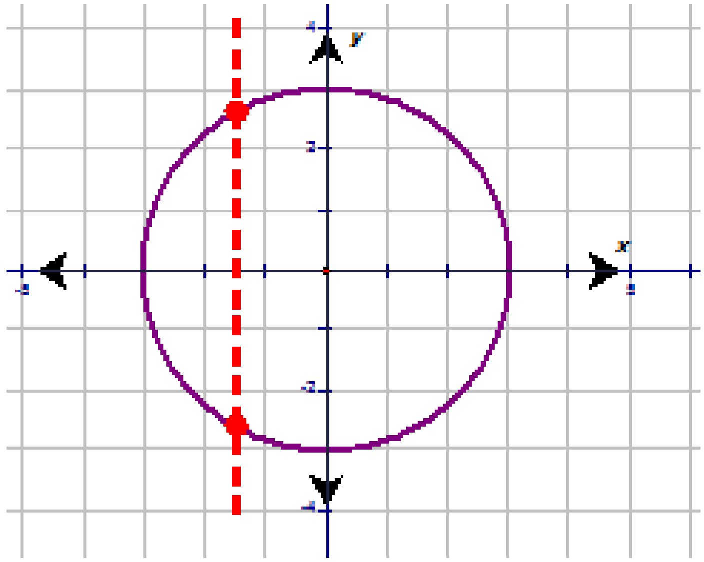




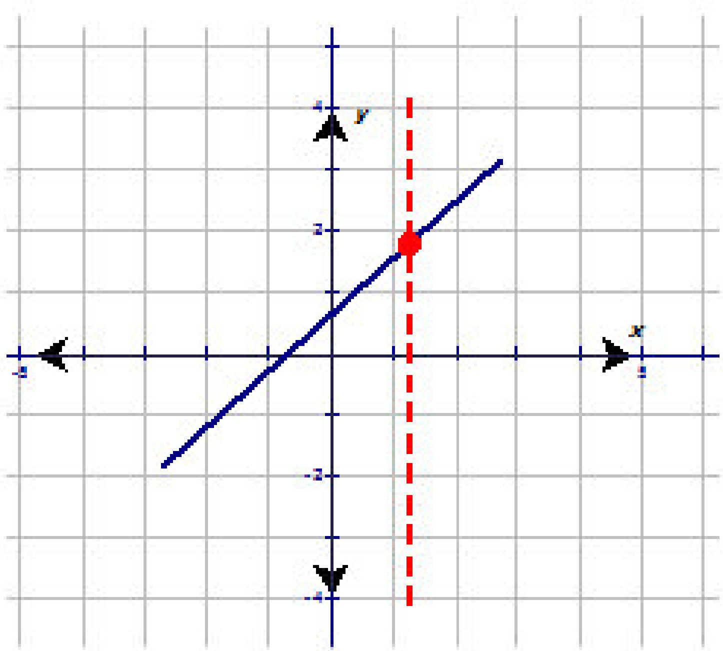


Solution:

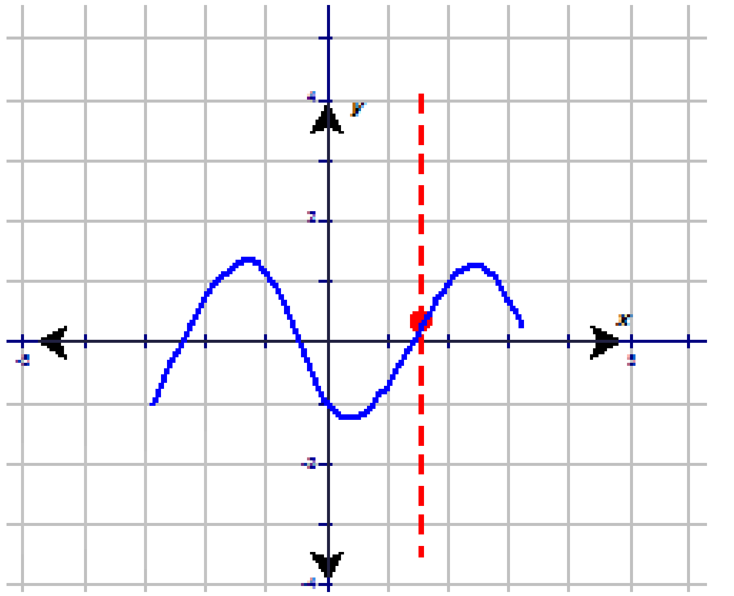
In order to answer this question, you need to use the vertical line test. A graph represents a function if no vertical line intersects the graph more than once. Let's look at the first graph. Draw a vertical line through the graph.



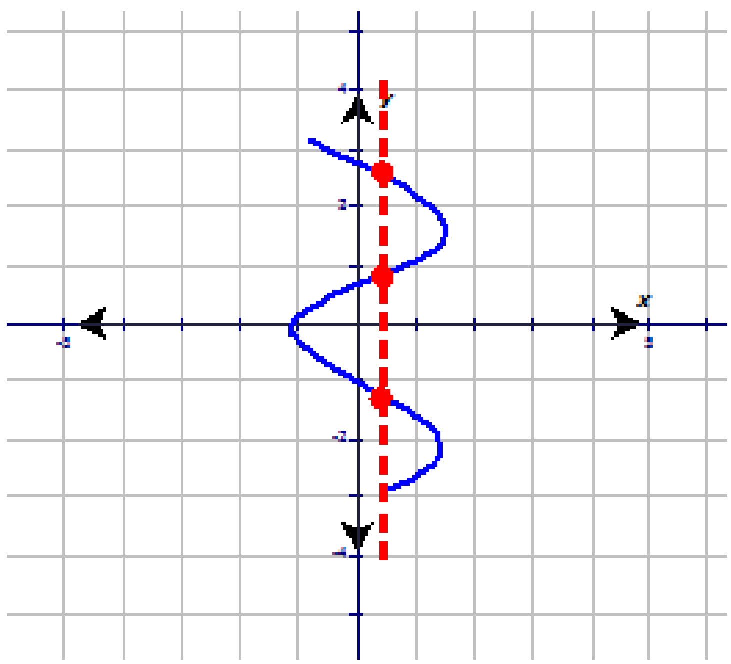
Since the vertical line hit the graph more than once (indicated by the two red dots), the graph ***does not*** represent a function.



Since the vertical line hit the graph only once (indicated by the one red dot), the graph ***does*** represent a function.



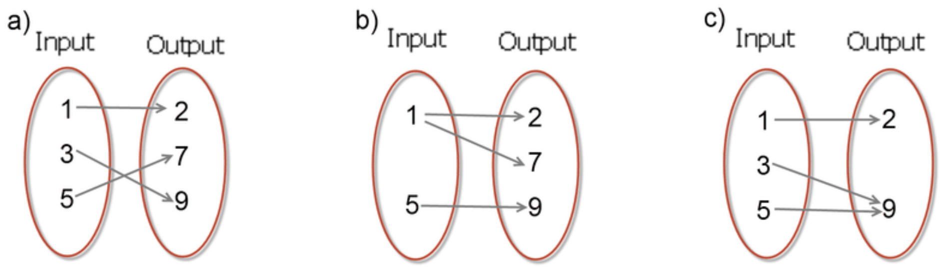
Since the vertical line hit the graph only once (indicated by the one red dot), the graph ***does*** represent a function.



Since the vertical line hit the graph more than once (indicated by the three red dots), the graph ***does not*** represent a function.

Example C

Which of the following represent functions?



Solution:

a) This is a function because every input has only one output.

b) This is not a function because one input (1) has two outputs (2 and 7).

c) This is a function because every input has only one output.

Concept Problem Revisited

Scenario 1

Are the two quantities related? Yes

The number of hours that a person works directly determines how much they get paid, so x and M are clearly related ($12 for 1 hour of work, $24 for 2 hours of work, etc). In fact, we can easily put together a table of values for them:

x	0	1	2	3	4	5	6...
M	0	12	24	36	48	60	72...

If we know the value of one quantity, can we find the value of the other? Yes

If someone works for 3 hours ($x = 3$) then we know they earned $36 ($M = 36$). There is no other possibility. This is true for any value of x.

Does this scenario describe a function? Yes

In this case, M is a function of x. It is not possible to be paid two different amounts for the same number of hours of work i.e. for each x there is exactly one M that it corresponds to.

Scenario 2

Are the two quantities related? Yes

You could certainly create a table containing the values for t and S for every student in the graduating class, so there is a relation between them.

If we know the value of one quantity, can we find the value of the other? No

If a student attended 8 semesters of classes, can we say exactly what her salary will be after she graduates? No, there are many other factors that will go into determining this. Similarly, simply knowing a student's salary is not enough information to determine how many semesters of classes they took.

Does this scenario describe a function? No

Each value of t may have several different values of S that is corresponds to. For example, many students may graduate in 8 semesters, but they will not all have the same salary afterwards.

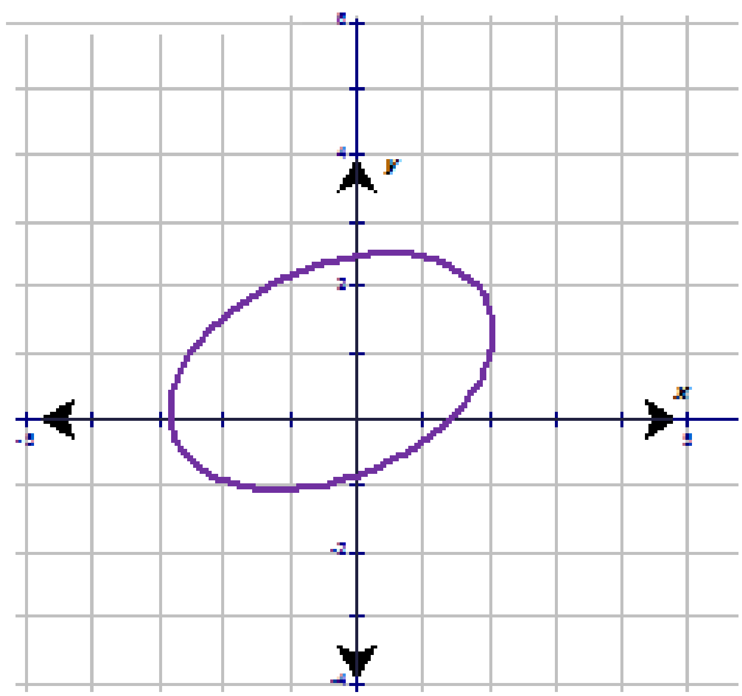
Overall, you can see that the relationship between x and M is clear-cut and well-defined, while the relationship between t and S is murkier and ill-defined. This illustrates why mathematicians prefer to work with functions: you can often answer questions with certainty when dealing with functions, and this is not often the case with relations in general.

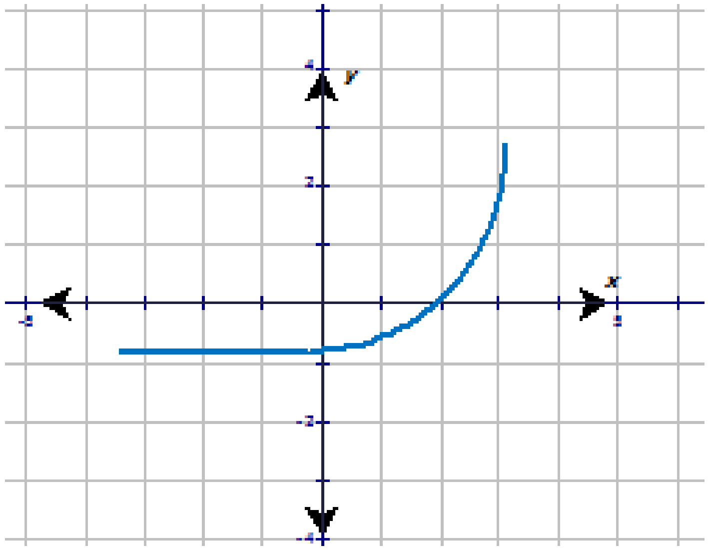
Guided Practice

1. Is the following a representation of a function? Explain.

$$s = \{(1,2),(2,2),(3,2),(4,2)\}$$

2. Which of the following relations represent a function? Explain.



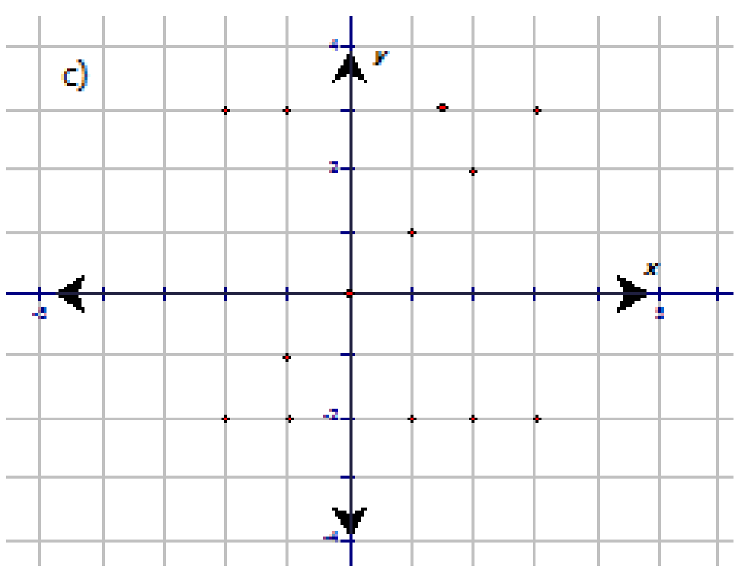


3. Which of the following relations represent a function? Explain.

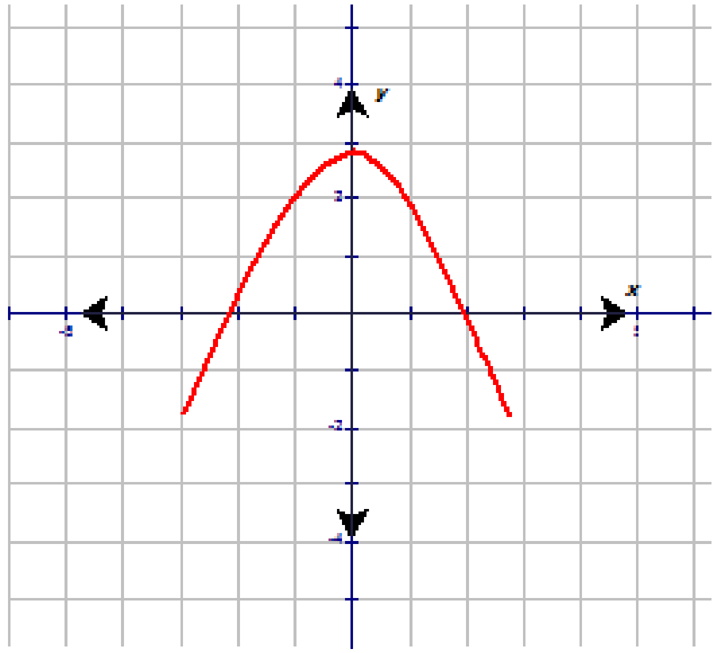
a)

x	2	4	6	8	10	12
y	3	7	11	15	19	23

b)



c)

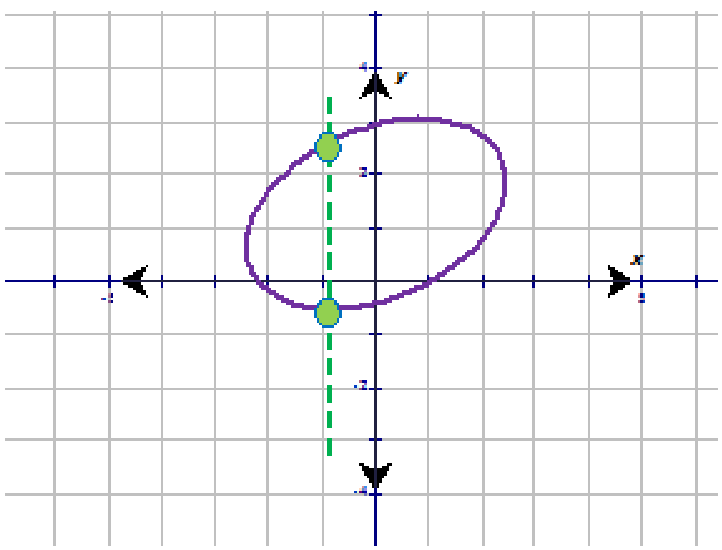


Answers:

1. $s = \{(1,2),(2,2),(3,2),(4,2)\}$

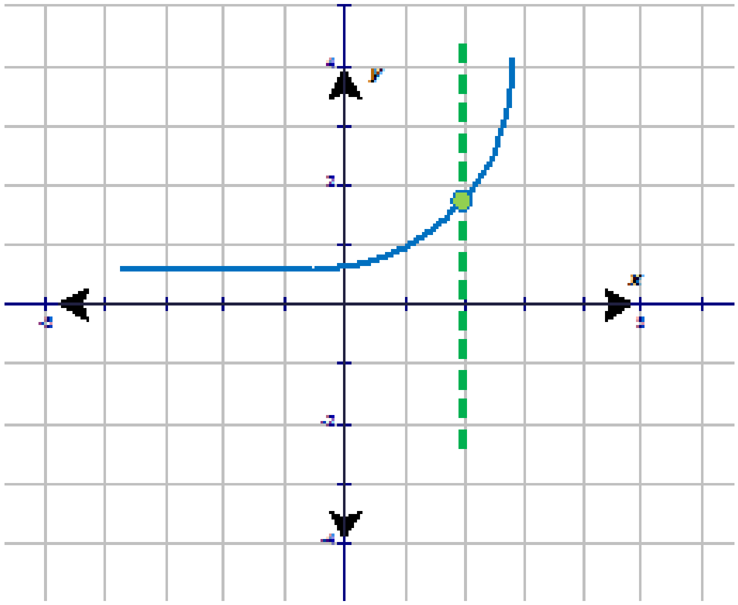
This is a function because there is one output for every input. In other words, if you think of these points as coordinate points (x, y), there is only one value for y given for every value of x.

2. a)



Since the vertical line hit the graph more than once (indicated by the two green circles), the graph ***does not*** represent a function.

b)

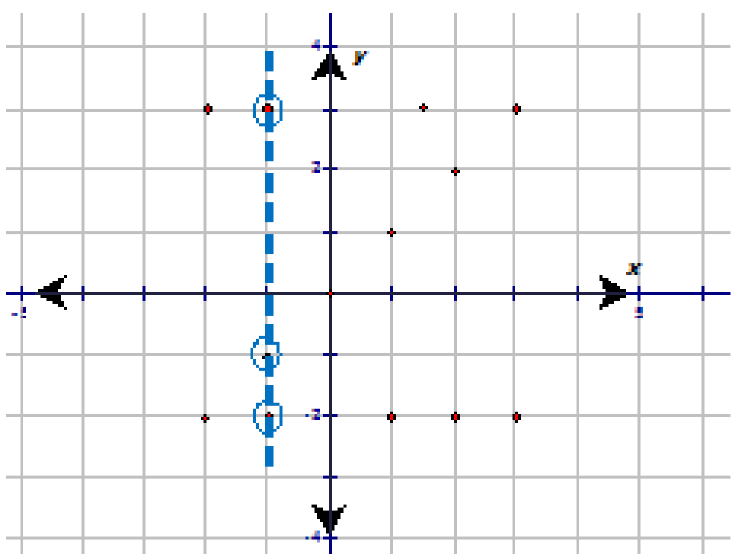


Since the vertical line hit the graph only once (indicated by the one green dot), the graph ***does*** represent a function.

3. a)

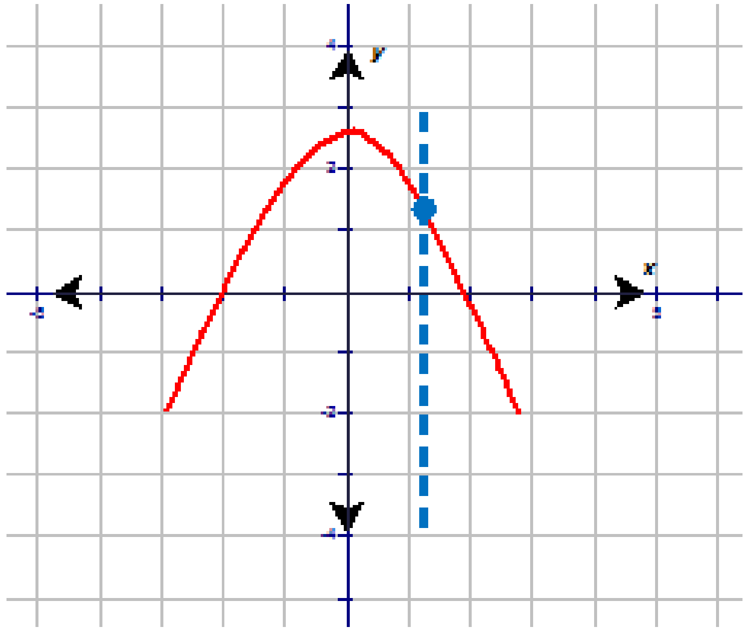
x	2	4	6	8	10	12
y	3	7	11	15	19	23

This ***is a function*** because there is only one output for a given input.
b)



Since the vertical line hit the graph more than once (indicated by the three blue circles), the graph ***does not*** represent a function.

c)

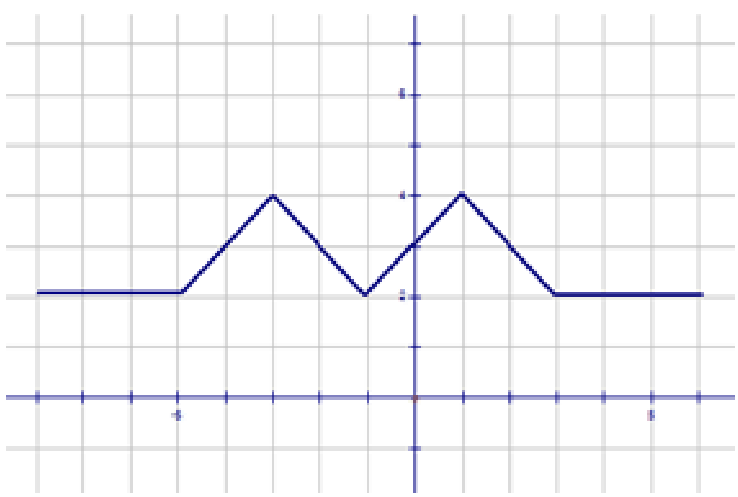


Since the vertical line hit the graph only once (indicated by the one blue dot), the graph ***does*** represent a function.

Explore More

Determine whether or not each relation is a function. Explain your reasoning.

1.



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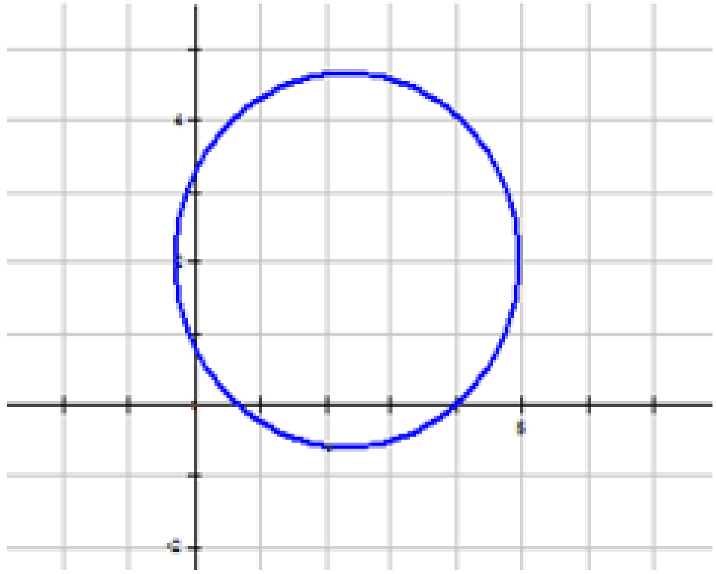

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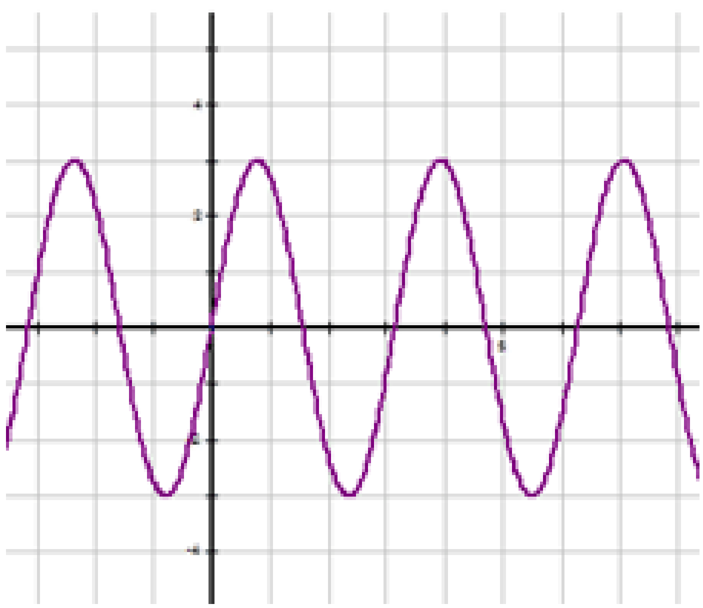
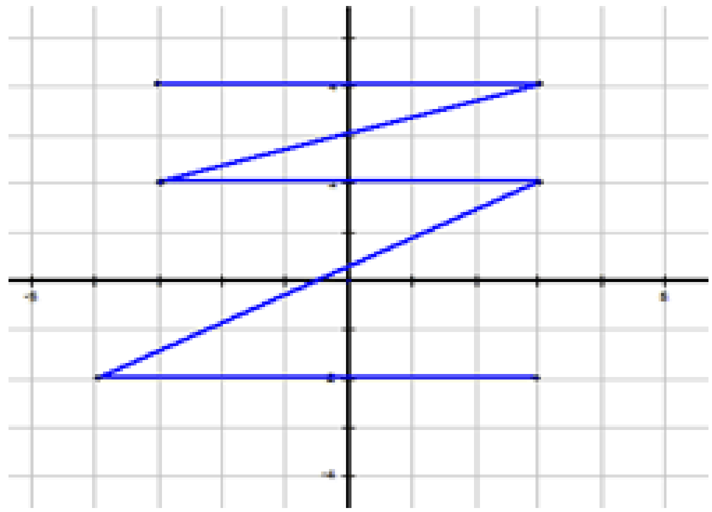


4.



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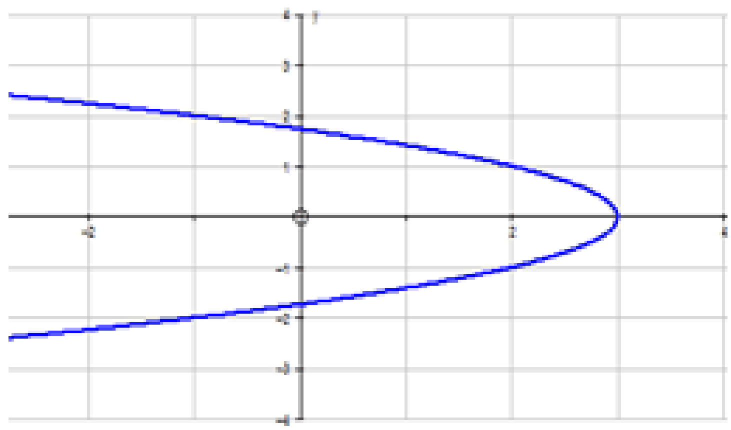


Which of the following relations represent a function? Explain.

6.

X	2	3	2	5
Y	3	-1	5	-4

7.

X	4	2	6	-1
Y	2	4	-3	5

8.

X	1	2	3	4
Y	5	8	5	8

9.

X	−6	−5	−4	−3
Y	4	4	4	4

10.

X	−2	0	−2	4
Y	6	4	4	6

Which of the following relations represent a function? Explain.

11. $s = \{(-3,3),(-2,-2),(-1,-1),(0,0),(1,1),(2,2),(3,3)\}$
12. $s = \{(1,1),(1,2),(1,3),(1,4),(1,5)\}$
13. $s = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$
14. $s = \{(-3,9),(-2,4),(-1,1),(1,1),(2,4)\}$
15. $s = \{(3,-3),(2,-2),(1,-1),(0,0),(-1,1),(-2,2)\}$

2.2 Function Notation

Concept Problem

Suppose the value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$.

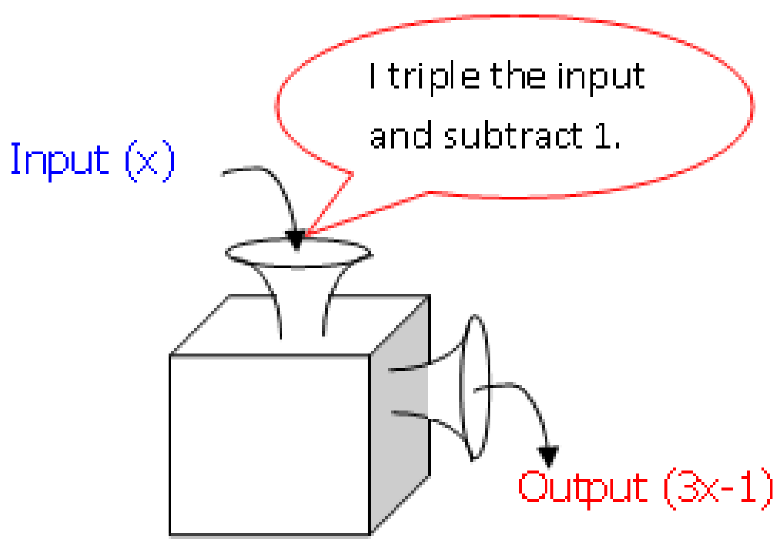
- Can you determine the value of $V(4)$ and explain what the solution means in the context of this problem?
- Can you determine the value of t when $V(t) = 525$ and explain what this situation represents?
- What was the original cost of the digital camera?

Watch This

Khan Academy - Function Notation

Guidance

A function machine shows how a function responds to an input. If I triple the input and subtract one, the machine will convert x into $3x-1$. So, for example, if the function is named f, and 3 is fed into the machine, $3(3)-1=8$ comes out.



When naming a function the symbol $f(x)$ is often used. The symbol $f(x)$ is pronounced as "f of x." This means that the name of the function is f and its equation is written in terms of the variable x. An example of such a function is $f(x) = 3x+4$. Functions can also be written using a letter other than f and a variable other than x. For example, $v(t) = 2t^2 - 5$ and $d(h) = 4h - 3$. In addition to representing a function as an equation, you can also represent a function:

- As a graph

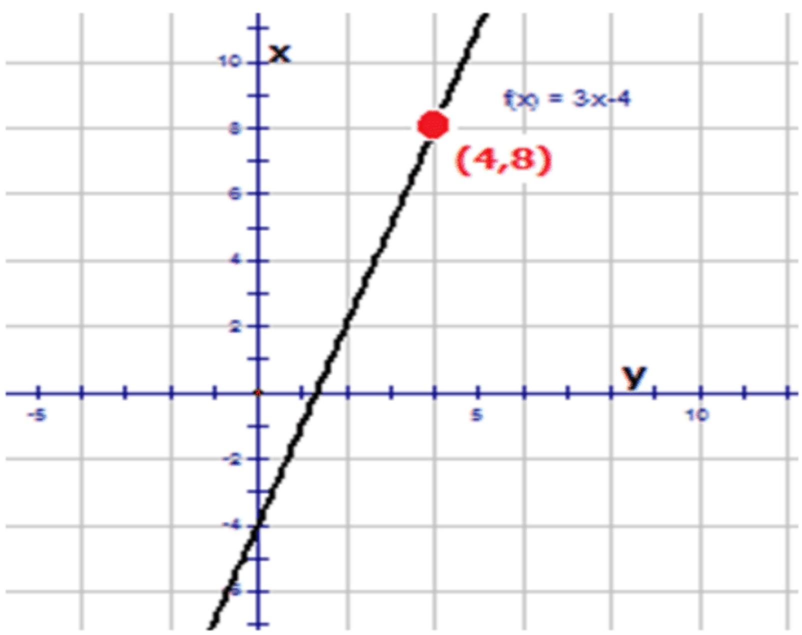
- As ordered pairs
- As a table of values
- As an arrow or mapping diagram

When dealing with two variables like x and y, we will sometimes use the expression $y = f(x)$. This indicates that y is a function of x.

When a function is represented as an equation, an ordered pair can be determined by evaluating various values of the assigned variable. Suppose $f(x) = 3x - 4$. To calculate $f(4)$, substitute:

$$\begin{aligned} f(4) &= 3(4) - 4 \\ f(4) &= 12 - 4 \\ f(4) &= 8 \end{aligned}$$

Graphically, if $f(4) = 8$, this means that the point (4, 8) is a point on the graph of $f(x)$..



Example A

If $f(x) = x^2 + 2x + 5$ find.

a) $f(2)$

b) $f(-7)$

c) $f(1.4)$

Solution:

To determine the value of the function for the assigned values of the variable, substitute the values into the function.

$$\begin{array}{lll} f(x) = x^2 + 2x + 5 & f(x) = x^2 + 2x + 5 & f(x) = x^2 + 2x + 5 \\ \downarrow \quad \downarrow \quad \searrow & \downarrow \quad \downarrow \quad \searrow & \downarrow \quad \downarrow \quad \searrow \\ f(2) = (2)^2 + 2(2) + 5 & f(-7) = (-7)^2 + 2(-7) + 5 & f(1.4) = (1.4)^2 + 2(1.4) + 5 \\ f(2) = 4 + 4 + 5 & f(-7) = 49 - 14 + 5 & f(1.4) = 1.96 + 2.8 + 5 \\ \boxed{f(2) = 13} & \boxed{f(-7) = 40} & \boxed{f(1.4) = 9.76} \end{array}$$

Example B

Functions can also be represented as mapping rules. If $g : x \rightarrow 5 - 2x$ find the following in simplest form:

a) $g(y)$

b) $g(y-3)$

c) $g(2y)$

Solution:

a) $g(y) = 5 - 2y$

b) $g(y-3) = 5 - 2(y-3) = 5 - 2y + 6 = 11 - 2y$

c) $g(2y) = 5 - 2(2y) = 5 - 4y$

Example C

Let $P(a) = \frac{2a-3}{a+2}$.

a) Evaluate

i) $P(0)$
ii) $P(1)$
iii) $P\left(-\frac{1}{2}\right)$

b) Find a value of 'a' where $P(a)$ does not exist.

c) Find $P(a-2)$ in simplest form

d) Find 'a' if $P(a) = -5$

Solution:

a)

$$P(a) = \frac{2a-3}{a+2}$$
$$P(0) = \frac{2(0)-3}{(0)+2}$$
$$\boxed{P(0) = \frac{-3}{2}}$$

$$P(a) = \frac{2a-3}{a+2}$$
$$P(1) = \frac{2(1)-3}{(1)+2}$$
$$P(1) = \frac{2-3}{1+2}$$
$$\boxed{P(1) = \frac{-1}{3}}$$

$$P(a) = \frac{2a-3}{a+2}$$
$$P\left(-\frac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right)-3}{\left(-\frac{1}{2}\right)+2}$$
$$P\left(-\frac{1}{2}\right) = \frac{{}^{1}\cancel{2}\left(-\frac{1}{\cancel{2}}\right)-3}{-\frac{1}{2}+\frac{4}{2}}$$
$$P\left(-\frac{1}{2}\right) = \frac{-1-3}{\frac{3}{2}}$$
$$P\left(-\frac{1}{2}\right) = -4 \div \frac{3}{2}$$
$$P\left(-\frac{1}{2}\right) = -4\left(\frac{2}{3}\right)$$
$$\boxed{P\left(-\frac{1}{2}\right)} = \frac{-8}{3}$$

b) The function will not exist if the denominator equals zero because division by zero is undefined.

$$a+2 = 0$$
$$a+2-2 = 0-2$$
$$\boxed{a = -2}$$

Therefore, if $a = -2$, then $P(a) = \frac{2a-3}{a+2}$ does not exist.

c)

$$P(a) = \frac{2a-3}{a+2}$$

$$P(a-2) = \frac{2(a-2)-3}{(a-2)+2} \quad \text{Substitue } a-2 \text{ for } a$$

$$P(a-2) = \frac{2a-4-3}{a-2+2} \quad \text{Remove parentheses}$$

$$P(a-2) = \frac{2a-7}{a} \quad \text{Combine like terms}$$

$$P(a-2) = \frac{2\cancel{a}}{\cancel{a}} - \frac{7}{a} \quad \text{Express the fraction as two separate fractions and reduce.}$$

$$\boxed{P(a-2) = 2 - \frac{7}{a}}$$

d)

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
-5 &= \frac{2a-3}{a+2} && \text{Let } P(a) = -5 \\
-5(a+2) &= \left(\frac{2a-3}{a+2}\right)(a+2) && \text{Multiply both sides by } (a+2) \\
-5a-10 &= \left(\frac{2a-3}{\cancel{a+2}}\right)(\cancel{a+2}) && \text{Simplify} \\
-5a-10 &= 2a-3 && \text{Solve the linear equation} \\
-5a-10-2a &= 2a-2a-3 && \text{Move } 2a \text{ to the left by subtracting} \\
-7a-10 &= -3 && \text{Simplify} \\
-7a-10+10 &= -3+10 && \text{Move 10 to the right side by addition} \\
-7a &= 7 && \text{Simplify} \\
\frac{-7a}{-7} &= \frac{7}{-7} && \text{Divide both sides by -7 to solve for } a. \\
\boxed{a = -1}
\end{aligned}$$

Concept Problem Revisited

The value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$

- Determine the value of $V(4)$ and explain what the solution means to this problem.
- Determine the value of t when $V(t) = 525$ and explain what this situation represents.
- What was the original cost of the digital camera?

Solution:

- The camera is valued at $675, 4 years after it was purchased.

$$\begin{aligned}
V(t) &= 875 - 50t \\
V(4) &= 875 - 50(4) \\
V(4) &= 875 - 200 \\
&\boxed{V(4) = \$675}
\end{aligned}$$

- The digital camera has a value of $525, 7 years after it was purchased.

$$\begin{aligned}
V(t) &= 875 - 50t && \text{Let } V(t) = 525 \\
525 &= 875 - 50t && \text{Solve the equation} \\
525 - 875 &= 875 - 875 - 50t \\
-350 &= -50t \\
\frac{-350}{-50} &= \frac{-50t}{-50} \\
\boxed{7 = t}
\end{aligned}$$

- The original cost of the camera was \$875.

$$\begin{aligned} V(t) &= 875 - 50t && \text{Let } t = 0. \\ V(0) &= 875 - 50(0) \\ V(0) &= 875 - 0 \\ &\boxed{V(0) = \$875} \end{aligned}$$

Guided Practice

1. If $f(x) = 3x^2 - 4x + 6$ find:

i) $f(-3)$
ii) $f(a-2)$

2. The emergency brake cable in a truck parked on a steep hill breaks and the truck rolls down the hill. The distance in feet, d, that the truck rolls is represented by the function $d = f(t) = 0.5t^2$. How far will the truck roll in 9 seconds?

3. Suppose that $8x + 2y = 14$. Solve for y as a function of x and express the result as a function called $g(x)$.

Answers:

1. $f(x) = 3x^2 - 4x + 6$

i)

$$\begin{aligned} f(x) &= 3x^2 - 4x + 6 && \text{Substitute } (-3) \text{ for } x \text{ in the function.} \\ f(-3) &= 3(-3)^2 - 4(-3) + 6 && \text{Perform the indicated operations.} \\ f(-3) &= 3(9) + 12 + 6 && \text{Simplify} \\ f(-3) &= 27 + 12 + 6 \\ f(-3) &= 45 \\ &\boxed{f(-3) = 45} \end{aligned}$$

ii)

$$\begin{aligned} f(x) &= 3x^2 - 4x + 6 \\ f(a-2) &= 3(a-2)^2 - 4(a-2) + 6 && \text{Write } (a-2)^2 \text{ in expanded form.} \\ f(a-2) &= 3(a-2)(a-2) - 4(a-2) + 6 && \text{Perform the indicated operations.} \\ f(a-2) &= (3a-6)(a-2) - 4(a-2) + 6 \\ f(a-2) &= 3a^2 - 6a - 6a + 12 - 4a + 8 + 6 && \text{Simplify} \\ f(a-2) &= 3a^2 - 16a + 26 \\ &\boxed{f(a-2) = 3a^2 - 16a + 26} \end{aligned}$$

2. $d = f(t) = 0.5t^2$

i)

$$\begin{aligned} d = f(t) &= 0.5t^2 && \text{Substitute 9 for } t. \\ f(9) &= 0.5(9)^2 && \text{Perform the indicated operations.} \\ f(9) &= 0.5(81) \\ \boxed{f(9) = 40.5 \ feet} \end{aligned}$$

After 9 seconds, the truck will roll 40.5 feet.

3. First, isolate y in the equation:

$$\begin{aligned} 8x + 2y &= 14 \\ 8x + 2y - 8x &= 14 - 8x \\ 2y &= -8x + 14 \\ \frac{2y}{2} &= \frac{-8x}{2} + \frac{14}{2} \\ y &= -4x + 7 \end{aligned}$$

Next, replace the y with $g(x)$ to complete the function notation:

$$g(x) = -4x + 7$$

Practice

If $g(x) = 4x^2 - 3x + 2$, find expressions for the following:

1. $g(a)$
2. $g(a-1)$
3. $g(a+2)$
4. $g(2a)$
5. $g(-a)$

If $f(y) = 5y - 3$, determine the value of 'y' when:

6. $f(y) = 7$
7. $f(y) = -1$
8. $f(y) = -3$
9. $f(y) = 6$
10. $f(y) = -8$

The value of a Bobby Orr rookie card n years after its purchase is $V(n) = 520 + 28n$.

11. Determine the value of $V(6)$ and explain what the solution means.
12. Determine the value of n when $V(n) = 744$ and explain what this situation represents.
13. Determine the original price of the card.

Let $f(x) = \frac{3x}{x+2}$.

14. When is $f(x)$ undefined?
15. For what value of x does $f(x) = 2.4$?

2.3 Domain and Range

Learning Objectives

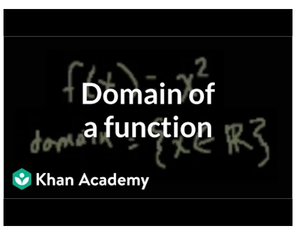
Here you'll learn how to find the domain and range of a relation.

Concept Problem

Suppose you have a 15-gallon container filled with maple syrup. As part of an experiment on fluid dynamics, you poke a hole in the bottom and observe how the syrup flows out. You notice that it takes 45 minutes for the container to drain completely. Let t represent the number of minutes that passed after you poked the hole and $S(t)$ represent the number of gallons of syrup remaining in the container. What values does t take in this scenario? What values does S take?

Watch This

Khan Academy Domain and Range of a Function



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58469

Guidance

In the last section we saw that a function can be thought of as a machine that takes input values and produces output values. It is often useful to talk about those sets of inputs and outputs separately.

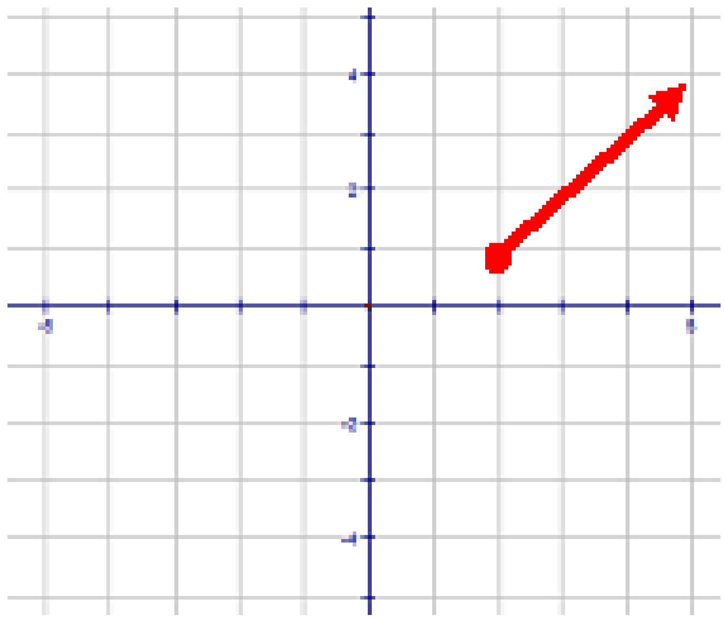
The **domain** of a function (or relation) is the set of all possible input values that the function can take. The **range** of a function (or relation) is the set of possible output values that the function can produce. We may write the domain and range of a relation in several different ways, including inequality notation, interval notation, set-builder notation, or set notation. Remember:

- $\mathbb{Z}(\text{integers}) = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{R}(\text{real numbers}) = \{\text{all rational and irrational numbers}\}$.

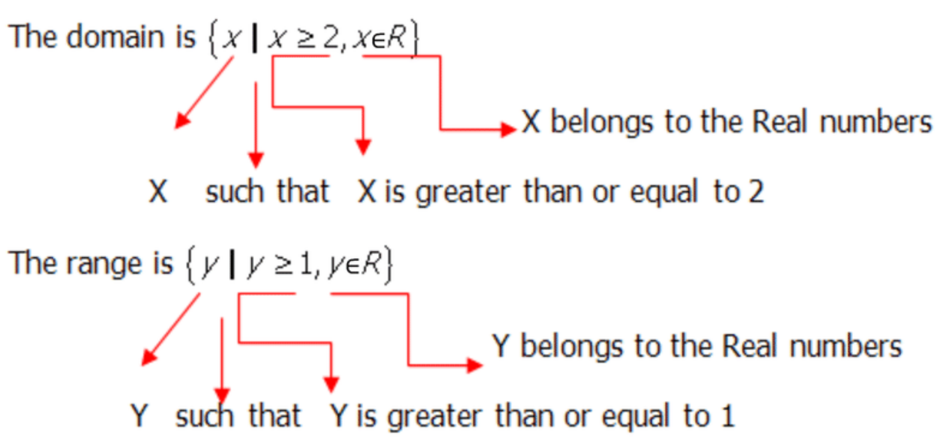
These number systems are very important when the domain and range of a relation are described using interval notation.

A relation is said to be **discrete** if there are a finite number of data points on its graph. A discrete relation may be expressed as a table of values or a list of points. Graphs of discrete relations appear as dots. A relation is said to be **continuous** if its graph is an unbroken curve with no "holes" or "gaps." A continuous relation may be expressed as an equation or graph. The graph of a continuous relation is represented by a line or a curve like the one below. Note that it is possible for a relation to be neither discrete nor continuous.

On a graph, inputs correspond to the horizontal axis, and outputs correspond to the vertical axis, so to find the domain and range we look at the horizontal and vertical separately.



The relation is a straight line that that begins at the point (2,1). Horizontally, the graph starts where $x = 2$ and then goes to the right indefinitely. Vertically, the graph starts where $y = 1$ and then goes up indefinitely. The domain and the range can be written in set-builder notation, as shown below:

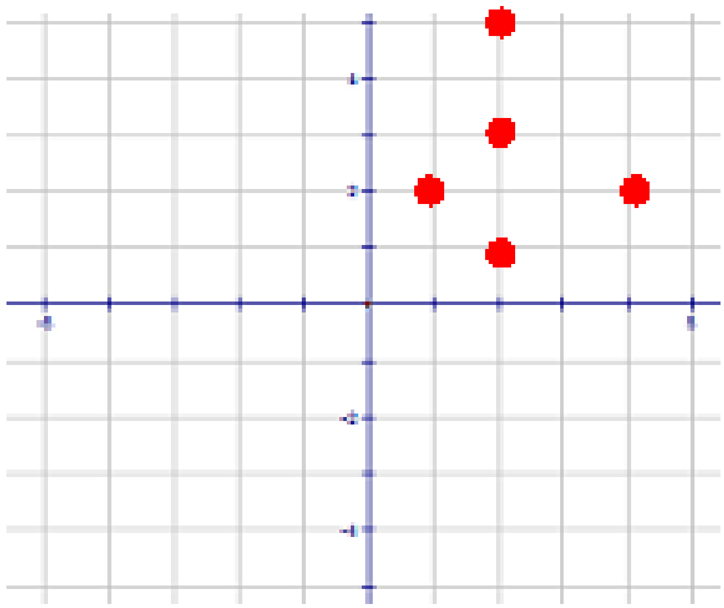


That's a lot of writing! When it is clear that we are working within the real numbers, we will drop the full set-builder notation and only state the inequality. So for the example above we would say the domain is $x \geq 2$ and the range is $y \geq 1$. In situations where every real number is a valid input/output, we say the the domain/range is "all real numbers".

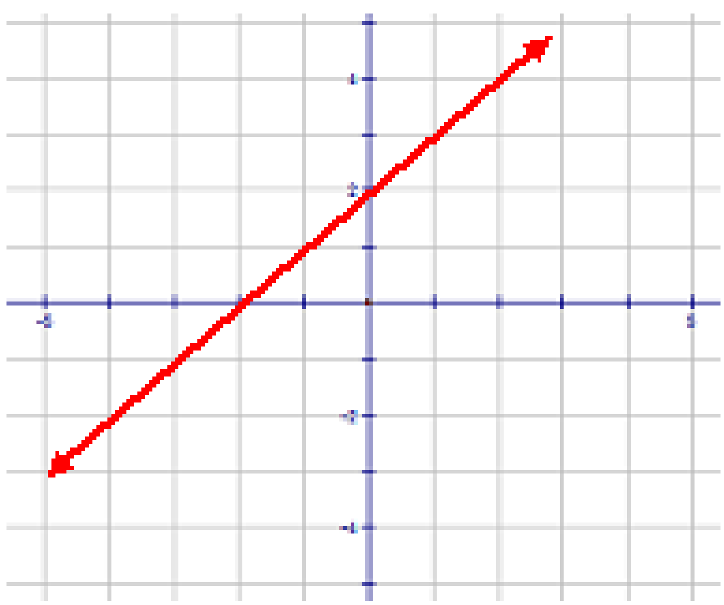
Example A

Which relations are discrete? Which relations are continuous? For each relation, find the domain and range.

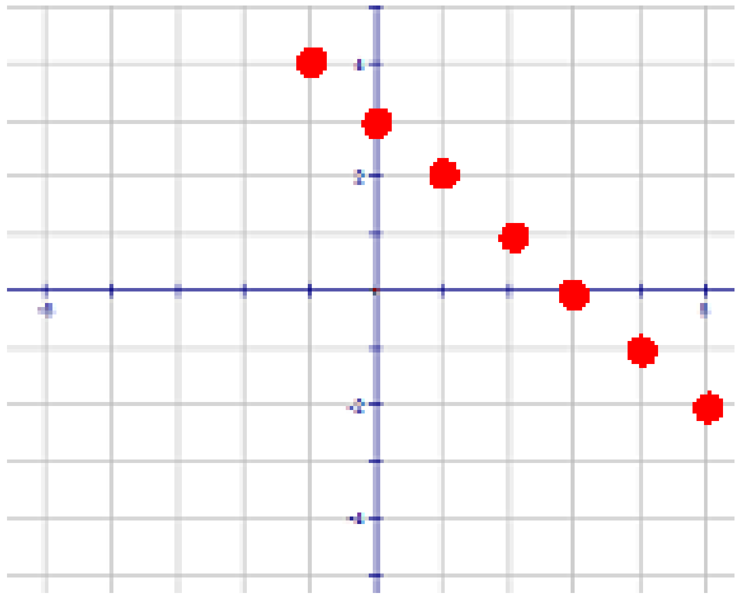
(i)



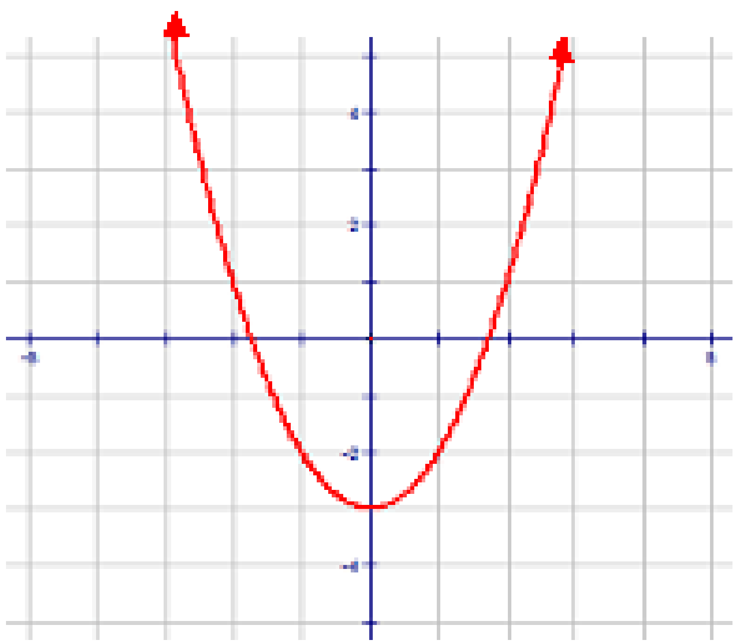
(ii)



(iii)



(iv)



Solution:

(i) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{1,2,4\}$. The range is $\{1,2,3,5\}$

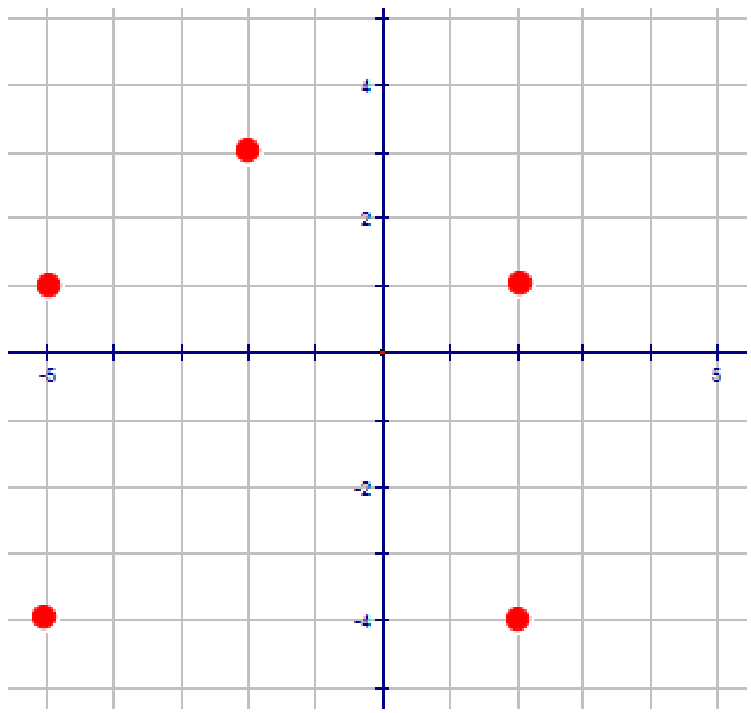
(ii) The graph appears as a continuous straight line. It extends indefinitely to the left and right, so the domain is all real numbers. It also extends indefinitely up and down, so the range is all real numbers.

(iii) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{-1,0,1,2,3,4,5\}$. The range is $\{-2,-1,0,1,2,3,4\}$

(iv) The graph appears as a continuous curve. Horizontally it extends to both the left and right indefinitely, so its domain is all real numbers. Vertically, however, the graph extends upwards indefinitely, but never goes below the value -3, so its range is $y \geq -3$

Example B

Can you state the domain and the range of the following relation?



Solution:

The points indicated on the graph are $\{(-5,-4),(-5,1),(-2,3),(2,1),(2,-4)\}$

The domain is $\{-5,-2,2\}$ and the range is $\{-4,1,3\}$.

Concept Problem Revisited

In the syrup example, we defined the function $S(t)$ which represented the number of gallons of syrup remaining in the container after t minutes of draining.

What values are valid for input variable t? The smallest value that makes sense for t is 0 (corresponding to the moment the hole was created). We stopped counting time after the container was empty, so the largest value of t that makes sense is 45. Putting these together we can say that the domain of $S(t)$ is $0 \leq t \leq 45$.

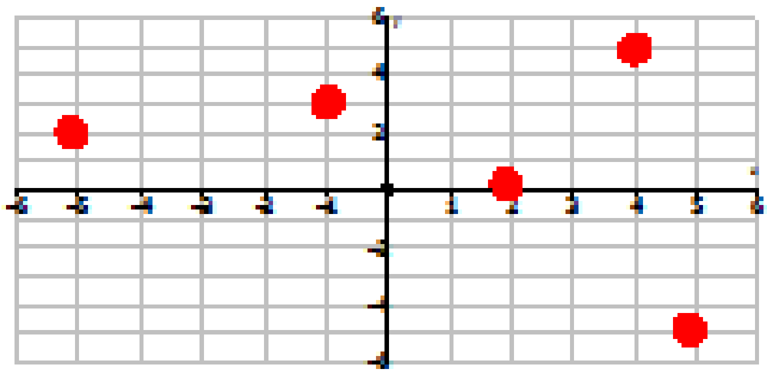
What values are valid for the output variable S? The container started out full, so the largest value of S that makes sense is 15 (the capacity of the container). On the other hand, the smallest value of S that makes sense is 0 (when the container was empty). Putting these together we can say that the range of $S(t)$ is $0 \leq S \leq 15$.

Guided Practice

1. Which relation is discrete? Which relation is continuous?

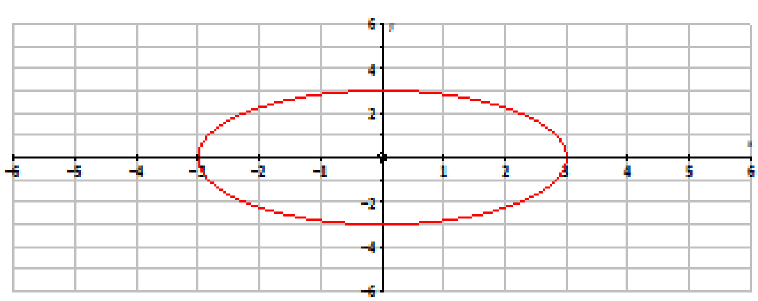
(i)

(ii)

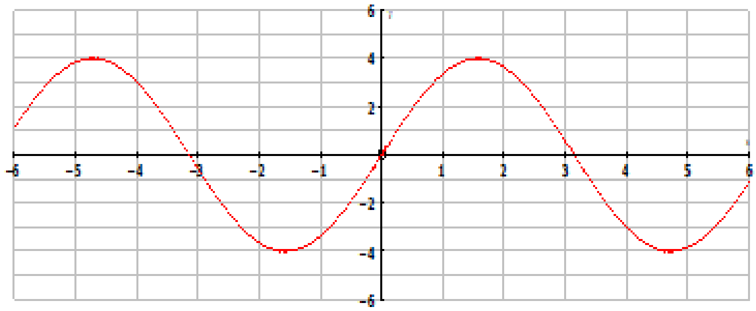


2. State the domain and the range for each of the following relations:

(i)



(ii)



3. A computer salesman's wage consists of a monthly salary of \$200 plus a bonus of \$100 for each computer sold.

(a) Complete the following table of values:

TABLE 2.4:

# computers sold	**0**	**2**	**5**	**10**	**18**
Wages for month (\$)	**?**	**?**	**?**	**?**	**?**

(b) Sketch the graph to represent the monthly salary (\$), against the number (N), of computers sold.
(c) Use the graph to write a suitable domain and range for the problem.

Answers:

1. (i) The graph clearly shows that the points are joined. Therefore the data is continuous.

(ii) The graph shows the plotted points as dots that are not joined. Therefore the data is discrete.

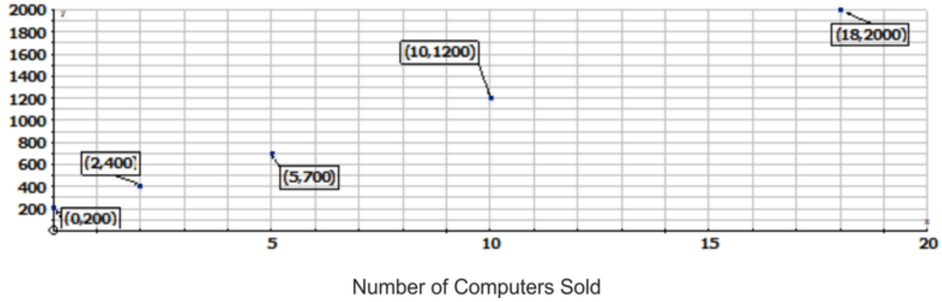
2. (i) The graph starts on the left at $x = -3$ and extends continuously to the right until $x = 3$, so the domain is $-3 \le x \le 3$. Vertically, the lowest point on the graph is at $y = -3$ and it extends upwards continuously to $y = 3$, so the range is $-3 \le y \le 3$.

(ii) The domain is all real numbers. The range is $-4 \le y \le 4$.

3.

TABLE 2.5:

# computers sold	**0**	**2**	**5**	**10**	**18**
Wages for month (\$)	**\$200**	**\$400**	**\$700**	**\$1200**	**\$2000**



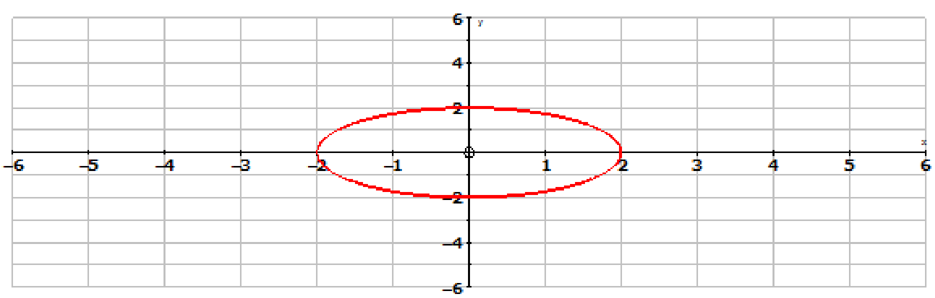
(c) The graph shows that the data is discrete. (The salesman can't sell a portion of a computer, so the data points can't be connected.) The number of computers sold and must be whole numbers. The wages must be natural numbers.

A suitable domain is $\{0,1,2,3,4,\ldots\}$

A suitable range is $\{200,300,400,500,\ldots\}$

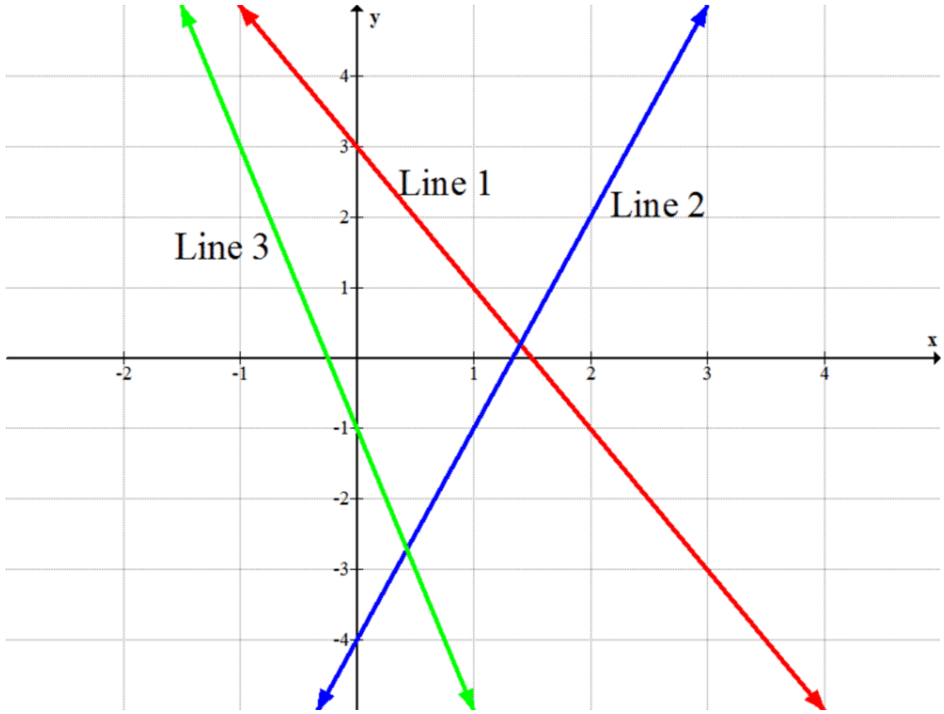
Practice Problems

Use the graph below for #1 and #2.



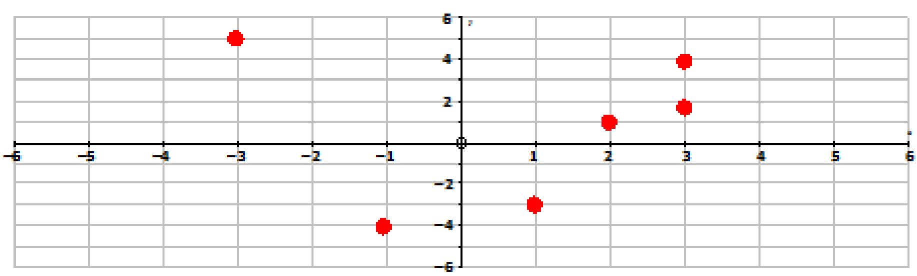
1. Is the relation discrete, continuous, or neither?
2. Find the domain and range for the relation.

Use the graph below for #3 and #4.



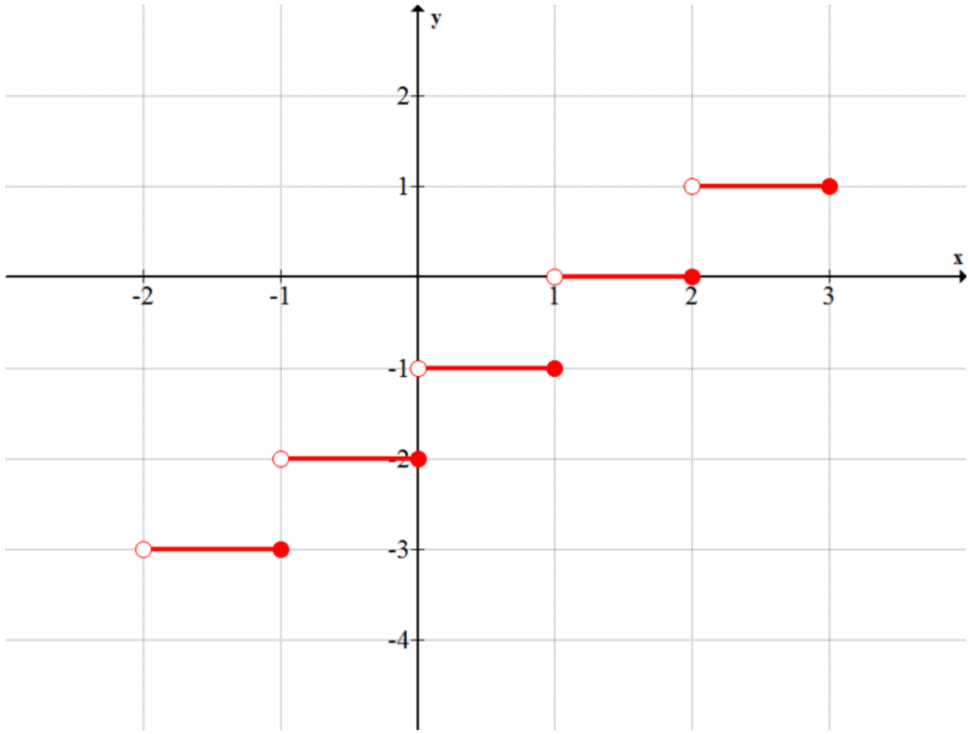
3. Is the relation discrete, continuous, or neither?
4. Find the domain and range for each of the three relations.

Use the graph below for #5 and #6.



5. Is the relation discrete, continuous, or neither?
6. Find the domain and range for the relation.

Use the graph below for #7 and #8.



7. Is the relation discrete, continuous, or neither?
8. Find the domain and range for the relation.

Examine the following pattern.

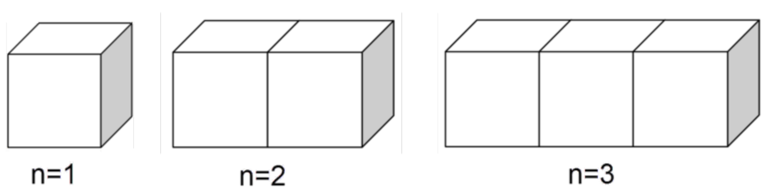


TABLE 2.6:

Number of Cubes (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**200**
Number of visible faces (f)	6	10	14						

9. Complete the table below the pattern.
10. Is the relation discrete, continuous, or neither?
11. Write a suitable domain and range for the pattern.

Examine the following pattern.

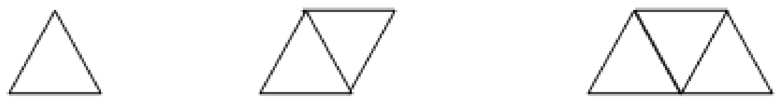


TABLE 2.7:

Number of triangles (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of toothpicks (t)									

12. Complete the table below the pattern.
13. Is the relation discrete, continuous, or neither?
14. Write a suitable domain and range for the pattern.

Examine the following pattern.

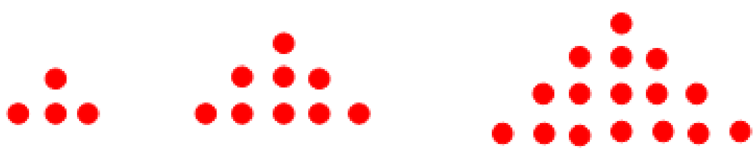


TABLE 2.8:

Pattern Number (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**100**
Number of dots (d)									

15. Complete the table below the pattern.
16. Is the relation discrete, continuous, or neither?
17. Write a suitable domain and range for the pattern.

2.4 Algebraic Equations to Represent Words

Learning Objectives

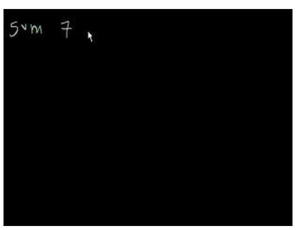
Here you will learn to translate words into algebraic equations.

Concept Problem

Peter's boss remarks to him "I noticed that the number of TPS reports you submitted this month is 7 less than twice the number you submitted last month. Good work!" Looking at his spreadsheet, Peter sees that this month he has turned in 49 TPS reports. How many did he submit last month?

Watch This

Khan Academy Problem Solving Word Problems 2



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58474

Guidance

To translate a problem from words into an equation, look for key words to indicate the operation used in the problem.

Once the equation is known, to solve the problem you use the same rules as when solving equations with one variable. Isolate the variable and then solve for it making sure that whatever you do to one side of the equals sign you do to the other side. Drawing a diagram is also helpful in solving some word problems.

Example A

The sum of two consecutive integers is 173. What are those numbers?

Solution: Let $x =$ integer #1

Then $x+1 =$ integer #2 (Because they are consecutive, they must be separated by only one number. For example: 1, 2, 3, 4,... all are consecutive.)

Translate the sentence into an equation and solve:

$$\begin{aligned}
x+(x+1) &= 173 \\
x+x+1 &= 173 && \text{(Remove the parentheses)} \\
2x+1 &= 173 && \text{(Combine like terms)} \\
2x+1-1 &= 173-1 && \text{(Subtract 1 from both sides to isolate the variable)} \\
2x &= 172 && \text{(Simplify)} \\
\frac{2x}{2} &= \frac{172}{2} && \text{(Divide both sides by 2 to solve for the variable)} \\
x &= 86 && \text{(Simplify)}
\end{aligned}$$

Therefore the first integer is 86 and the second integer is $(86+1) = 87$. Check: $86+87 = 173$.

Example B

When a number is subtracted from 35, the result is 11. What is the number?

Solution: Let $x =$ the number

Translate the sentence into an equation and solve:

$$\begin{aligned}
35-x &= 11 \\
35-35-x &= 11-35 && \text{(Subtract 35 from both sides to isolate the variable)} \\
-x &= -24 && \text{(Simplify)} \\
\frac{-x}{-1} &= \frac{-24}{-1} && \text{(Divide both sides by } -1 \text{ to solve for the variable)} \\
x &= 24 && \text{(Simplify)}
\end{aligned}$$

Therefore the number is 24.

Example C

When one third of a number is subtracted from one half of that number, the result is 14. What is the number?

Solution: Let $x =$ the number

Translate the sentence into an equation and solve:

$\frac{1}{2}x - \frac{1}{3}x = 14$

You need to get a common denominator in this problem in order to solve it. For this problem, the denominators are 2, 3, and 1. The LCD is 6. Multiply both sides of the equation by 6 to removed the fractions:

$$\begin{aligned}
(6)\frac{1}{2}x - (6)\frac{1}{3}x &= (6)14 \\
3x - 2x &= 84 && \text{(Simplify)}
\end{aligned}$$

Now, solve by combining like terms:

$$\begin{aligned}3x - 2x &= 84\\ x &= 84 && \text{(Combine like terms)}\end{aligned}$$

Therefore the number is 84.

Concept Problem Revisited

It is important to get a handle on precisely what question we are trying to answer. Then, we can define a variable to represent that answer. In this case, we are asked how many TPS reports Peter submitted last month, we can let

$x =$ number of reports submitted last month

Now, let's see what we can say about x. We can translate the verbal information in the problem into an equation that we can solve:

$$\underbrace{\text{\# TPS reports this month}}_{49}\ \underbrace{is}_{=}\ \underbrace{7}_{7}\ \underbrace{\text{less than}}_{\text{taken away from}}\ \underbrace{\text{twice the \# from last month}}_{2x}$$

$$\downarrow$$

$$49 = 2x - 7$$

Now, we solve the equation for x :

$$\begin{aligned}49{+7} &= 2x - 7{+7}\\ 56 &= 2x\\ \frac{56}{2} &= \frac{2x}{2}\\ 28 &= x\\ x &= 28\end{aligned}$$

So we know that Peter submitted 28 TPS reports last month.

Guided Practice

1. What is a number that when doubled would equal sixty?

2. The sum of two consecutive odd numbers is 176. What are these numbers?

3. The perimeter of a square frame is 48 in. What are the lengths of each side?

Answers:

1. The number is 30.

$$\begin{aligned}2x &= 60\\ \frac{2x}{2} &= \frac{60}{2} && \text{(Divide by 2 to solve for the variable)}\\ x &= 30 && \text{(Simplify)}\end{aligned}$$

2. The first number is 87 and the second number is $(87+2) = 89$.

$$\begin{aligned}
x+(x+2) &= 176 \\
x+x+2 &= 176 && \text{(Remove parentheses)} \\
2x+2 &= 176 && \text{(Combine like terms)} \\
2x+2-2 &= 176-2 && \text{(Subtract 2 from both sides of the equals sign to isolate the variable)} \\
2x &= 174 && \text{(Simplify)} \\
\frac{2x}{2} &= \frac{174}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &= 87
\end{aligned}$$

3. The side length is 12 inches.

$$\begin{aligned}
s+s+s+s &= 48 && \text{(Write initial equation with four sides adding to the perimeter)} \\
4s &= 48 && \text{(Simplify)} \\
\frac{4s}{4} &= \frac{48}{4} && \text{(Divide by 4 to solve for the variable)} \\
s &= 12
\end{aligned}$$

Practice

1. The sum of two consecutive numbers is 159. What are these numbers?
2. The sum of three consecutive numbers is 33. What are these numbers?
3. A new computer is on sale for 30% off. If the sale price is \$500, what was the original price?
4. Jack and his three friends are sharing an apartment for the next year while at university (8 months). The rent costs \$1200 per month. How much does Jack have to pay if they split the cost evenly?
5. You are designing a triangular garden with an area of 168 square feet and a base length of 16 feet. What would be the height of the triangular garden shape?
6. If four times a number is added to six, the result is 50. What is that number?
7. This week, Emma earned ten more than half the number of dollars she earned last week babysitting. If this week, she earned 100 dollars, how much did she earn last week?
8. Three is twenty-one divided by the sum of a number plus five. What is the number?
9. Five less than three times a number is forty-six. What is the number?
10. Hannah had \$237 in her bank account at the start of the summer. She worked for four weeks and now she has \$1685 in the bank. How much did Hannah make each week in her summer job?
11. The formula to estimate the length of the Earth's day in the future is found to be twenty-four hours added to the number of million years divided by two hundred and fifty. In five hundred million years, how long will the Earth's day be?
12. Three times a number less six is one hundred twenty-six. What is the number?
13. Sixty dollars was two-thirds the total money spent by Jack and Thomas at the store. How much did they spend total?
14. Ethan mowed lawns for five weekends over the summer. He worked ten hours each weekend and each lawn takes an average of two and one-half hours. How many lawns did Ethan mow?
15. The area of a rectangular pool is found to be 280 square feet. If the base length of the pool is 20 feet, what is the width of the pool?
16. A cell phone company charges a base rate of \$10 per month plus 5¢ per minute for any long distance calls. Sandra gets her cell phone bill for \$21.20. How many long distance minutes did she use?

Summary

At this point you should have a good understanding of how functions operate and their notation. In this chapter you learned how to evaluate functions are particular inputs given an equation, graph, or table.

You also learned the concepts of domain and range. You should be able to identify the domain and range of a table from a graph or function and identity any restrictions on the domain of a function from its symbolic form.

Now that we have these broad notions under our belt, it is time to get more specific. In the next chapter (and several others as well) we will narrow our focus and look at a specific type of function. What do its graphs look like? What properties does it have? What kind of questions can it answer?

CHAPTER 3 Units 3 and 4 - Linear Functions

Chapter Outline

Introduction

Here you'll learn all about lines. You will learn how to find the slope of a line and the equation of a line. You will learn to graph a line from its equation. You will learn about parallel and perpendicular lines and their relationship with slope. Finally, you will learn how to use your knowledge of lines to model and solve real-world problems. In the process, you will be introduced to a useful technique for dealing with fractions in equations. We will revisit this technique later in the discussion of rational expressions.

3.1 Introduction to Slope and Linear Functions

Learning Objectives

Here you'll learn what is meant by the "slope" of a line. You will learn how to find the slope of a line from its graph.

Concept Problem

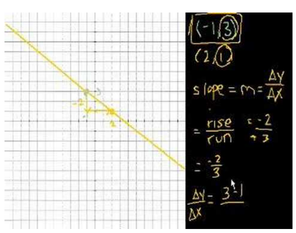
Peter works as a data processor for Initech. On Thursday afternoon, he decides that he would like to know how much money he will be paid this week, so that he can budget for the weekend. Being data-minded, he has a table to track his work hours and the pay he has received so far this week. Here is what he has recorded:

Hours	5	10	15	20	25	30
Pay ($)	80	160	240	320	400	480

Peter expects that he will work 45 hours this week. Is it possible for him to determine what his total pay for the week will be?

Watch This

Khan Academy Slope and Rate of Change



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Guidance

In order to make predictions, we have to know how quantities are going to behave. The simplest behavior that something can exhibit is to always change by the same amount. For instance, if you know that your car goes 30 miles on 1 gallon of gas, then we can predict that it will go 60 miles on 2 gallons of gas. Nothing is different between the first gallon and the second gallon. This is the central notion behind linear functions.

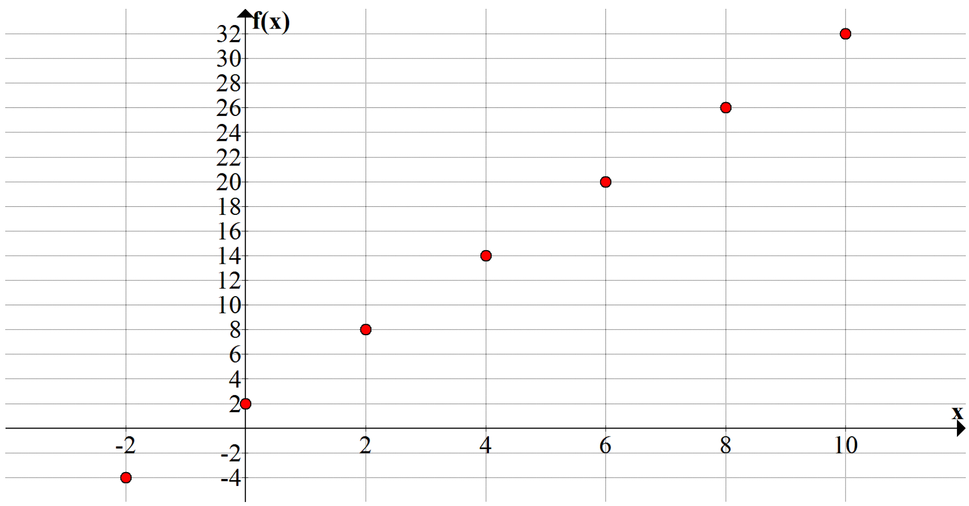
A **linear function** is a function that has a constant rate of change i.e. a change in the input value will produce a proportional change in the output value.

The table below gives some values for a linear function $f(x)$:

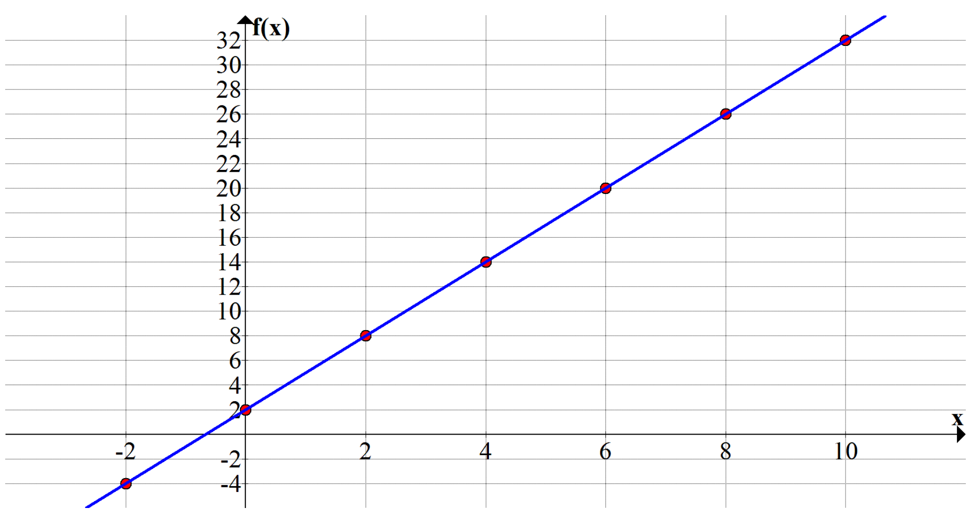
x	-2	0	2	4	6	8	10
$f(x)$	-4	2	8	14	20	26	32

Notice that at each entry, the value of x increases by 2 while the value of $f(x)$ increases by 6. It is the same whether x goes from 0 to 2 or from 8 to 10. This is a central characteristic of linear functions.

What does the graph of a linear function look like? Let's plot the points in the table above and see:

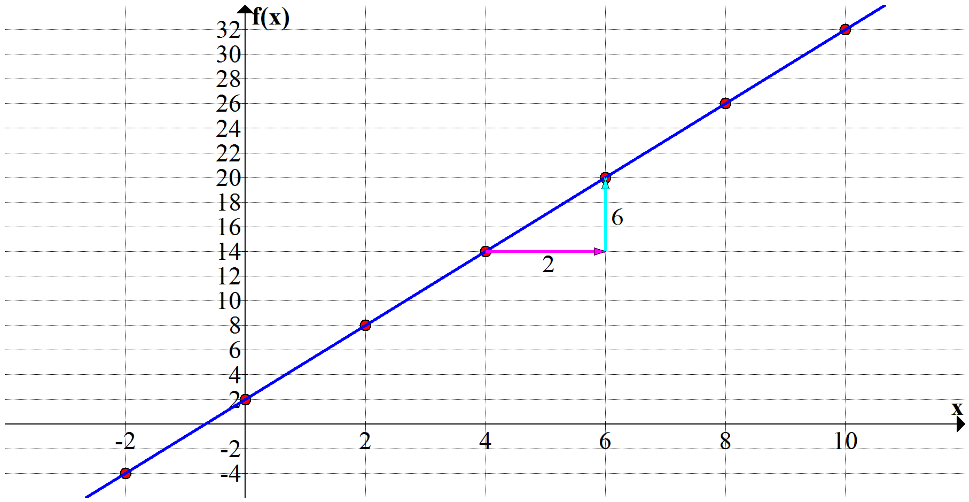


It's a straight line! We can connect the dots to see this more clearly:

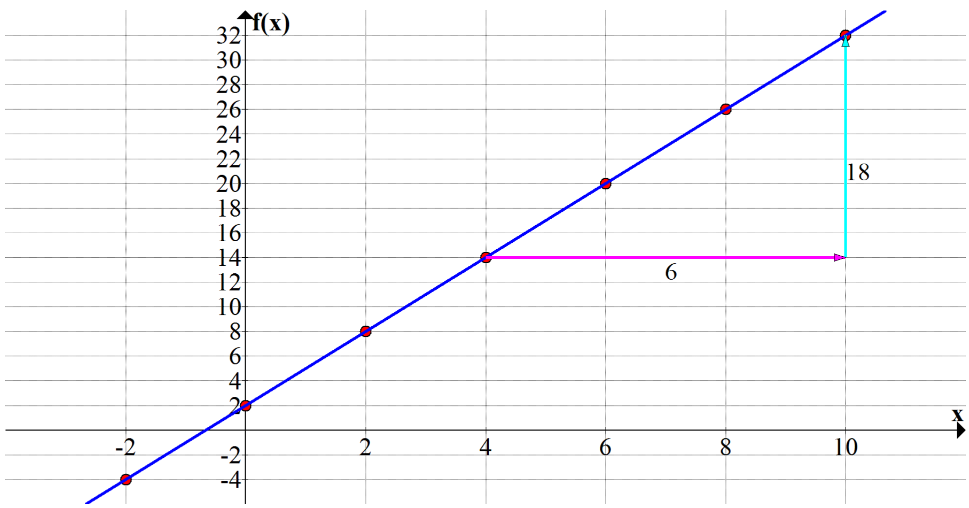


In fact, the graph of any linear function will be a line. This is hinted at in the name itself: **line**ar.

The fact we noticed above in the table can also be seen in the graph: for every 2 units the graph moves to the right, it moves up by 6 units. (Note that the graph is not to scale)



Similarly, for every 6 units the graph moves to the right, it moves up by 18 units:



In fact, these two statements are expressing the same fact in two different ways. We can see this if we look at the ratios of the changes:

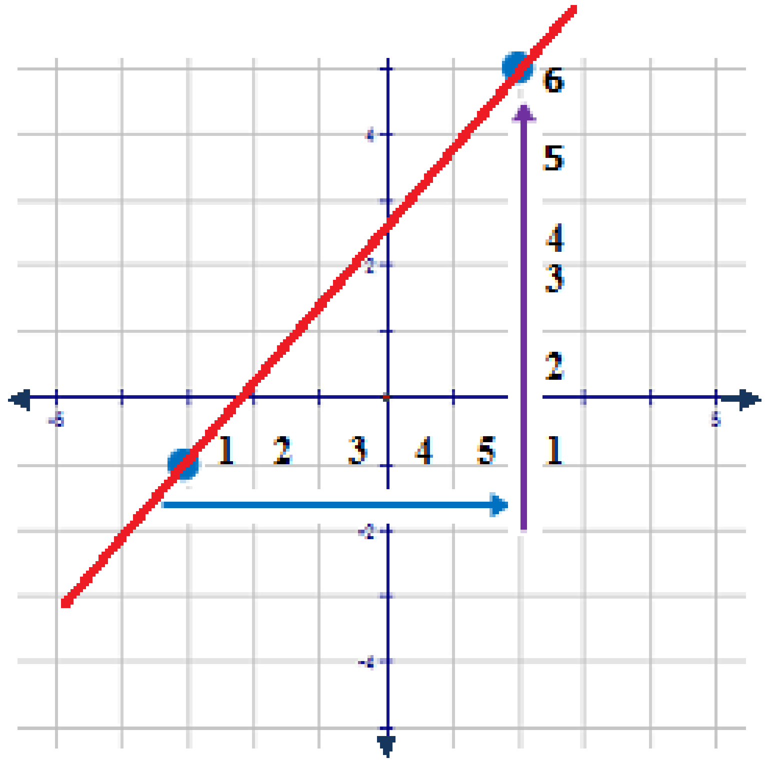
$$\frac{6 \text{ up}}{2 \text{ over}} = \frac{6}{2} = \frac{3}{1} = 3$$

$$\frac{18 \text{ up}}{6 \text{ over}} = \frac{18}{6} = \frac{3}{1} = 3$$

This ratio will be the same no matter how you move along the line; it is a fundamental characteristic of the linear function $f(x)$ and is called the *slope* of $f(x)$.

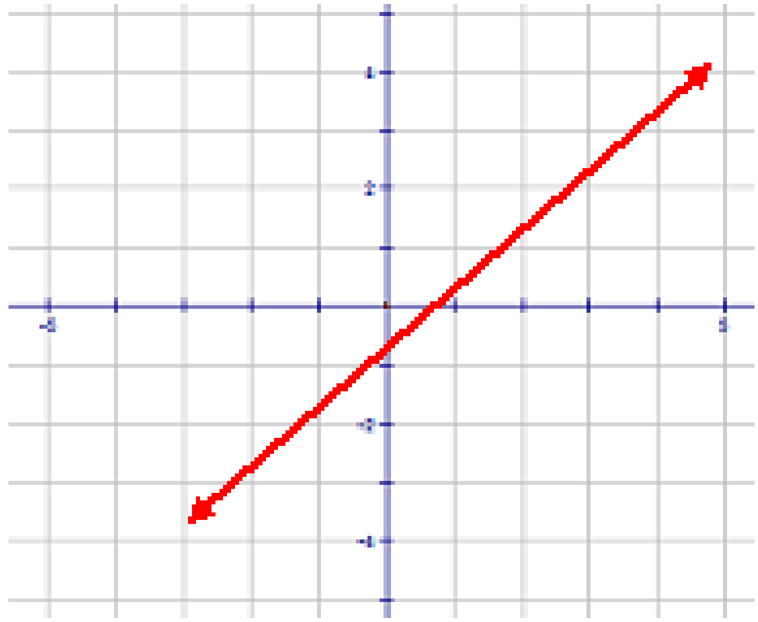
The **slope** of a linear function is the ratio of the change in its output to the change in its input. Graphically, it is the steepness, slant or gradient of the function's graph. Slope is often stated as $\frac{\text{rise}}{\text{run}}$ (rise over run). The slope of a line is represented by the letter 'm' and its value is a real number.

You can determine the slope of a line from a graph by counting. Choose two points on the line that are exact points on the Cartesian grid. Exact points mean points that are located on the corner of a box or points that have coordinates that do not have to be estimated. On the graph below, two exact points are indicated by the blue dots.



Begin with the point that is farthest to the left and RUN to the right until you are directly below (in this case) the second indicated point. Count the number of spaces that you had to run to be below the second point and place this value in the run position in the denominator of the slope. Next count the number of spaces you have to move to reach the second point. In this case you have to rise upward which indicates a positive move. This value must be placed in the rise position in the numerator of the slope.

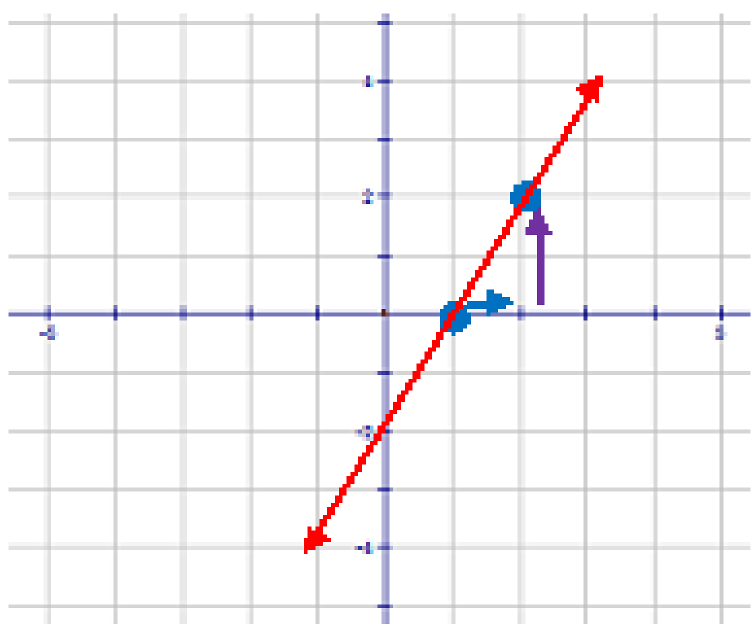
Here, you had to run 5 spaces to the right, which indicates moving 5 spaces in a positive direction. You now have $m = \frac{\text{rise}}{5}$. To reach the point directly above involved moving upward 6 spaces in a positive direction. You now have $m = \frac{6}{5}$. The slope of the above line is $\frac{6}{5}$.



In the above graph, there are not two points on the line that are exact points on the Cartesian grid. Therefore, the slope of the line cannot be determined by counting. The coordinates of points on this line would only be estimated values. When this occurs, the task of calculating the slope of the line must be presented in a different way. The slope would have to be determined from two points that are on the line and these points would have to be given.

Example A

What is the slope of the following line?

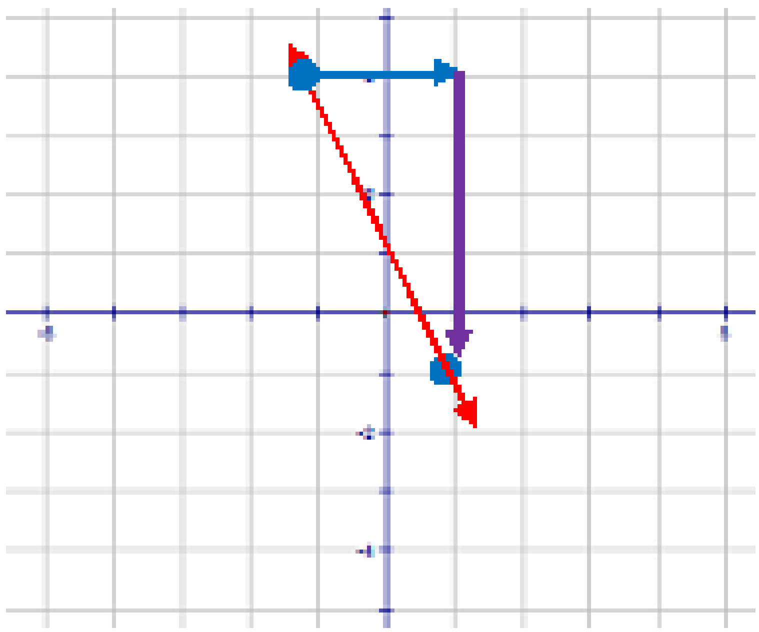


Solution: Two points have been indicated. These points are exact values on the graph. From the point to the left, run one space in a positive direction and rise upward 2 spaces in a positive direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{2}{1}$$

Example 2

What is the slope of the following line?



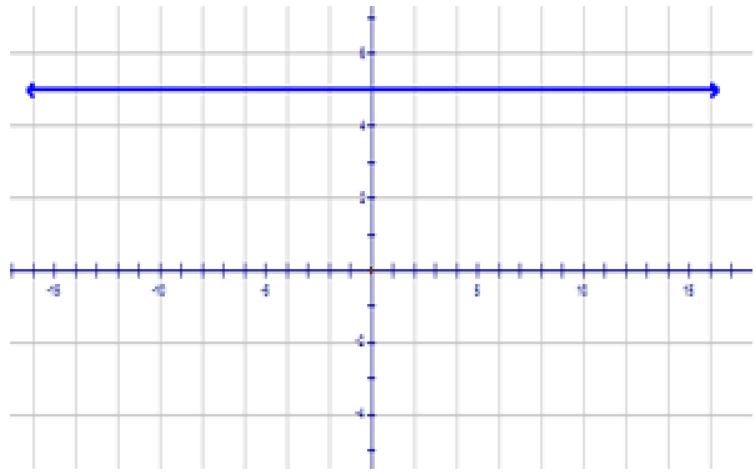
Solution: Two points have been indicated. These points are exact values on the graph. From the point to the left, run two spaces in a positive direction and move downward 5 spaces in a negative direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-5}{2}$$

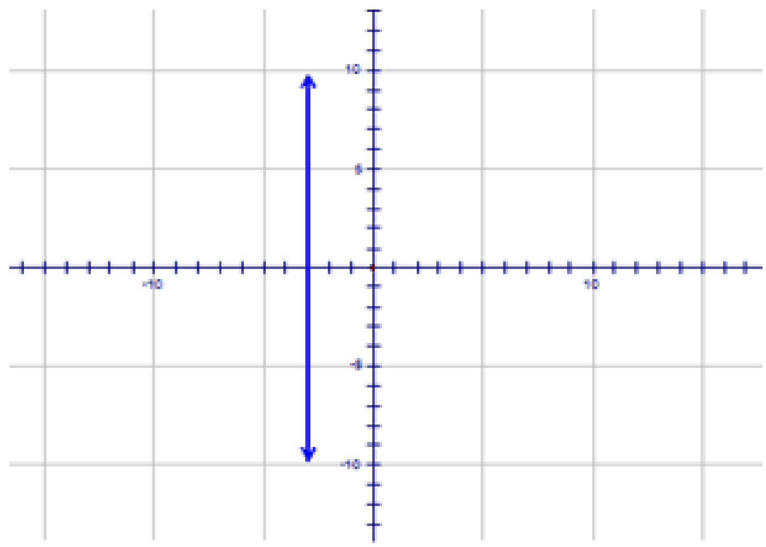
Example 3

Find the slope of each of the following lines:

(a)

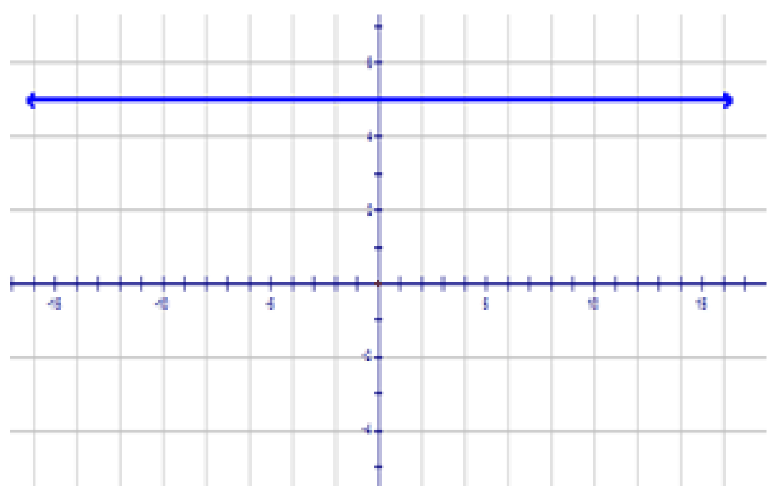


(b)



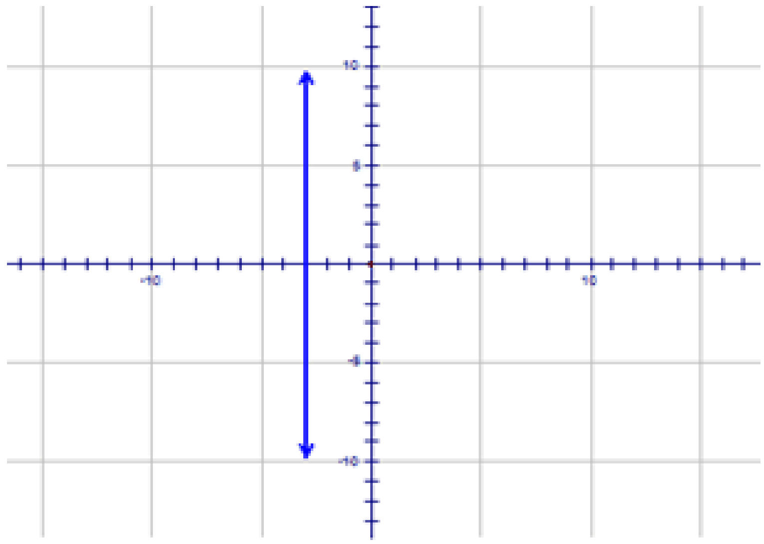
Solution:

(a)



Two points on this line are (-5, 5) and (4, 5). The rise is 0 and the run is 9. The slope is $m = \frac{0}{9} = 0$.

(b)



Two points on this line are (-3, 5) and (-3, -10). The rise is 15 and the run is 0. The slope is $m = \frac{15}{0}$. Since division by 0 is not defined, the slope of this line is *undefined*

Note that having a slope of 0 is different from having a slope that is undefined.

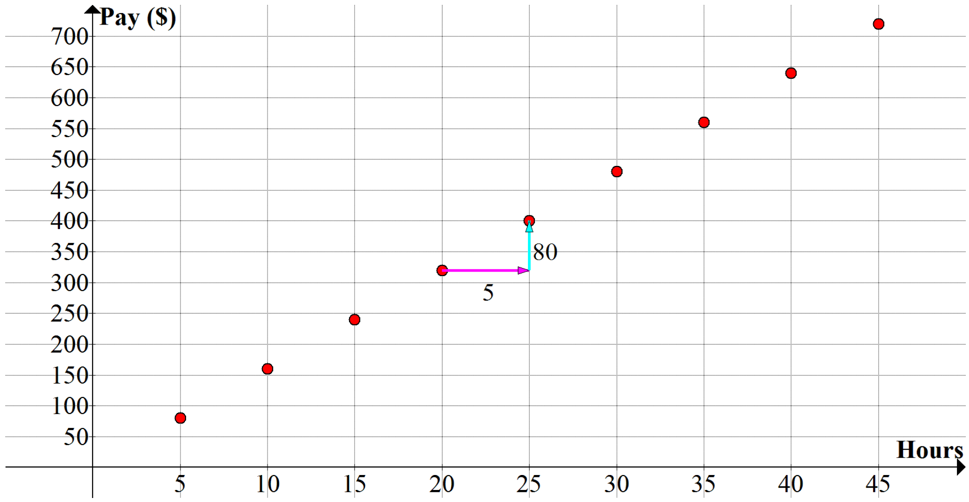
Concept Problem Revisited

We can see the pattern in Peter's table: For every 5 hours that he works he receives an additional $80. We can continue this pattern to complete the table:

Hours	5	10	15	20	25	30	35	40	45
Pay ($)	80	160	240	320	400	480	560	640	720

We can see directly that at the end of his 45 hours of work, Peter should have earned $720.

But wait, there's more! Let's plot these points and find the slope of the line:



Use the ratio to find the slope: $\frac{80}{5} = \frac{16}{1} = 16$. This tells us that for every hour Peter works, he earns \$16.

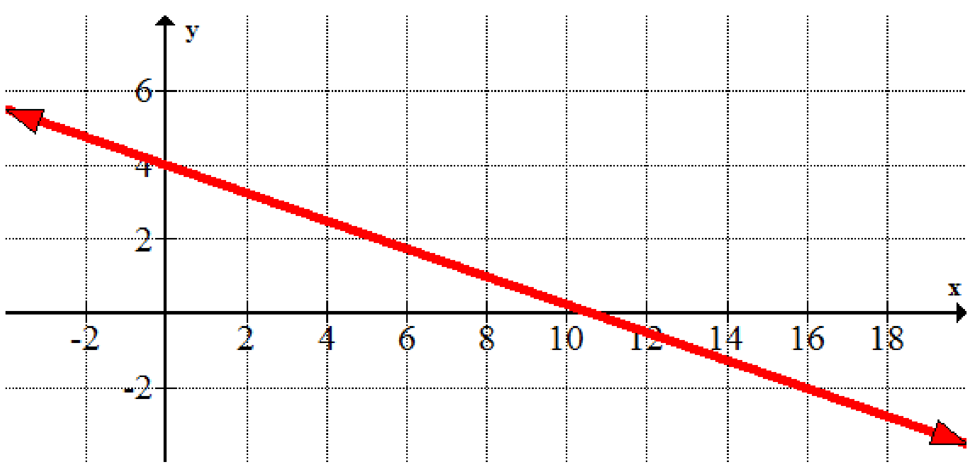
Vocabulary

Slope

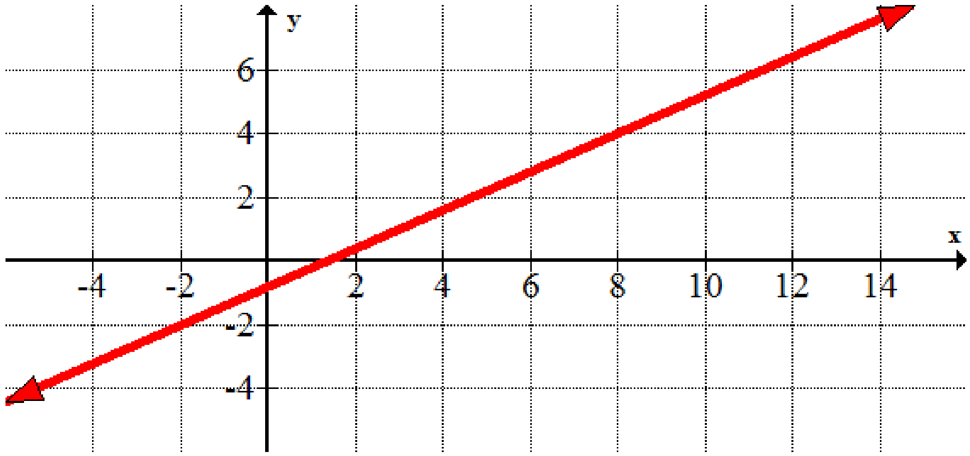
The ***slope*** of a linear function is the ratio of the change in its output to the change in its input. A graphical formula for slope is $\frac{\text{rise}}{\text{run}}$.

Practice Problems

1. Identify the slope for the following graph.



2. Identify the slope for the following graph.

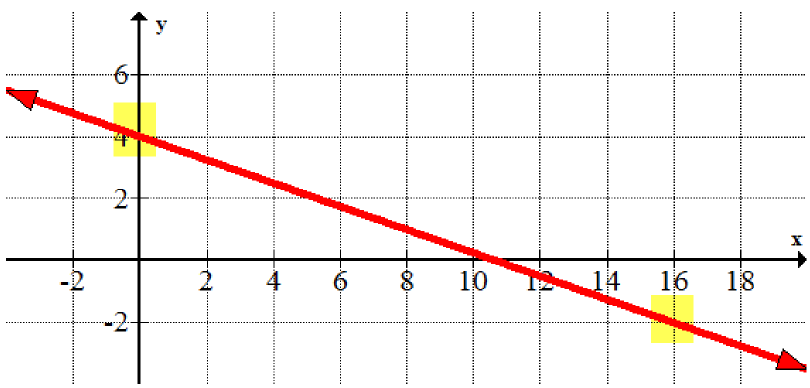


3. What is the slope of the line that passes through the point (2, 4) and is perpendicular to the x-axis?

4. What is the slope of the line that passes through the point (-6, 8) and is perpendicular to the y-axis?

Answers:

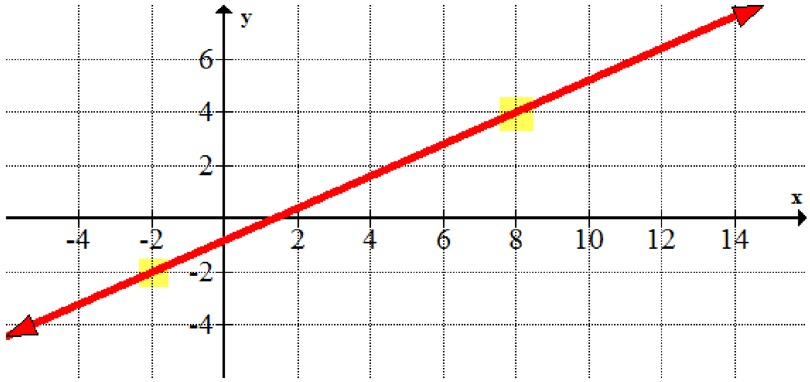
1.



Two exact points on the above graph are (0, 4) and (16, -2). From the point to the left, run sixteen spaces in a positive direction and move downward six spaces in a negative direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-6}{16}$$
$$m = \frac{-3}{8}$$

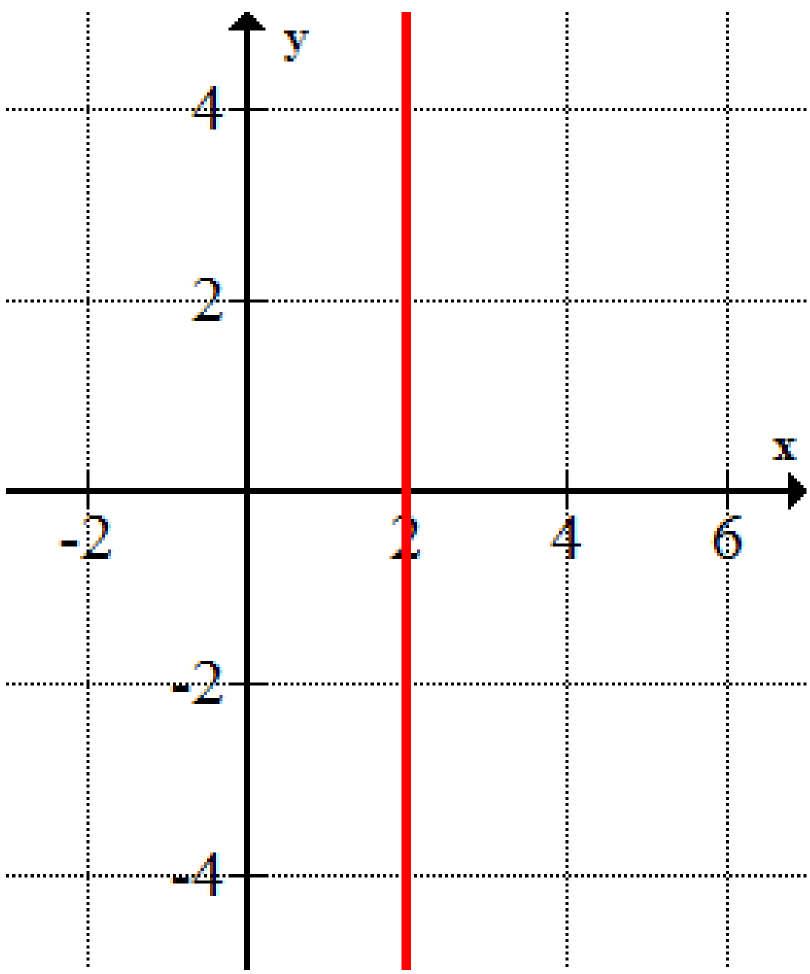
2.



Two exact points on the above graph are (-2, -2) and (8, 4). From the point to the left, run ten spaces in a positive direction and move upward six spaces in a positive direction.

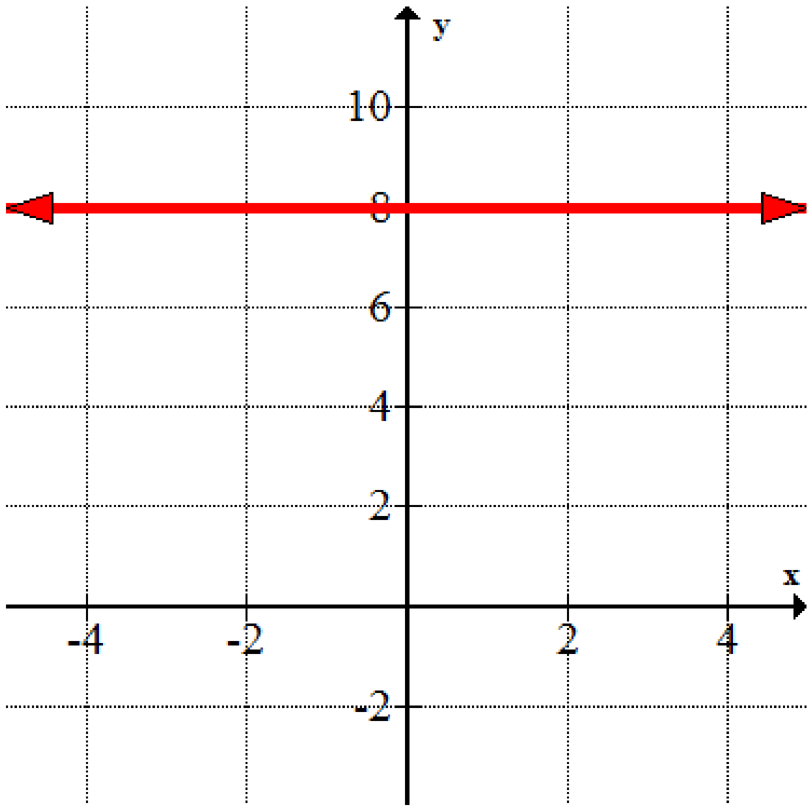
$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{6}{10}$$
$$m = \frac{3}{5}$$

3. You are not given the coordinates of two points. Sketch the graph according the information given.



A line that is perpendicular to the x-axis is parallel to the y-axis. The slope of a line that is parallel to the y-axis has a slope that is undefined.

4. You are not given the coordinates of two points. Sketch the graph according the information given.



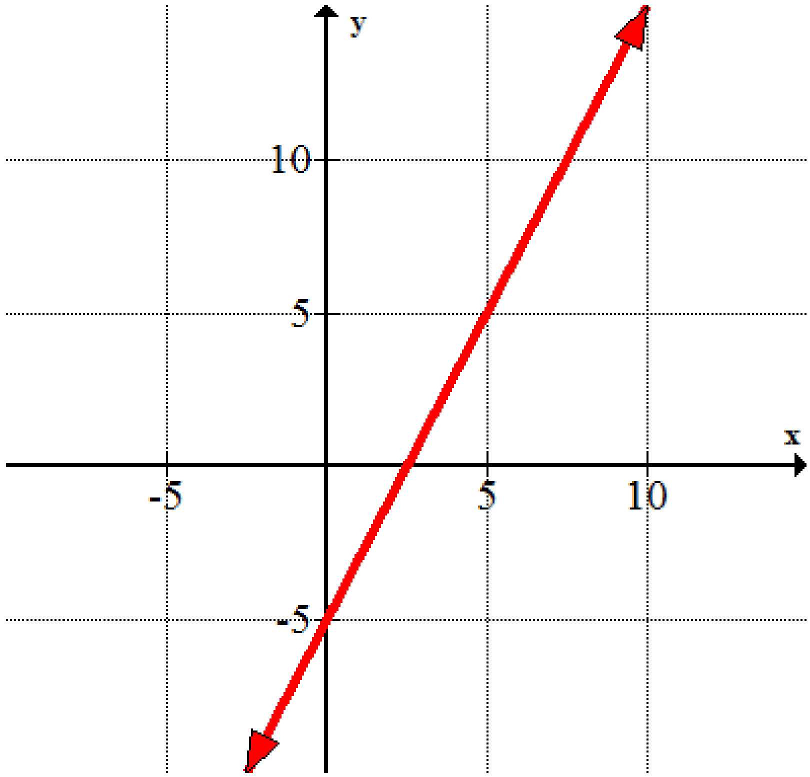
A line that is perpendicular to the y-axis is parallel to the x-axis. The slope of a line that is parallel to the x-axis has a slope that is zero.

Practice

1. Explain how to find the slope of a line from its graph.
2. What does the slope of a line represent?
3. From left to right, a certain line points upwards. Is the slope of the line positive or negative?
4. How can you tell by looking at a graph if its slope is positive or negative?
5. What is the slope of a horizontal line?
6. What is the slope of a vertical line?

Find the slope of each of the following lines.

7.



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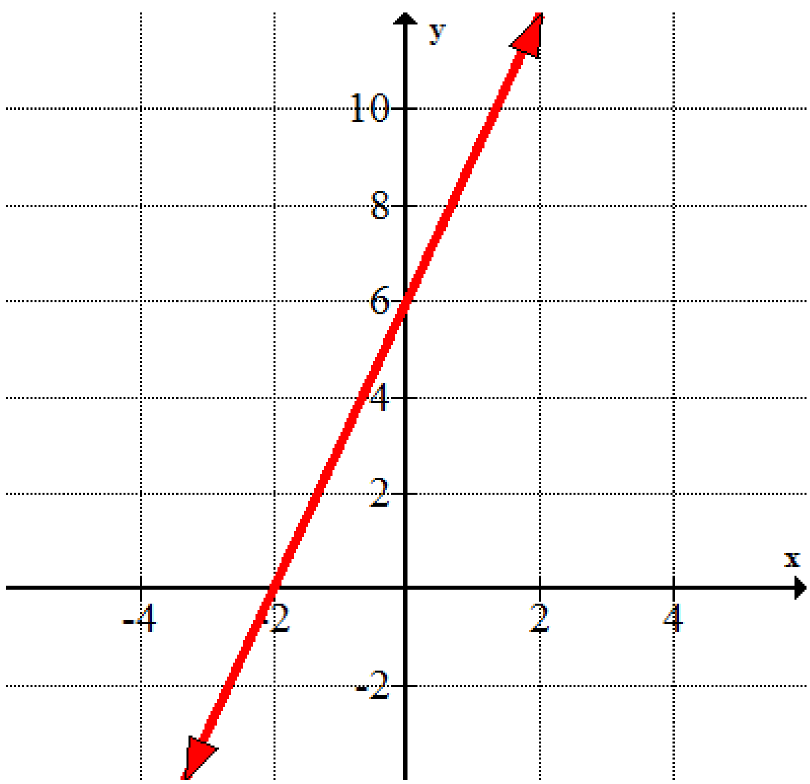

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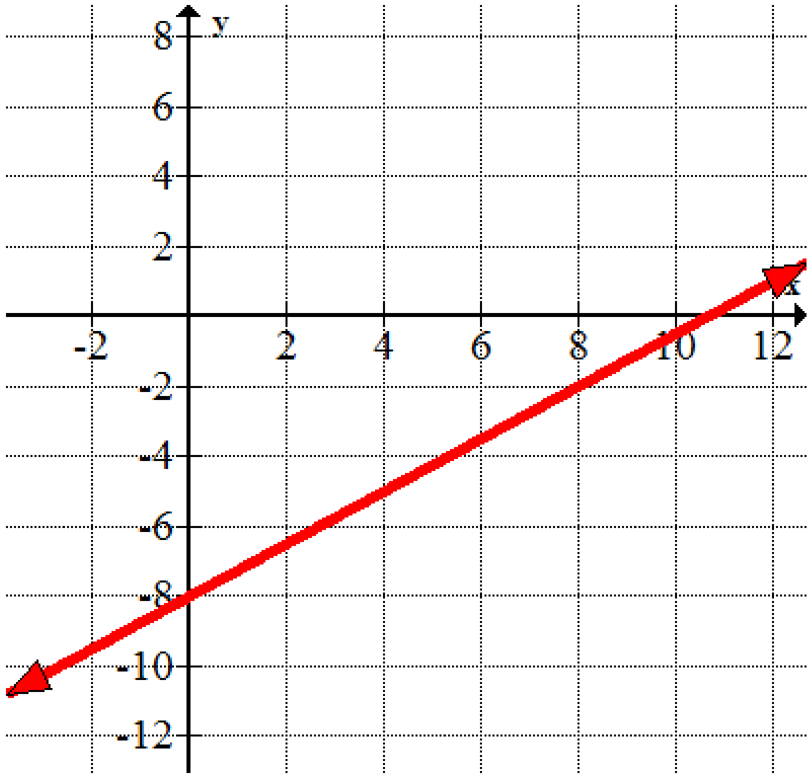

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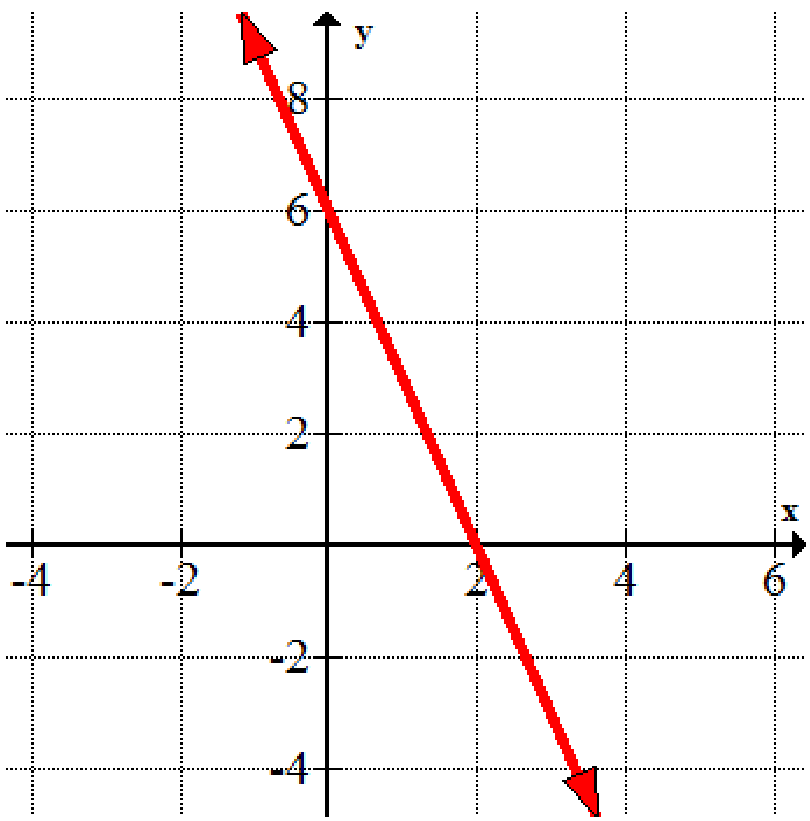

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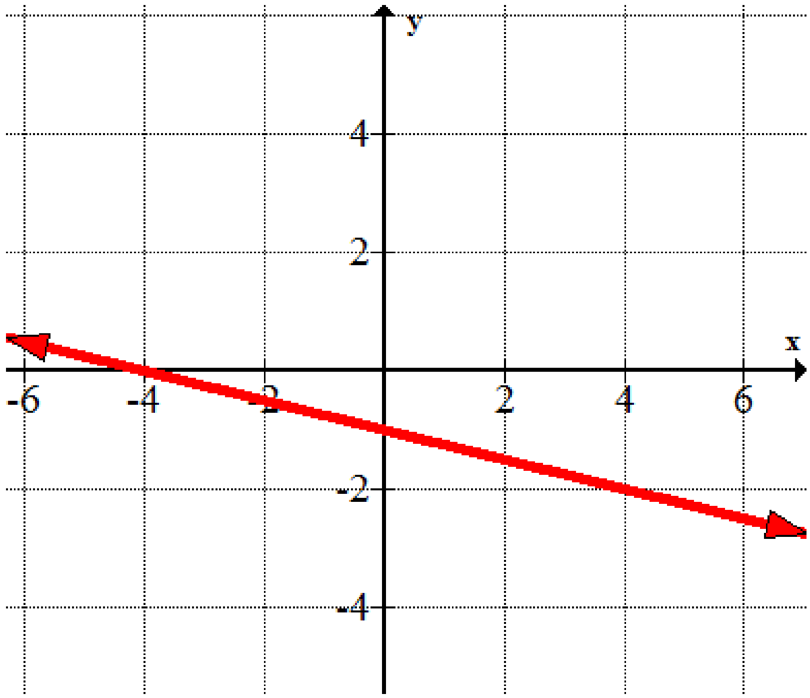

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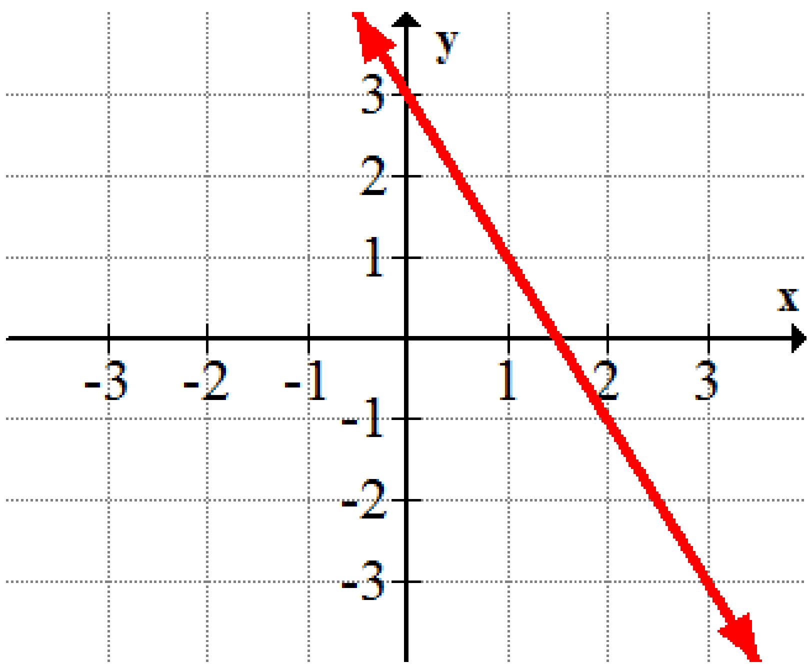

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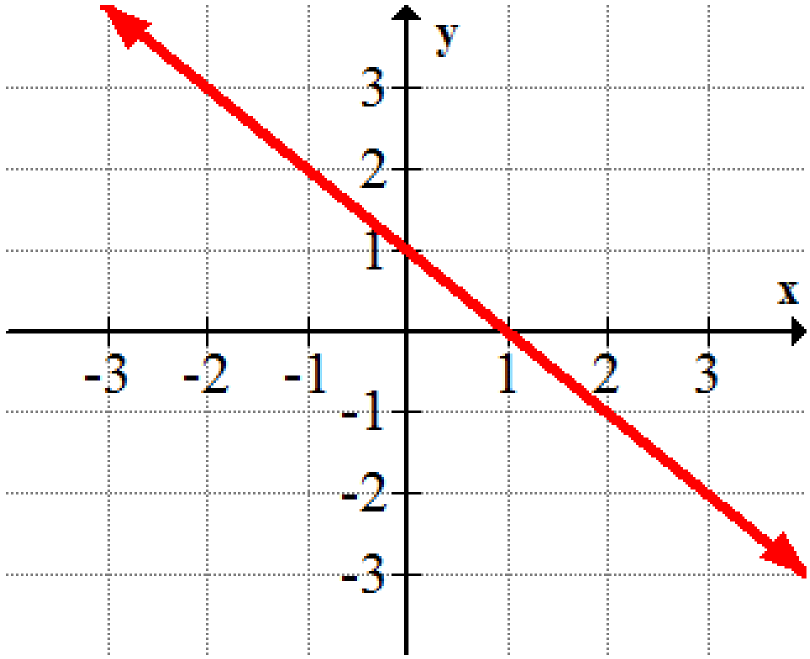
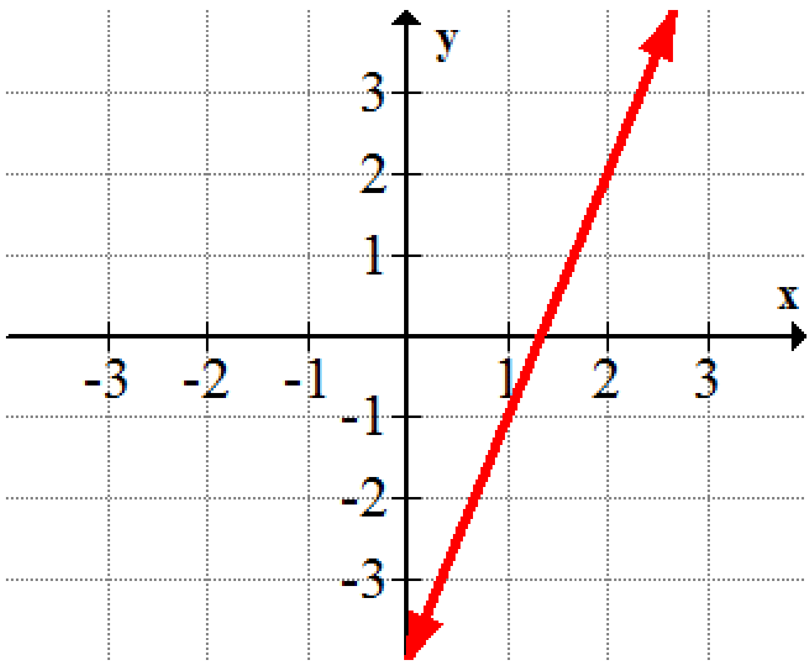


14.



Plot the pairs of points and then find the slope of the line connecting the points. Can you come up with a way to find the slope without graphing?

15. $(2,4)$ and $(-1,3)$
16. $(-4,-2)$ and $(2,7)$

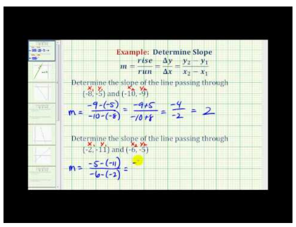
3.2 Slopes of Linear Functions from Two Points

Concept Problem

You fill your car's 24-gallon gas tank before a road trip. On the drive, you notice that the car has 21 gallons remaining after you have driven 84 miles, and that it has 15 gallons of gas remaining after you have traveled 252 miles. Can we use this information to measure the gas efficiency of your car i.e. it's miles per gallon?

Watch This

James Sousa: Ex. Determine the Slope of a Line Given Two Points on the Line]



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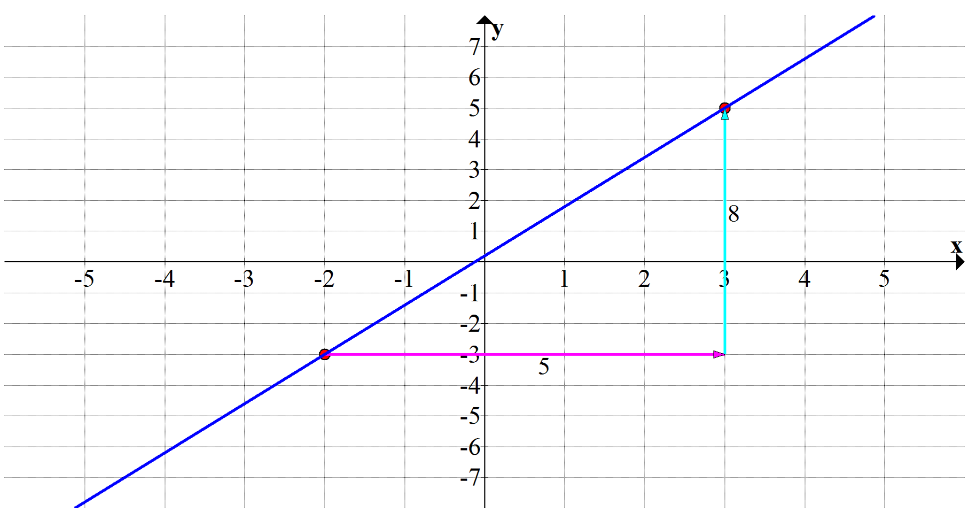
Guidance

The **slope** of a linear function is its rate of change and also measure the steepness of the line. Slope is defined as $\frac{\text{rise}}{\text{run}}$ (rise over run) or $\frac{\Delta y}{\Delta x}$ (change in y over change in x). Whatever definition of slope is used, they all mean the same.

Let's calculate the slope of a line by using the coordinates of two points on the line.

Example A

Consider a line that passes through the points $A(-2,-3)$ and $B(3,5)$. Find the slope of the line. Let's look at the picture and calculate the rise and the run.



For the slope we have $\frac{\text{rise}}{\text{run}} = \frac{8}{5}$.

The Slope Formula

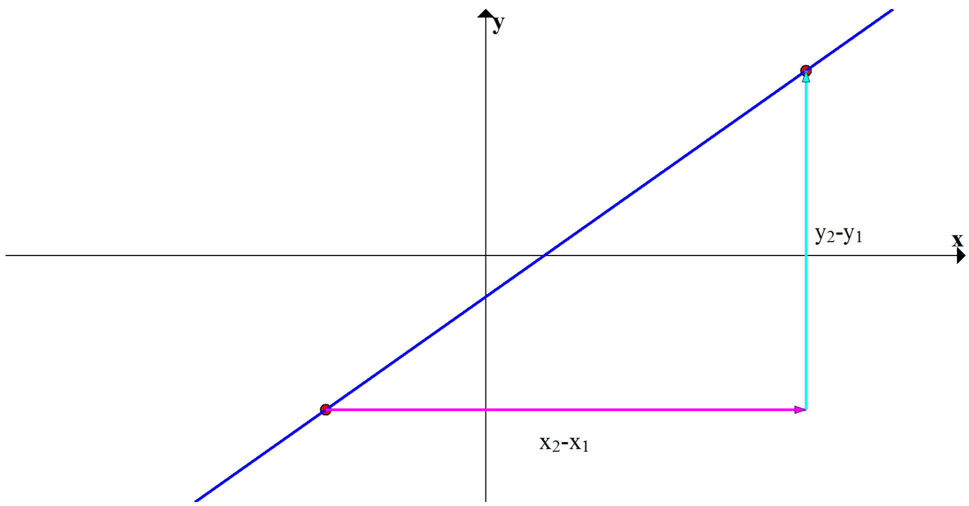
Is it possible for us to find the slope in Example 1 without drawing the picture? Where did the numbers 8 and 5 come from?

The 8 is the vertical distance between point A and point B i.e. the difference in the y-coordinates: $5-(-3)$

The 5 is the horizontal distance between point A and point B i.e. the difference in the x-coordinates: $3-(-2)$.

Rewriting the slope in these terms we have $\frac{\text{rise}}{\text{run}} = \frac{5-(-3)}{3-(-2)}$

We can use this same principle on two arbitrary points $A(x_1, y_1)$ and $B(x_2, y_2)$. Draw the picture:



For the slope we get the following expression:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is a formula that will work for any two points, where (x_1, y_1) are the coordinates of the first point and (x_2, y_2) are the coordinates of the second point. The choice of the first and second point will not affect the result.

Example B

Determine the slope of the line passing through the pair of points(-3, -8) and (5, 8).

Solution: To determine the slope of a line from two given points, the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ can be used. Don't forget to designate your choice for the first and the second point. Designating the points will reduce the risk of entering the values in the wrong location of the formula.

$$\begin{pmatrix} x_1, & y_1 \\ -3, & -8 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 5, & 8 \end{pmatrix}$$

Substitute the values into the formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$
	$m = \frac{8 - -8}{5 - -3}$
Simplify	$m = \frac{8+8}{5+3}$
Calculate	$m = \frac{16}{8}$
Simplify	$m = 2$

Example C

Determine the slope of the line passing through the pair of points $(9,5)$ and $(-1,6)$.

Solution:

$$\begin{pmatrix} x_1, & y_1 \\ 9, & 5 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -1, & 6 \end{pmatrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{6-5}{-1-9}$$

$$m = -\frac{1}{10}$$

Example D

Determine the slope of the line passing through the pair of points $(-2,7)$ and $(-3,-1)$.

Solution:

$$\begin{pmatrix} x_1, & y_1 \\ -2, & 7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -3, & -1 \end{pmatrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1-7}{-3--2}$$

$$m = \frac{-1-7}{-3+2}$$

$$m = \frac{-8}{-1}$$

$$m = 8$$

Concept Problem Revisited

Let's translate the information we are given into two points:

$$21 \text{ gallons and } 84 \text{ miles} \Rightarrow (21, 84)$$
$$15 \text{ gallons and } 252 \text{ miles} \Rightarrow (15, 252)$$

Now, use the formula to find the slope:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{252 - 84}{15 - 21} = \frac{168}{-6} = -28$$

So we conclude that for every 28 miles you drive, the gas in the tank will decrease by 1 gallon i.e. your car gets 28 miles per gallon.

Guided Practice

Calculate the slope of the line that passes through the following pairs of points:

1. (5, -7) and (16, 3)
2. (-6, -7) and (-1, -4)
3. (5, -12) and (0, -6)
4. The local *Wine and Dine Restaurant* has a private room that can serve as a banquet facility for up to 200 guests. When the manager quotes a price for a banquet she includes the cost of the room rent in the price of the meal. The price of a banquet for 80 people is $900 while one for 120 people is $1300.

i) Plot a graph of cost versus the number of people.
ii) What is the slope of the line and what meaning does it have for this situation?

Answers:

1. The slope is $\frac{10}{11}$.

$$\begin{pmatrix} x_1, & y_1 \\ 5, & -7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 16, & 3 \end{pmatrix}$$ Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3 - -7}{16 - 5}$$ Fill in the values

$$m = \frac{3 + 7}{16 - 5}$$ Simplify the numerator and denominator (if possible)

$$m = \frac{10}{11}$$ Calculate the value of the numerator and the denominator

2. The slope is $\frac{3}{5}$.

$$\begin{pmatrix} x_1, & y_1 \\ -6, & -7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -1, & -4 \end{pmatrix}$$ Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4 - -7}{-1 - -6}$$ Fill in the values

$$m = \frac{-4+7}{-1+6}$$ Simplify the numerator and denominator (if possible)

$$m = \frac{3}{5}$$ Calculate the value of the numerator and the denominator

3. The slope is $-\frac{6}{5}$.

$$\begin{pmatrix} x_1, & y_1 \\ 5, & -12 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 0, & -6 \end{pmatrix}$$ Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

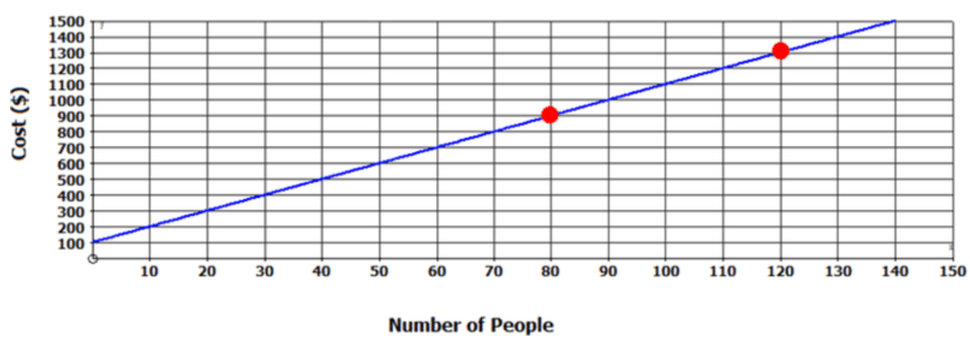
$$m = \frac{-6 - -12}{0 - 5}$$ Fill in the values

$$m = \frac{-6+12}{0-5}$$ Simplify the numerator and denominator (if possible)

$$m = \frac{6}{-5}$$ Calculate the value of the numerator and the denominator

$$m = -\frac{6}{5}$$

4.



The domain for this situation is N. However, to demonstrate the slope and its meaning, it is more convenient to draw the graph as $x \varepsilon R$ instead of showing just the points on the Cartesian grid. The x-axis has a scale of 10 and the y-axis has a scale of 100. The slope can be calculated by counting to determine $\frac{\text{rise}}{\text{run}}$.

From the point to the left, run four spaces (40) in a positive direction and move upward four spaces (400) in a positive direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{400}{40}$$
$$m = \frac{10}{1}$$
$$m = \frac{10 \; dollars}{1 \; person}$$

The slope represents the cost of the meal for each person. It will cost $10 per person for the meal.

Practice Problems

Calculate the slope of the line that passes through the following pairs of points:

1. (3, 1) and (-3, 5)
2. (-5, -57) and (5, -5)
3. (-3, 2) and (7, -1)
4. (-4, 2) and (4, 4)
5. (-1, 5) and (4, 3)
6. (0, 2) and (4, 1)
7. (12, 15) and (17, 3)
8. (2, -43) and (2, -14)
9. (-16, 21) and (7, 2)

The cost of operating a car for one month depends upon the number of miles you drive. According to a recent survey completed by drivers of midsize cars, it costs $124/month if you drive 320 miles/month and $164/month if you drive 600 miles/month.

10. Plot a graph of distance/month versus cost/month.
11. What is the slope of the line and what does it represent?

A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for $1950 and 15 in another for $2850. The number of computers and the cost forms a linear relationship.

12. Plot a graph of number of computers sold versus cost.
13. What is the slope of the line and what does it represent?

Shop Rite sells one-quart cartons of milk for $1.65 and two-quart cartons for $2.95. Assume there is a linear relationship between the volume of milk and the price.

14. Plot a graph of volume of milk sold versus cost.
15. What is the slope of the line and what does it represent?

Some college students, who plan on becoming math teachers, decide to set up a tutoring service for high school math students. One student was charged $25 for 3 hours of tutoring. Another student was charged $55 for 7 hours of tutoring. The relationship between the cost and time is linear.

16. Plot a graph of time spent tutoring versus cost.
17. What is the slope of the line and what does it represent?

3.3 Equations of Lines

Learning Objectives

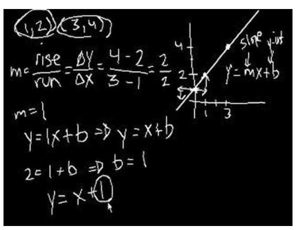
Here you'll learn how to find the equation of a line in slope intercept form or standard form.

Concept Problem

A supermarket stocks 120 boxes of Chips Ho! cookies. Six days later, the inventory manager counts 78 boxes remaining. Supposing that the same number of boxes are sold every day, he would like to create an equation that he can use to predict when the store will need to restock. Is it possible to do this?

Watch This

Khan Academy Equation of a Line



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Guidance

So far we have dealt with linear functions using tables and graphs. These are good for getting 'the big picture' for a function, but to answer questions accurately we need a symbolic way to represent them. There are several different ways to express a linear function using an equation, but the the primary one is...

Slope-Intercept Form

The **slope-intercept** form of a line looks like this:

$$y = mx + b \qquad \text{(y= notation)}$$

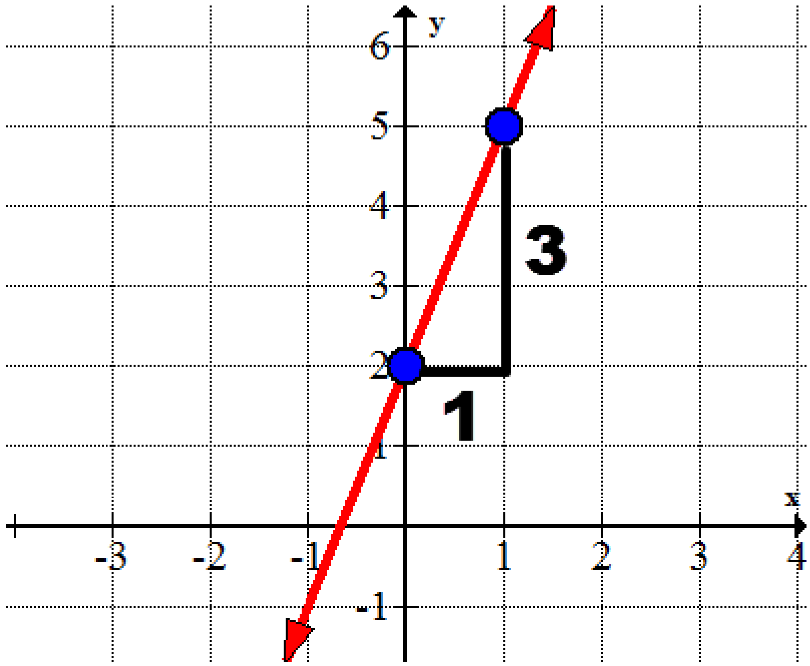
OR

$$f(x) = mx + b \qquad \text{(function notation)}$$

In this formula m is the slope of the line and b is the y-intercept (or vertical intercept) of its graph.

Example A

Find the equation of the line graphed below:



In the above graph, the line crosses the y-axis at the point (0, 2). This is the y-intercept of the line, so $b = 2$. The slope is $m = \frac{\text{rise}}{\text{run}} = \frac{3}{1} = 3$. Now that you know the values for m and b, you can write the equation of the line:

$$\boxed{y = 3x + 2.}$$

Example B

A line passes through the points $(7,9)$ and $(0,5)$. Find its equation in slope-intercept form.

Solution: We can find the slope of the line using the slope formula: $m = \frac{9-5}{7-0} = \frac{4}{7}$. Now what? Consider the second point, $(0,5)$. Where does it lie? Since its x-coordinate is 0, it must sit right on the y-axis... It's the y-intercept! So the y-coordinate 5 is the value of b.

Putting it all together we get $y = \frac{4}{7}x + 5$.

Example C

A linear function $g(t)$ has $g(0) = 4$ and $g(-3) = 10$. Find the formula for $g(t)$ in slope-intercept form.

Solution: We can translate the information given in function notation into points. If $g(-3) = 10$ then the point$(-3, 10)$ lies on the graph of $g(t)$. Similarly, the point $(0,4)$ lies on the graph. Notice that this point is the y-intercept, so we can say that $b = 4$.

Next, we can find the slope using the slope formula: $m = \frac{4-10}{0-(-3)} = \frac{-6}{3} = -2$.

Putting it together we have $g(t) = -2t + 4$.

Example D

Use the table below to find an equation for y in terms of x in slope-intercept form.

x	-12	-8	-4	0	4
y	16	13	10	7	4

This tables gives us five points that lie on the line. We can find the slope by taking two of them (any two) and using the slope formula. Let's use $(4,4)$ and $(-8,13)$.

The slope formula gives us: $m = \frac{4-13}{4-(-8)} = \frac{-9}{12} = -\frac{3}{4}$.

Now, what about b? We have five points on the table. Is one of them the y-intercept?

Yes! The y-intercept happens when the value of x is 0, and that point is given on the table: $(0,7)$. So now we know that $b = 7$.

Putting it all together we get $y = -\frac{3}{4}x + 7$.

Concept Problem Revisited

Solution: Let's see if we can convert the information in the problem into mathematical objects. First, we need to define some variables:

$$t = \text{\# of days}$$
$$C = \text{\# of boxes of cookies remaining}$$

Since the same number of boxes are sold every day, there is a linear relationship between t and C.

Let's find some points: There were 120 boxes initially, so we have the point $(0,120)$. Similarly, there were 78 boxes left 6 days later, so we also have the point $(6,78)$.

Notice that the y-intercept is one of the points, so we have $b = 120$.

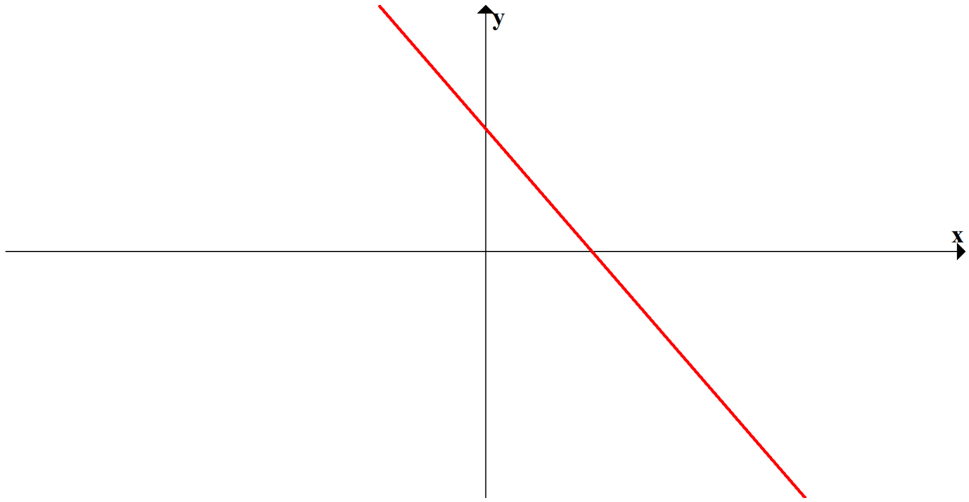
Now, the slope formula: $m = \frac{78-120}{6-0} = \frac{-42}{6} = -7$. This means that the store sells 7 boxes of cookies each day.

Putting it all together our equation is $C = -7t + 120$.

Example E

Sketch the graph of $y = mx + b$ where $m < 0$ and $b > 0$.

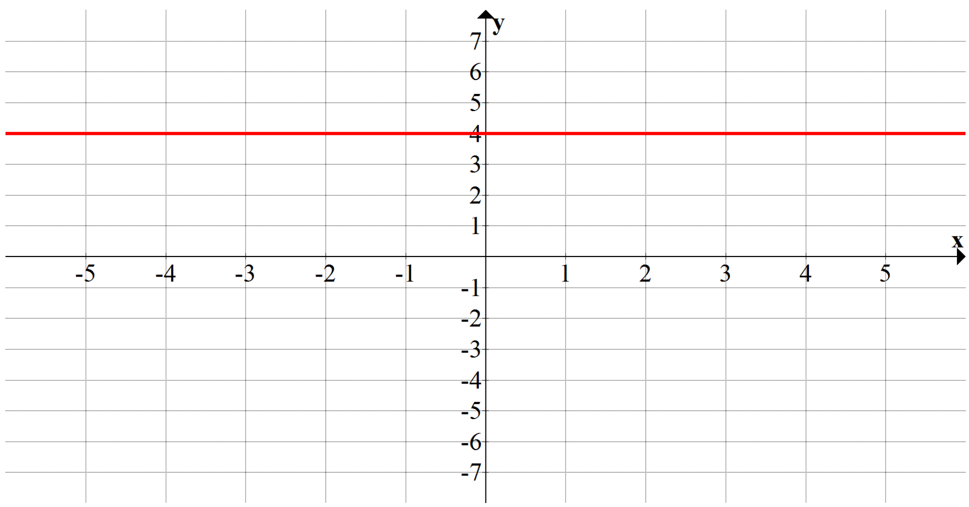
Solution: We don't have any numbers, so how are we supposed to make a graph? Instead of being specific, we can think in general terms. If $m < 0$ that means the slope of the graph is negative, so it will go down as the graph moves from left to right. On the other hand, since $b > 0$ we know that the y-intercept is positive, so the graph will hit the y-axis at a positive value. Putting those two facts together we can make this sketch:



Horizontal and Vertical Lines

Horizontal Lines

A **horizontal line** is a line that runs only left and right and does not move vertically. The graph of a horizontal line looks like this:



What is the equation of this line? We can see that the y-intercept is 4, so we just need to find the slope. We can pick any two points on the line and use the slope formula to figure it out. Let's use $(1,4)$ and $(3,4)$.

The slope is $m = \frac{4-4}{3-1} = \frac{0}{2} = 0$, which means the equation is:

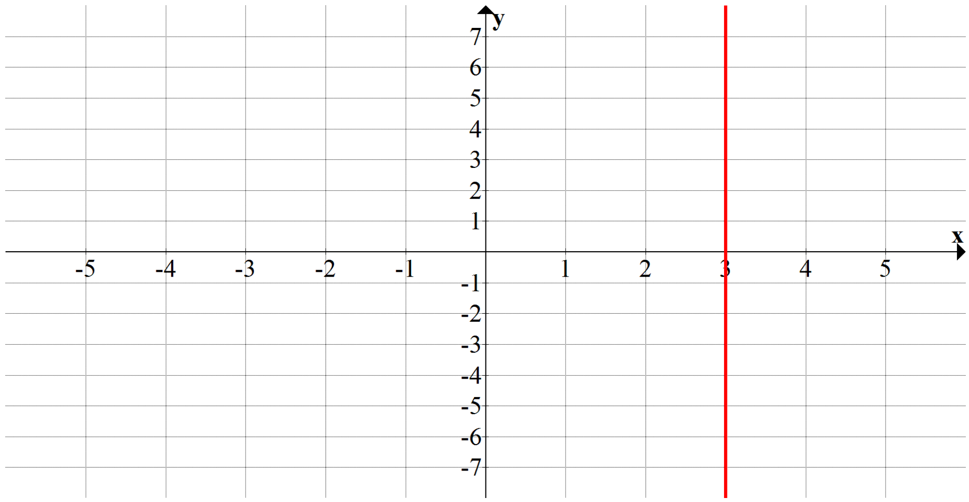
$$y = 0x + 4$$
$$y = 4$$

This will happen with any horizontal line.

- The slope of a horizontal line is 0.
- The equation of a horizontal line will have the form $y = b$ where b is the y-intercept of the graph.

Vertical Lines

A **vertical line** is a line that only runs up and down, it does not move horizontally. The graph of a vertical line looks like this:



What is the equation of this line? Let's pick two points and use the slope formula as we did with the horizontal line. In this case, let's use $(3,1)$ and $(3,5)$:

$$m = \frac{5-1}{3-3} = \frac{4}{0} = ???$$

We've hit a roadblock. Division by zero is undefined! What does this mean about the slope?

A vertical line does not have a slope in the sense that we have been using so far. Does that mean there is no equation for our line?

You can find an equation for a vertical line, but it looks different than other equations we have seen so far. The situation is analogous to what we saw with horizontal lines:

- The slope of a vertical line is undefined
- The equation of a vertical line is $x = a$ where a is the x-intercept of the line

Example F

(a) Write the equation of the horizontal line passing through the point (6, -4).

(b) Write the equation of the vertical line passing through the point (3, -2).

Solution:

(a) A line that is horizontal has the equation $y = a$, where a is the $y-$coordinate of the point through which the line passes. Therefore, the equation of this line is

$$\boxed{y = -4}$$

.

(b) A line that is vertical has the equation $x = c$, where c is the $x-$coordinate of the point through which the line passes. Therefore, the equation of this line is

$$\boxed{x = 3}$$

Point-Slope Form of a Line

In the examples we have done so far, the y-intercept has been given to us in one form or another. What if we are given two points but neither of them is the y-intercept?

In this case, there is another form of the linear equation that we can use, called the **point-slope** form. There are two equivalent versions of this form. Here they are:

$$y = m(x - x_1) + y_1$$

OR

$$y - y_1 = m(x - x_1)$$

There's a lot going on there! What does it all mean??

- As you might have guessed, m is the slope of the line, just like in the slope-intercept form.
- x_1 and y_1 are the coordinates of a point on the line. Any point at all will do as long as it is on the line.
- The variables of the equation are x and y. Usually, we will leave them as variables and not plug numbers in for them.

Example G

Find the equation of the line passing through the points $(7,12)$ and $(10,19)$ in point-slope form.

Solution: We need a slope, so let's use the slope formula on our points: $m = \frac{19-12}{10-7} = \frac{7}{3}$.

Now, we need a point. There were two given to us in the problem; does it matter which one we choose? Nope! Either one will get the job done. Let's use $(7,12)$ so $x_1 = 7$ and $y_1 = 12$.

Plug the values of $m, x_1,$ and y_1 into the point-slope form to produce the equation:

$$y = \frac{7}{3}(x - 7) + 12$$

Steps for Finding a Linear Equation

We usually use the point-slope form as a stepping stone on the way to the slope-intercept form. The point-slope form will give us *an* equation for our line, and the we clean it up into the form that we like. Here are some steps your can follow to find the equation of a line from two points.

1. Use the slope formula to find the slope of the line
2. Plug the slope and the coordinates of one of the points into the point-slope form
3. 'Clean up' the equation into slope-intercept form i.e. solve for y, combine like terms

Example H

Write the equation for the line that passes through the points $A(3,4)$ and $B(8,2)$.

Solution:

1. Determine the slope of the line:

$$\begin{pmatrix} x_1, & y_1 \\ 3, & 4 \end{pmatrix} \qquad \begin{pmatrix} x_2, & y_2 \\ 8, & 2 \end{pmatrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{2-4}{8-3}$$
$$m = -\frac{2}{5}$$

2. Pick a point and plug into the point-slope form. In this case we will use $(3,4)$

$$y = m(x - x_1) + y_1$$
$$y = -\frac{2}{5}(x - 3) + 4 \qquad \text{Use (3,4) for } (x,y) \text{ and } \left(-\frac{2}{5}\right) \text{ for } m.$$

3. Clean up the equation into slope-intercept form:

$$y = -\frac{2}{5}x + \frac{6}{5} + 4 \qquad \text{Distribute}$$
$$y = -\frac{2}{5}x + \frac{26}{5} \qquad \text{Add numbers together}$$

The equation for the line is

$$\boxed{y = -\frac{2}{5}x + \frac{26}{5}}$$

Standard Form of a Line

There is one more form of the linear equation, known as standard form. Standard form is

$$Ax + By = C$$

where $A, B,$ and C are integers and $A \geq 0$. For instance, $4x - 7y = 13$ is in standard form.

You can rewrite the equation of a line given in standard form as an equation in slope-intercept form by solving for y. This will allow you to determine the slope and y-intercept of the line.

Example I

Rewrite the equation $3x+2y=8$ in slope-intercept form

$$\begin{aligned}3x+2y&=8\\3x-3x+2y&=8-3x\\2y&=-3x+8\\\frac{2y}{2}&=\frac{-3x}{2}+\frac{8}{2}\\\boxed{y=\frac{-3}{2}x+4}\end{aligned}$$

The equation has been solved for y and is now in the form $y=mx+b$. You can now see that the slope of the line is $m=-\frac{3}{2}$ and its y-intercept is 4.

Finding and Equation in Standard Form

If you are given two points and asked to find the equation in standard form, the procedure is the same as finding the equation in slope-intercept form with two more steps at the end:

1. Multiplying by the denominator of m to clear the fractions
2. Getting x and y on the same side of the equation.
3. If necessary, multiply by -1 to make the coefficient on x positive.

Example J

Find the standard form equation of the line passing through the points $(3,5)$ and $(6,-2)$.

Solution: We need to find the slope-intercept form of the line before we get to standard form, so let's go through those steps first:

1. Find the slope: $m=\frac{-2-5}{6-3}=\frac{-7}{3}=-\frac{7}{3}$
2. Plug into the point-slope form. We will use $(6,-2)$ as our point:

   $y=m(x-x_1)+y_1$
3. Clean up the equation to put it into slope-intercept form:

$$\begin{aligned}y&=-\frac{7}{3}(x-6)-2\\y&=-\frac{7}{3}+14-2 && \text{Distribute } -\frac{7}{3}\\y&=-\frac{7}{3}+12 && \text{Combine like terms}\end{aligned}$$

Now that we have the slope-intercept form, we convert it to standard form:

1. Multiply both sides of the equation by 3 to remove the fractions:

$$y = -\frac{7}{3}x + 12$$
$$3 \cdot y = 3 \cdot \left(-\frac{7}{3}x + 12\right) \qquad \text{Multiply by 3}$$
$$3y = -7x + 36$$

2. The last step is to move the x term to the left side of the equation:

$$3y = -7x + 12$$
$$7x + 3y = -7x + 12 + 7x \qquad \text{Add 7x to both sides}$$
$$7x + 3y = 12$$

So in standard form the equation of the line is $7x + 3y = 12$

Example K

Write the equation of the line that passes through the point (-2, 5) and has the same y-intercept as: $-3x + 6y = -18$.

Solution: In order to find the equation of any line, you can figure out the slope and the intercept. First, find the y-intercept.

$$-3x + 6y = -18 \qquad \text{Want to solve for y}$$
$$+3x - 3x + 6y = +3x - 18 \qquad \text{Add 3x}$$
$$6y = 3x - 18$$
$$\frac{6y}{6} = \frac{3}{6}x - \frac{18}{6} \qquad \text{Divide by 6}$$
$$y = \frac{1}{2}x - 3$$

You can see now that $b = -3$. The line also passes through the point (-2, 5). You can use this point along with the y-intercept to help find the slope. Let's use that point and the y-intercept $(0, -3)$ in the slope formula to find the slope:

$$m = \frac{5 - (-3)}{0 - 2} = \frac{8}{-2} = -4$$

So the equation of the line is

$$\boxed{y = -4x - 3}$$

Vocabulary

Slope - Intercept Form

The ***slope-intercept form*** is $y = mx + b$ where m refers to the slope and b identifies the y-intercept.

Point-Slope Form

The ***point-slope form*** is $y = m(x - x_1) + y_1$ where m refers to the slope and (x_1, y_1) is a point on the line

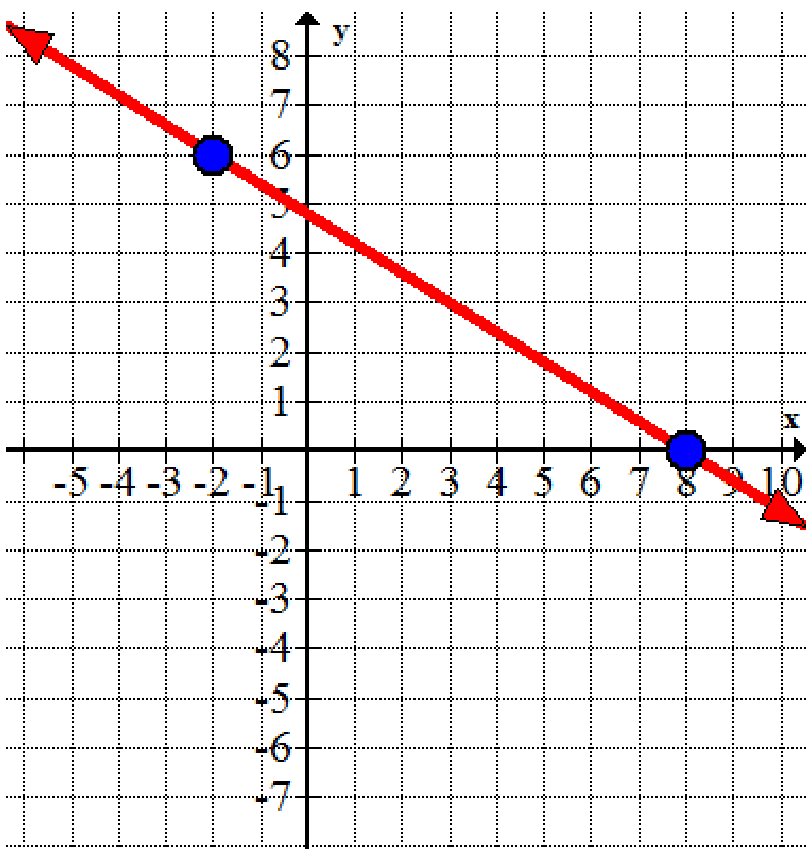
Standard Form

The ***standard form*** is another method for writing the equation of a line. The standard form is $Ax + By = C$ where A, B, and C are integers and $A \geq 0$

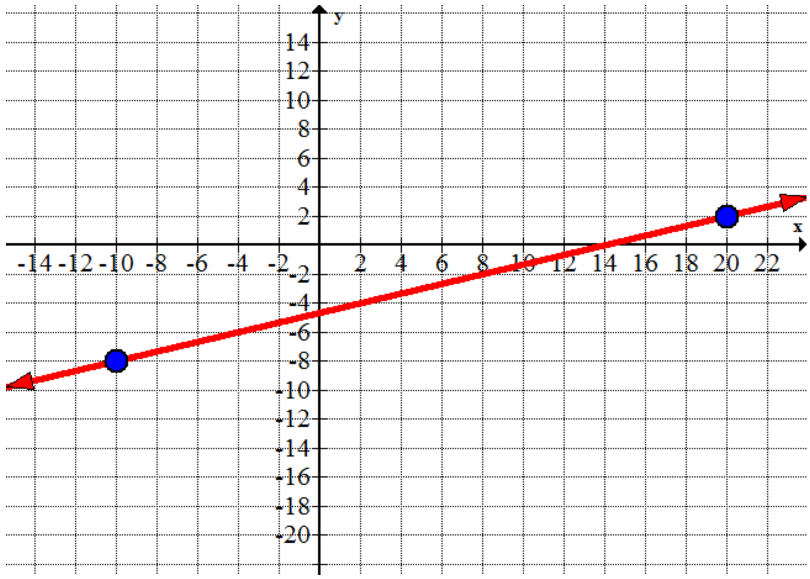
Guided Practice

Write the equation for each of the lines graphed below in slope-intercept form and in standard form.

1.



2.



3. Write the equation for the line that passes through the point (-4, 7) and is perpendicular of the *y*-axis.

Answers:

1. Two points on the graph are (-2, 6) and (8, 0). First use the formula to determine the slope of this line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{0-6}{8--2}$$
$$m = \frac{0-6}{8+2}$$
$$m = -\frac{6}{10}$$
$$m = -\frac{3}{5}$$

Use the slope and one of the points in the point-slope form:

$$y = m(x - x_1) + y_1$$
$$y = -\frac{3}{5}(x - (-2)) + 6 \quad \text{Plug in}$$
$$y = -\frac{3}{5}x - \frac{6}{5} + 6 \quad \text{Distribute}$$
$$y = -\frac{3}{5}x + \frac{24}{5}$$

The equation for the line in slope-intercept form is

$$\boxed{y = -\frac{3}{5}x + \frac{24}{5}}$$

To express the equation in standard form, multiply each term by 5 and move the variables to the left-hand side:

$$
\begin{aligned}
y &= -\frac{3}{5}x + \frac{24}{5} \\
5(y) &= 5\left(-\frac{3}{5}x\right) + 5\left(\frac{24}{5}\right) && \text{Multiply both sides by 5} \\
5(y) &= \cancel{5}\left(-\frac{3}{\cancel{5}}x\right) + \cancel{5}\left(\frac{24}{\cancel{5}}\right) \\
5y &= -3x + 24 && \text{Add 3x to both sides} \\
5y + 3x &= -3x + 3x + 24 \\
3x + 5y &= 24
\end{aligned}
$$

$$\boxed{3x + 5y = 24}$$

2. Two exact points on the graph are (-10, -8) and (20, 2). The slope of the line can be calculated by counting to determine the value of $m = \frac{\text{rise}}{\text{run}}$.

$$
\begin{aligned}
m &= \frac{\text{rise}}{\text{run}} \\
m &= \frac{10}{30} \\
m &= \frac{1}{3}
\end{aligned}
$$

Now, use the slope and one of the points in the point-slope form:

$$
\begin{aligned}
y &= \frac{1}{3}(x - 20) + 2 && m = \frac{1}{3} \text{ and } \begin{pmatrix} x, & y \\ 20, & 2 \end{pmatrix} \\
y &= \frac{1}{3}x - \frac{20}{3} + 2 && \text{Distribute} \\
y &= \frac{1}{3}x - \frac{14}{3} && \text{Add}
\end{aligned}
$$

The equation of the line in slope-intercept form is

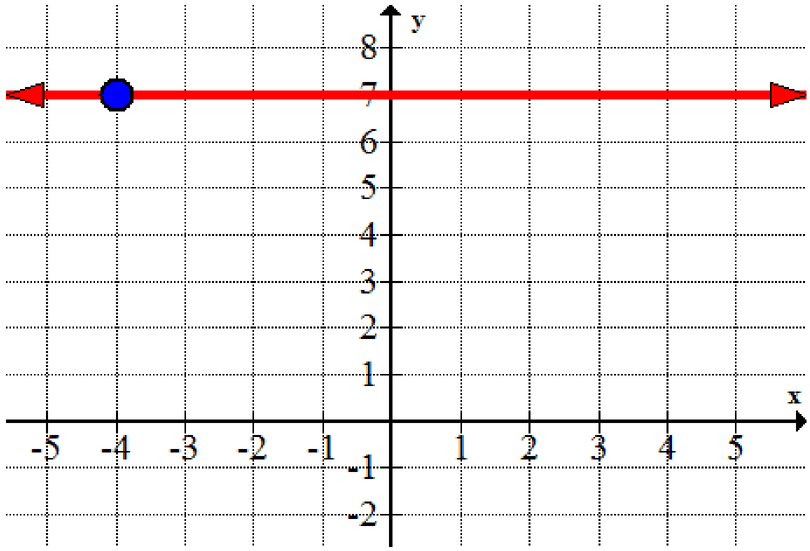
$$\boxed{y = \frac{1}{3}x - \frac{14}{3}}$$

. Multiply the equation by 3 and set the equation equal to zero to write the equation in standard form. The equation of the line in standard form is

$$\boxed{x - 3y = 14}$$

.

3. Begin by sketching the graph of the line.



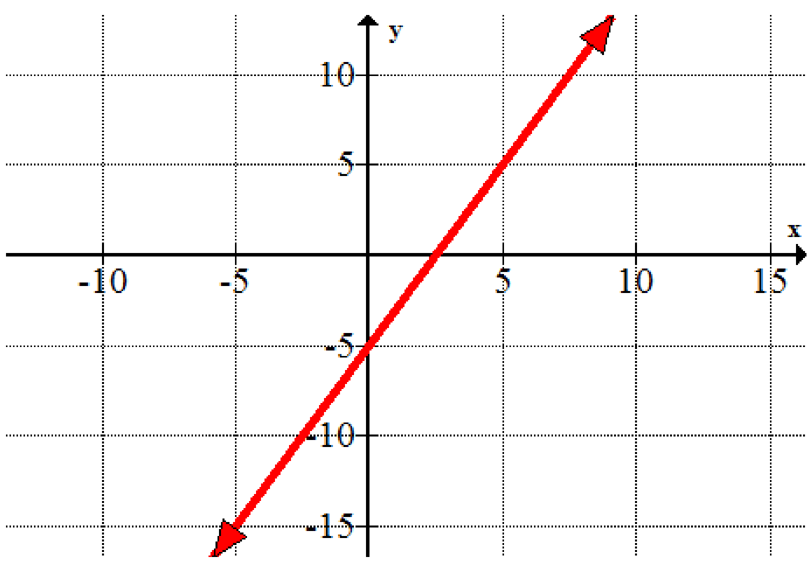
A line that is perpendicular to the y-axis is parallel to the x-axis. The slope of such a line is zero. The equation of this line is

$$\boxed{y = 7}$$

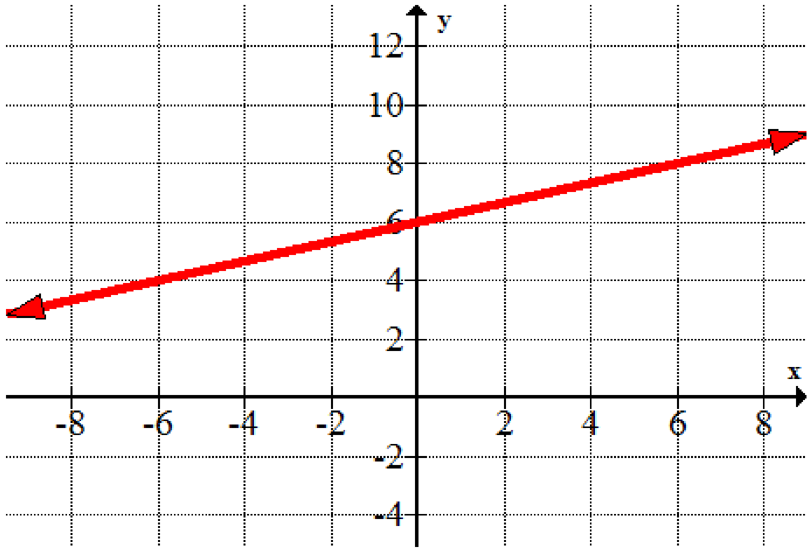
Practice Problems

For each of the following graphs, write the equation of the line in slope-intercept form.

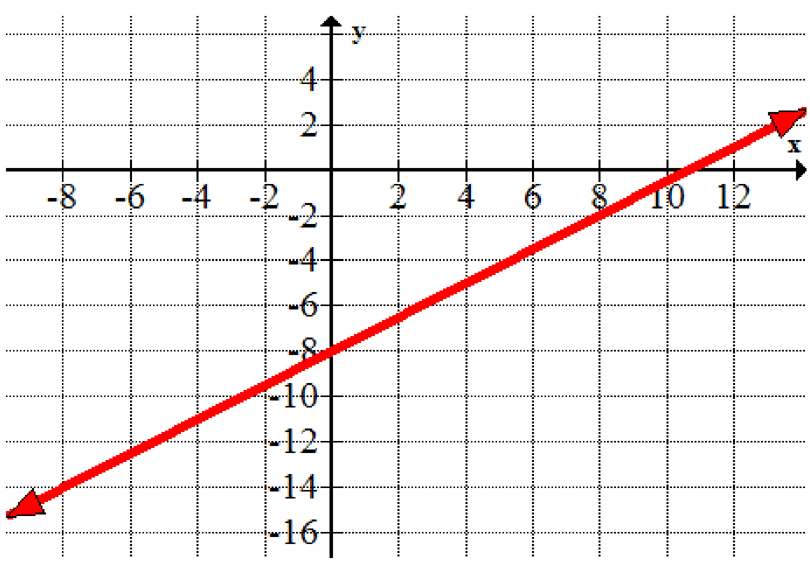
1.



2.



3.



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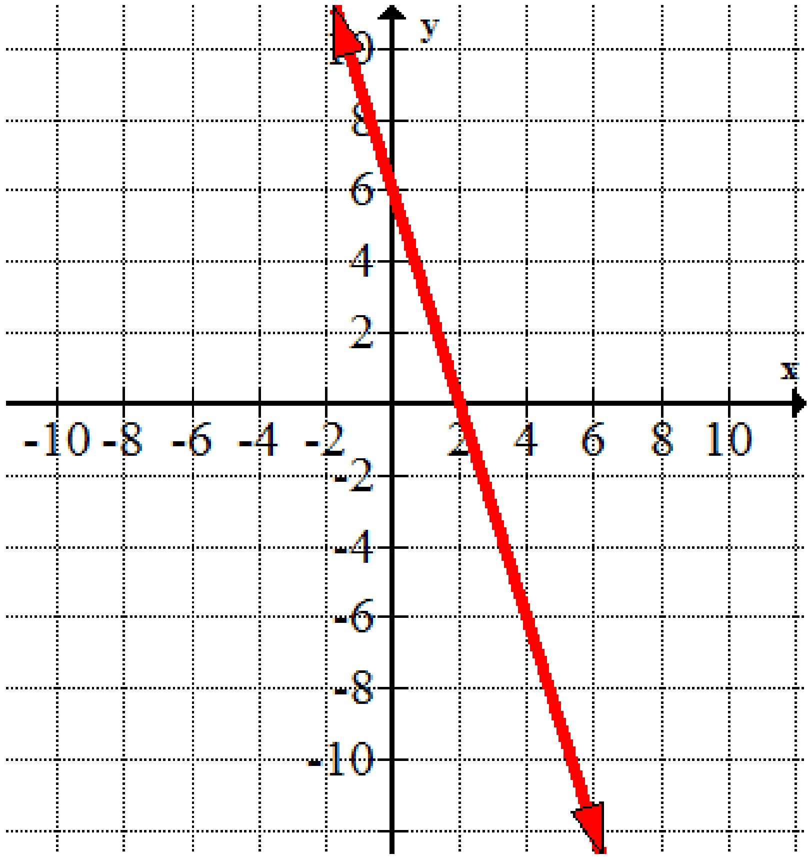
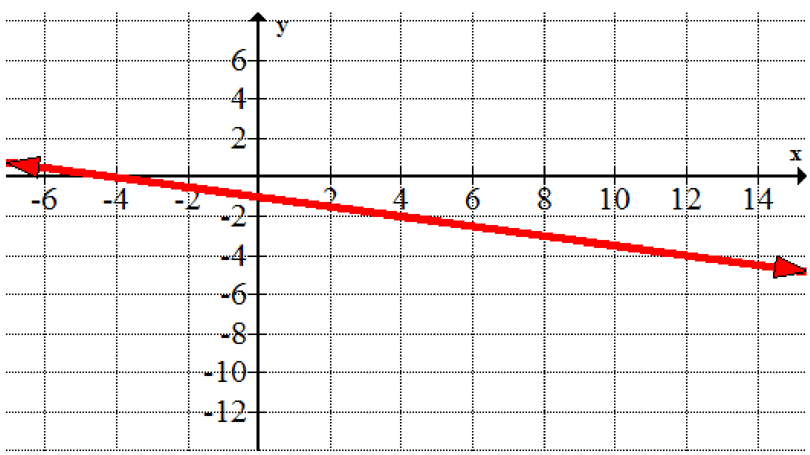


5.



Determine the equation of the line that passes through the following pairs of points:

6. (-3, 1) and (-3, -7)
7. (-5, -5) and (10, -5)
8. (-8, 4) and (2, -6)
9. (14, 8) and (4, 4)
10. (0, 5) and (4, -3)
11. (4, 7) and (2, -5)

For each of the following real world problems, write the linear equation in standard form that would best model the problem.

12. The cost of operating a car for one month depends upon the number of miles you drive. According to a recent survey completed by drivers of midsize cars, it costs \$124/month if you drive 320 miles/month and \$164/month if you drive 600 miles/month.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
13. A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for \$1950 and 15 in another for \$2850. The number of computers and the cost forms a linear relationship.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
14. Shop Rite sells a one-quart carton of milk for \$1.65 and a two-quart carton for \$2.95. Assume there is a linear relationship between the volume of milk and the price.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
15. Some college students who plan on becoming math teachers decide to set up a tutoring service for high school math students. One student was charged \$25 for 3 hours of tutoring. Another student was charged \$55 for 7 hours of tutoring. The relationship between the cost and time is linear.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?

3.4 Graphs of Lines from Equations

Learning Objectives

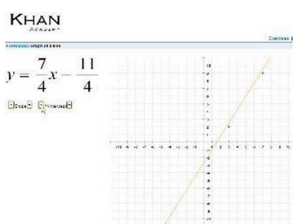
Here you will learn how to graph a linear function from its equation without first making a table of values.

Concept Problem

Can you graph the linear function $4y-5x=16$?

Watch This

Khan Academy Slope and y-Intercept Intuition



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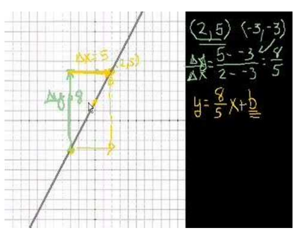
Khan Academy Slope 2



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Guidance

To graph a linear function, you only need to plot two points and then connect them with a line. There are different ways to find your two points, but the most common one is the use the slope-intercept form of the linear equation.

Here are steps for using the slope-intercept form to create a graph

- Step 1: Solve the equation for y if it is not already in the form $y=mx+b$.
- Step 2: Plot the y-intercept as the first point.
- Step 3: Use the slope to find another point on the line. Starting from the y-intercept, interpret the slope m as $\frac{\text{rise}}{\text{run}}$. The numerator tells you how far up or down to go; the denominator tells you how far left or right to go.
- Step 4: Connect these two points to form a line and extend the line.

Note: You can repeat Step 3 multiple times in order to find more points on the line if you wish.

Example A

For the following linear function, state the y-intercept and the slope: $4x - 3y - 9 = 0$.

Solution:

The first step is to rewrite the equation in the form $y = mx + b$. To do this, solve the equation for 'y'.

$$\begin{aligned}
4x - 3y - 9 &= 0 && \text{Apply the zero principle to move } x \text{ to the right side of the equation.}\\
4x \color{red}{-4x} - 3y - 9 &= 0 \color{red}{-4x} &&\\
-3y - 9 &= -4x && \text{Apply the zero principle to move } -9 \text{ to the right of the equation.}\\
-3y - 9 \color{red}{+9} &= -4x \color{red}{+9} &&\\
-3y &= -4x + 9 && \text{Divide all terms by the coefficient of } y\text{. Divide by } -3.\\
\frac{-3y}{\color{red}{-3}} &= \frac{-4x}{\color{red}{-3}} + \frac{9}{\color{red}{-3}} &&\\
\frac{\cancel{-3}y}{\cancel{-3}} &= \frac{-4x}{-3} + \frac{9}{-3} &&\\
\boxed{y = \frac{4}{3}x - 3} &&&
\end{aligned}$$

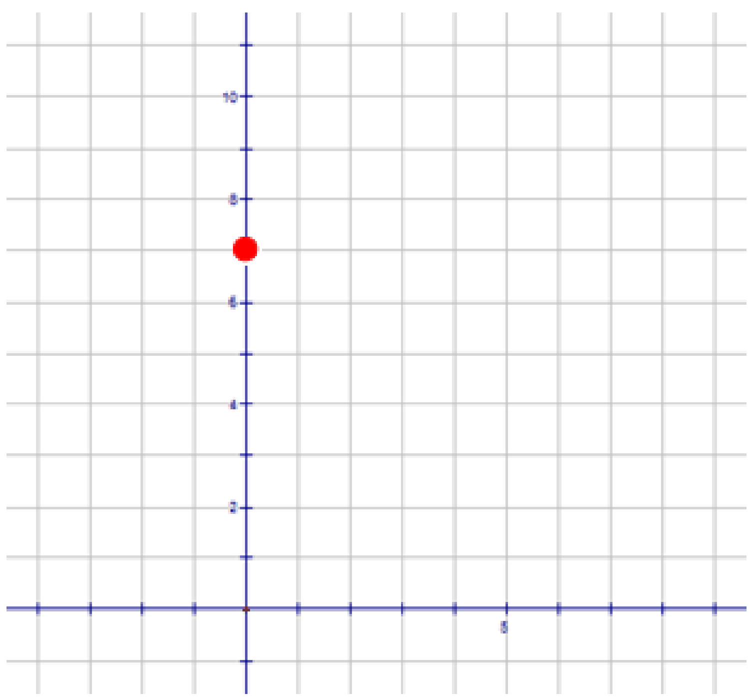
The y-intercept is (0, -3) and the slope is $\frac{4}{3}$.

Example B

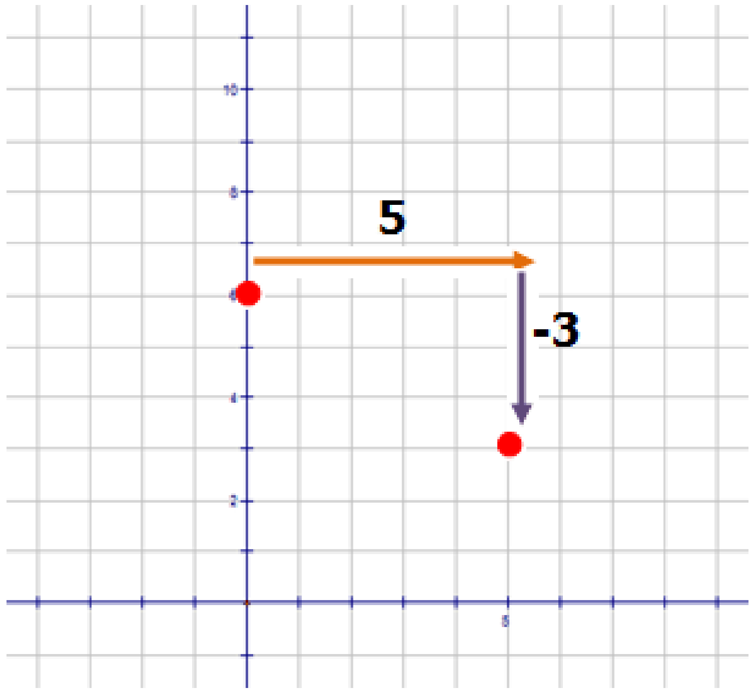
Graph the linear function $y = \frac{-3}{5}x + 7$ on a Cartesian grid.

Solution:

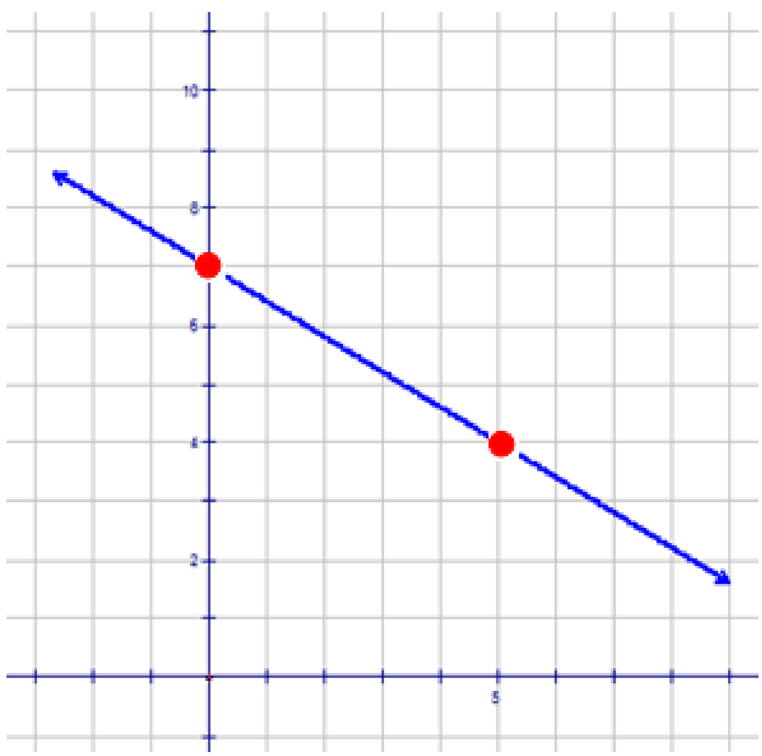
The y-intercept is (0, 7) and the slope is $\frac{-3}{5}$. Begin by plotting the y-intercept on the grid.



From the *y*-intercept, move to the right (run) 5 units and then move downward (rise) 3 units. Plot a point here.



Join the points with a straight line. Use a straight edge to draw the line.



Graphing Horizontal and Vertical Lines

Because the equations of horizontal and vertical lines are special, these types of lines can be graphed differently:

- The graph of a horizontal line will have an equation of the form $y = a$ where a is the y-intercept of the line. You can simply draw a horizontal line through the y-intercept to sketch the graph.
- The graph of a vertical line will have an equation of the form $x = c$, where x is the x-intercept of the line. You can simply draw a vertical line through the x-intercept to sketch the graph.

Example C

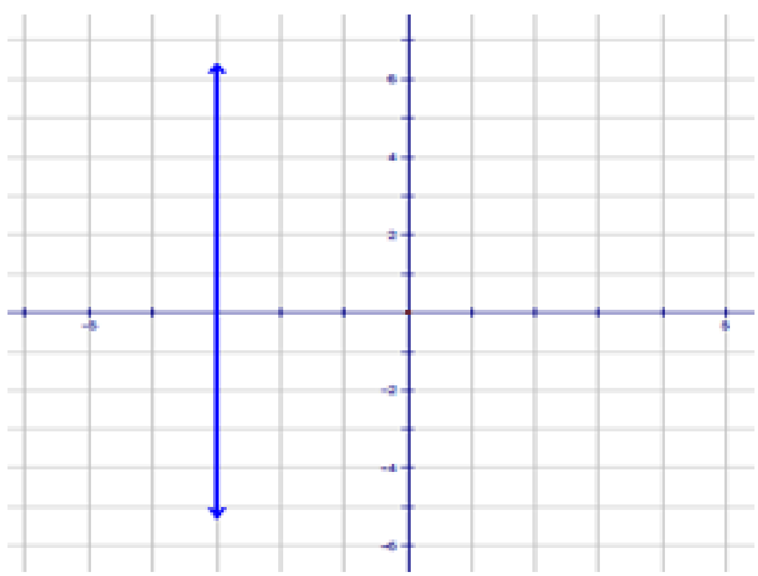
Plot the following linear equations on a Cartesian grid.

i) $x = -3$

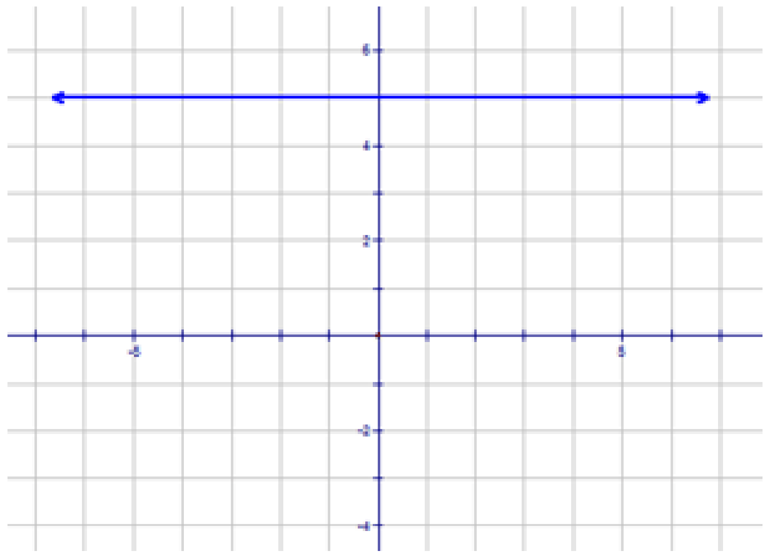
ii) $y = 5$

Solution:

i) A line that has $x = -3$ as its equation passes through all points that have -3 as the $x-$coordinate. The line also has a slope that is undefined. This line is parallel to the y-axis.



ii) A line that $y = 5$ has as its equation passes through all points that have 5 as the $y-$coordinate. The line also has a slope of zero. This line is parallel to the x-axis.



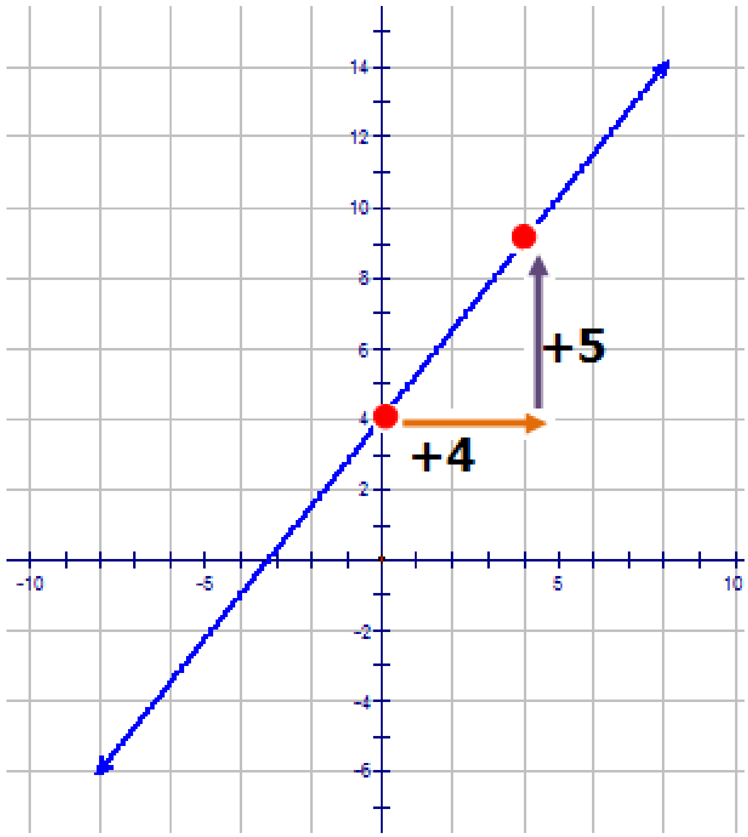
Concept Problem Revisited

Plot the linear function $4y-5x=16$ on a Cartesian grid.

The first step is to rewrite the function in slope-intercept form.

$4y-5x=16$	Apply the zero principle to move $5x$ to the right side of the equation.
$4y-5x+5x=16+5x$	
$4y=16+5x$	Divide every term 4.
$\frac{4y}{4}=\frac{16}{4}+\frac{5x}{4}$	
$y=4+\frac{5}{4}x$	Write the equation in the form $y=mx+b$.
$\boxed{y=\frac{5}{4}x+4}$	

The slope of the line is $\frac{5}{4}$ and the y-intercept is (0, 4)



Plot the y-intercept at (0, 4). From the y-intercept, move to the right 4 units and then move upward 5 units. Plot the point. Using a straight edge, join the points.

Guided Practice

1. Using the slope-intercept method, graph the linear function $y = -\frac{3}{2}x - 1$

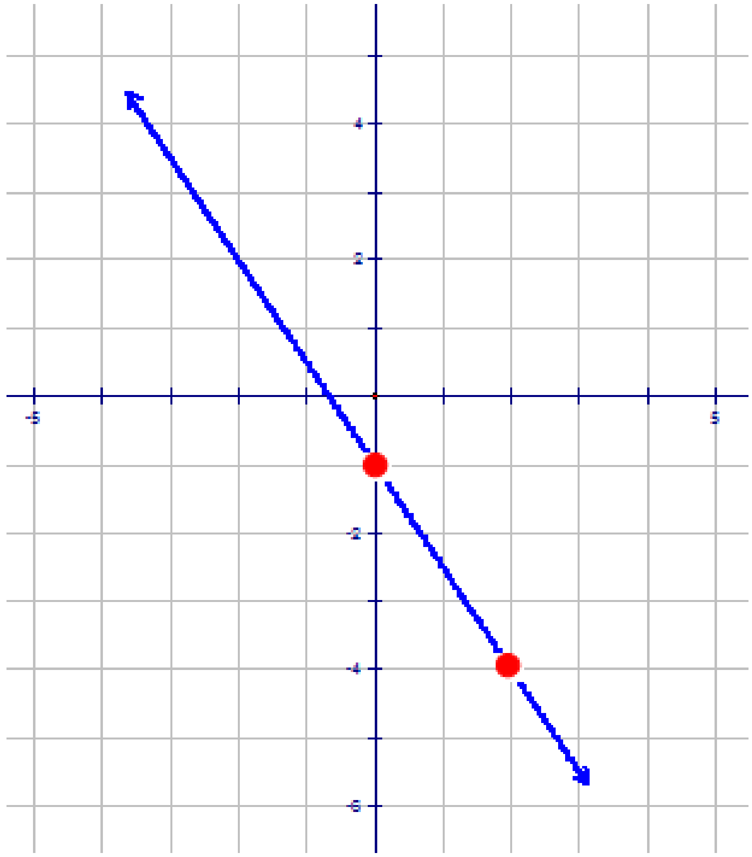
2. Using the slope-intercept method, graph the linear function $7x - 3y - 15 = 0$

3. Graph the following lines on the same Cartesian grid. What shape is formed by the graphs?

(a) $y = -3$
(b) $x = 4$
(c) $y = 2$
(d) $x = -6$

Answers:

1. The slope of the line is $-\frac{3}{2}$ and the y-intercept is (0, -1). Plot the y-intercept. Apply the slope to the y-intercept. Use a straight edge to join the two points.

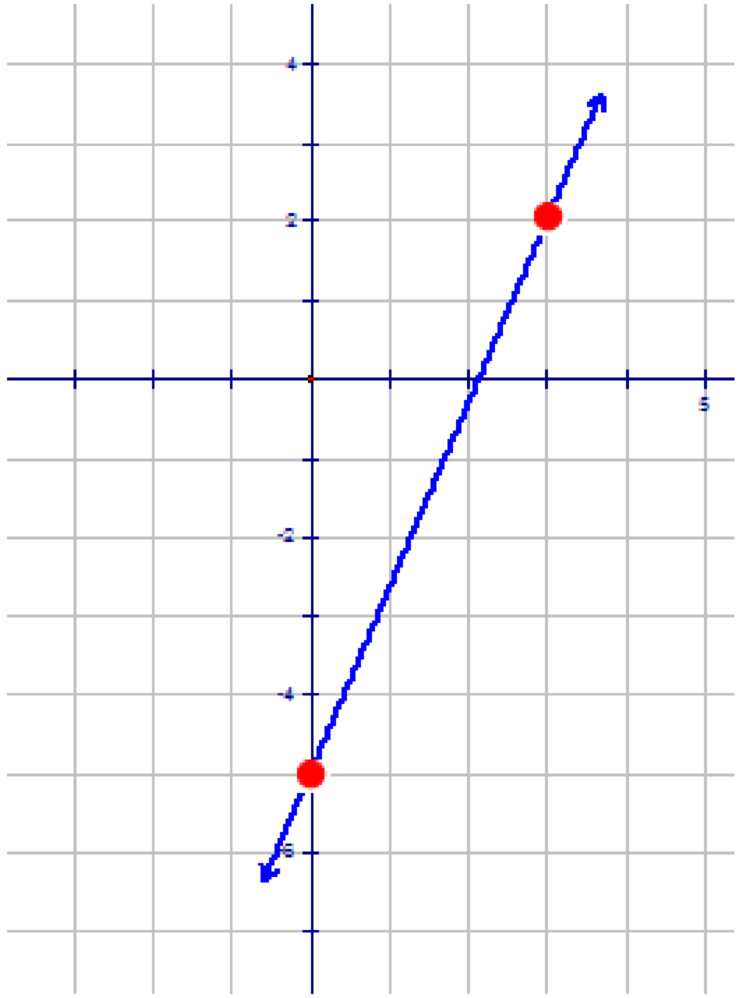


2. Write the equation in slope-intercept form.

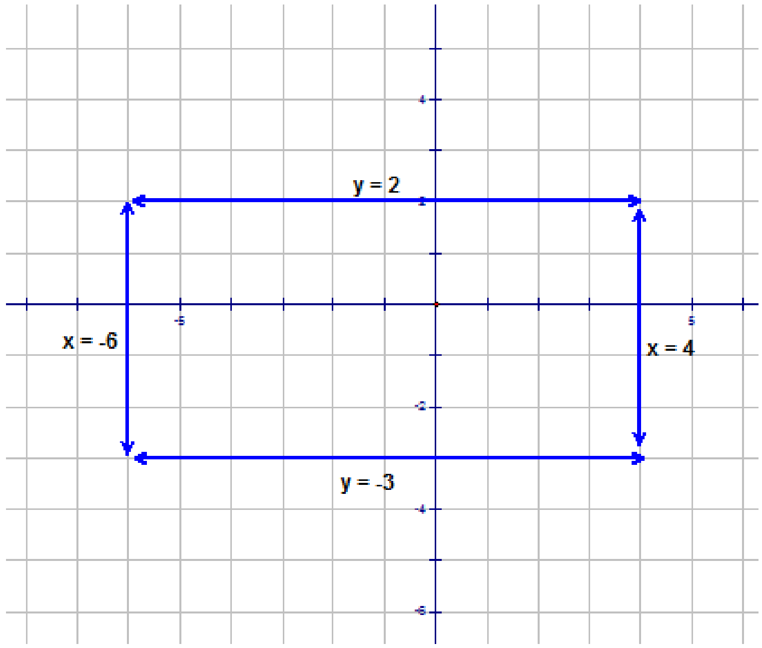
$$7x - 3y - 15 = 0$$
$$7x - 7x - 3y - 15 = 0 - 7x$$
$$-3y - 15 = -7x$$
$$-3y - 15 + 15 = -7x + 15$$
$$-3y = -7x + 15$$
$$\frac{-3y}{-3} = \frac{-7x}{-3} + \frac{15}{-3}$$
$$\boxed{y = \frac{7}{3}x - 5}$$

Solve the equation for the variable y.

The slope is $\frac{7}{3}$ and the y-intercept is (0, -5). Plot the y-intercept. Apply the slope to the y-intercept. Use a straight edge to join the two points.



3. There are four lines to be graphed. The lines a and c are lines with a slope of zero and are parallel to the x-axis. The lines b and d are lines that have a slope that is undefined and are parallel to the x-axis. The shape formed by the intersections of the lines is a rectangle.



Practice Problems

For each of the following linear functions, state the slope and the y-intercept:

1. $y = \frac{5}{8}x + 3$
2. $4x + 5y - 3 = 0$
3. $4x - 3y + 21 = 0$
4. $y = -7$
5. $9y - 8x = 27$

Using the slope-intercept method, graph the following linear functions:

6. $3x + y = 4$
7. $3x - 2y = -4$
8. $2x + 6y + 18 = 0$
9. $3x + 7y = 0$
10. $4x - 5y = -30$
11. $6x - 2y = 8$

Graph the following linear equations and state the slope of the line:

12. $x = -5$
13. $y = 8$
14. $y = -4$
15. $x = 7$

3.5 Graphs of Linear Functions from Intercepts

Learning Objectives

Here you will learn how to graph a linear function by first finding the x and y intercepts.

Concept Problem

What are the intercepts of $4x+2y=8$? How could you use the intercepts to quickly graph the function?

Watch This

Khan Academy X and Y Intercepts

MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/58470

Guidance

To graph a linear function, you need to plot only two points. These points can then be joined by a straight line. While any two points can be used to graph a linear function, two points that are often easy to use are the x-intercept and the y-intercept.

The x-intercept is where the graph crosses the x-axis. Its coordinates are $(x,0)$. Because all x-intercepts have a y-coordinate equal to 0, you can find an x-intercept by substituting 0 for y in the equation and solving for x.

The y-intercept is where the graph crosses the y-axis. Its coordinates are $(0,y)$. Because all y-intercepts have a x-coordinate equal to 0, you can find an y-intercept by substituting 0 for x in the equation and solving for y.

Example A

Identify the $x-$ and y-intercepts for each line.

(a) $2x+y-6=0$

(b) $\frac{1}{2}x-4y=4$

Solution:

(a)

Let $y = 0$. Solve for 'x'.

$2x + y - 6 = 0$

$2x + (0) - 6 = 0$

$2x - 6 = 0$

$2x - 6 + 6 = 0 + 6$

$2x = 6$

$\frac{2x}{2} = \frac{6}{2}$

$x = 3$

The x-intercept is $(3, 0)$

Let $x = 0$. Solve for 'y'.

$2x + y - 6 = 0$

$2(0) + y - 6 = 0$

$y - 6 = 0$

$y - 6 + 6 = 0 + 6$

$y = 6$

The y-intercept is $(0, 6)$

(b)

Let $y = 0$. Solve for 'x'.

$\frac{1}{2}x - 4y = 4$

$\frac{1}{2}x - 4(0) = 4$

$\frac{1}{2}x - 0 = 4$

$\frac{1}{2}x = 4$

$\cancel{2}\left(\frac{1}{\cancel{2}}\right)x = 2(4)$

$x = 8$

The x-intercept is $(8, 0)$

Let $x = 0$. Solve for 'y'.

$\frac{1}{2}x - 4y = 4$

$\frac{1}{2}(0) - 4y = 4$

$0 - 4y = 4$

$-4y = 4$

$\frac{-4y}{-4} = \frac{4}{-4}$

$y = -1$

The y-intercept is $(0, -1)$

Example B

Use the intercept method to graph $2x - 3y = -12$.

Solution:

Let $y = 0$. Solve for 'x'.

$2x - 3y = -12$

$2x - 3(0) = -12$

$2x - 0 = -12$

$2x = -12$

$\frac{2x}{2} = \frac{-12}{2}$

$x = -6$

The x-intercept is $(-6, 0)$

Let $x = 0$. Solve for 'y'.

$2x - 3y = -12$

$2(0) - 3y = -12$

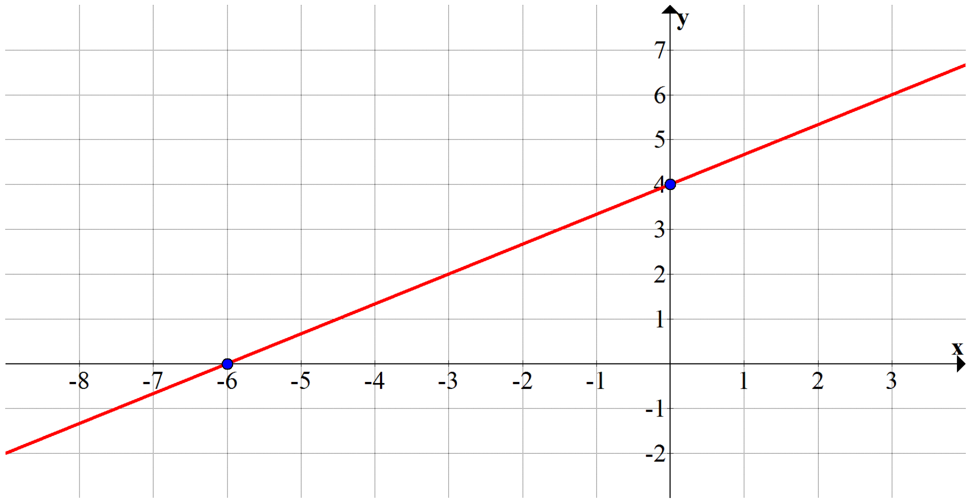
$0 - 3y = -12$

$-3y = -12$

$\frac{-3y}{-3} = \frac{-12}{-3}$

$y = 4$

The y-intercept is $(0, 4)$



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Example C

Use the $x-$ and y-intercepts to graph the linear function.

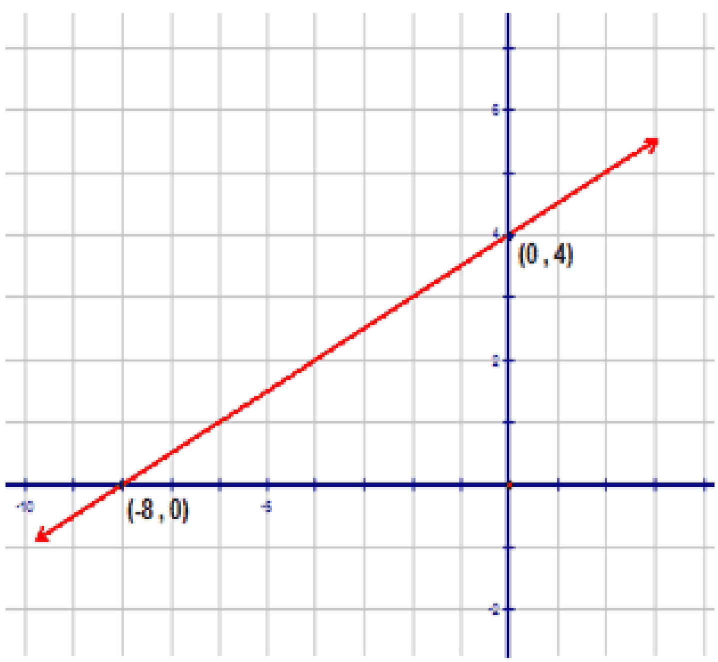
$$x-2y+8=0$$

Solution:

x intercept: $x-2(0)+8=0 \rightarrow x=-8$

y intercept: $0-2y+8=0 \rightarrow y=4$

The x-intercept is (-8, 0) and the y-intercept is (0, 4).



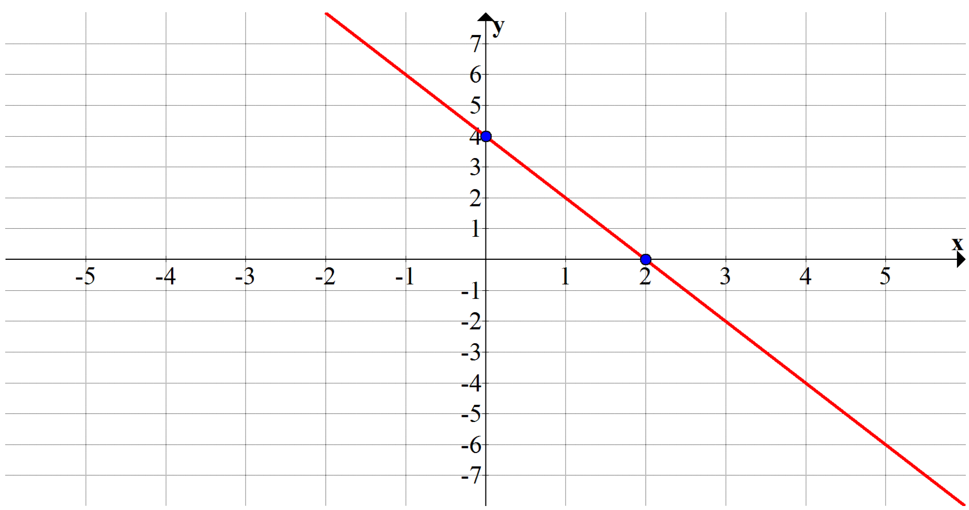
Concept Problem Revisited

The linear function $4x+2y=8$ can be graphed by using the intercept method.

To determine the x-intercept, let $y=0$.
Solve for 'x'.
$4x+2y=8$
$4x+2(0)=8$
$4x+0=8$
$4x=8$
$\frac{4x}{4}=\frac{8}{4}$
$x=2$
The x-intercept is $(2,0)$

To determine the y-intercept, let $x=0$.
Solve for 'y'.
$4x+2y=8$
$4(0)+2y=8$
$0+2y=8$
$2y=8$
$\frac{2y}{2}=\frac{8}{2}$
$y=4$
The y-intercept is $(0,4)$

Plot the x-intercept on the x-axis and the y-intercept on the y-axis. Join the two points with a straight line.



Guided Practice

1. Identify the $x-$ and y-intercepts of the following linear functions:

(i) $2(x-3)=y+4$
(ii) $3x+\frac{2}{3}y-3=0$

2. Use the intercept method to graph the following relation:

(i) $5x+2y=-10$

3. Use the $x-$ and y-intercepts of the graph, to match the graph to its function.

$4x-3y-12=0$

Answers:

1. (i)

$$\begin{aligned} 2(x-3) &= y+4 && \text{Simplify the equation} \\ 2x-6 &= y+4 \\ 2x-6+6 &= y+4+6 \\ 2x &= y+10 \\ 2x-y &= y-y+10 \\ 2x-y &= 10 \end{aligned}$$

Now find the intercepts:

Let $y=0$. Solve for x.
$2x-y=10$
$2x-(0)=10$
$2x=10$
$\frac{2x}{2}=\frac{10}{2}$
$x=5$
The x-intercept is $(5,0)$

Let $x=0$. Solve for y.
$2x-y=10$
$2(0)-y=10$
$0-y=10$
$\frac{-y}{-1}=\frac{10}{-1}$
$y=-10$
The y-intercept is $(0,-10)$

(ii)

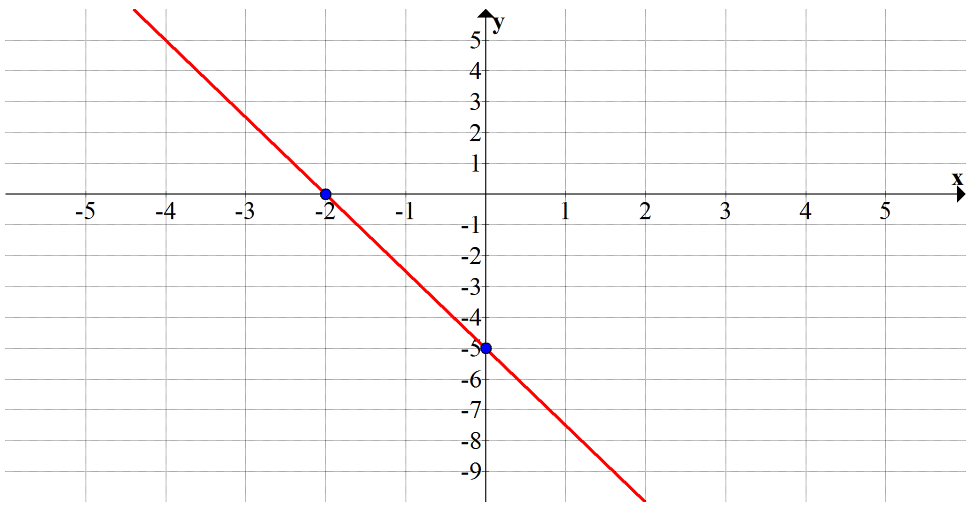
$$\begin{aligned} 3x+\frac{2}{3}y-3 &= 0 && \text{Simplify the equation.} \\ 3(3x)+3\left(\frac{2}{3}\right)y-3(3) &= 3(0) && \text{Multiply each term by 3.} \\ 3(3x)+\cancel{3}\left(\frac{2}{\cancel{3}}\right)y-3(3) &= 3(0) \\ 9x+2y-9 &= 0 \\ 9x+2y-9+9 &= 0+9 \\ 9x+2y &= 9 \end{aligned}$$

Let $y=0$. Solve for x.
$9x+2y=9$
$9x+2(0)=9$
$9x+0=9$
$\frac{9x}{9}=\frac{9}{9}$
$x=1$
The x-intercept is $(1,0)$

Let $x=0$. Solve for y.
$9x+2y=9$
$9(0)+2y=9$
$0+2y=9$
$\frac{2y}{2}=\frac{9}{2}$
$y=4.5$
The y-intercept is $(0,4.5)$

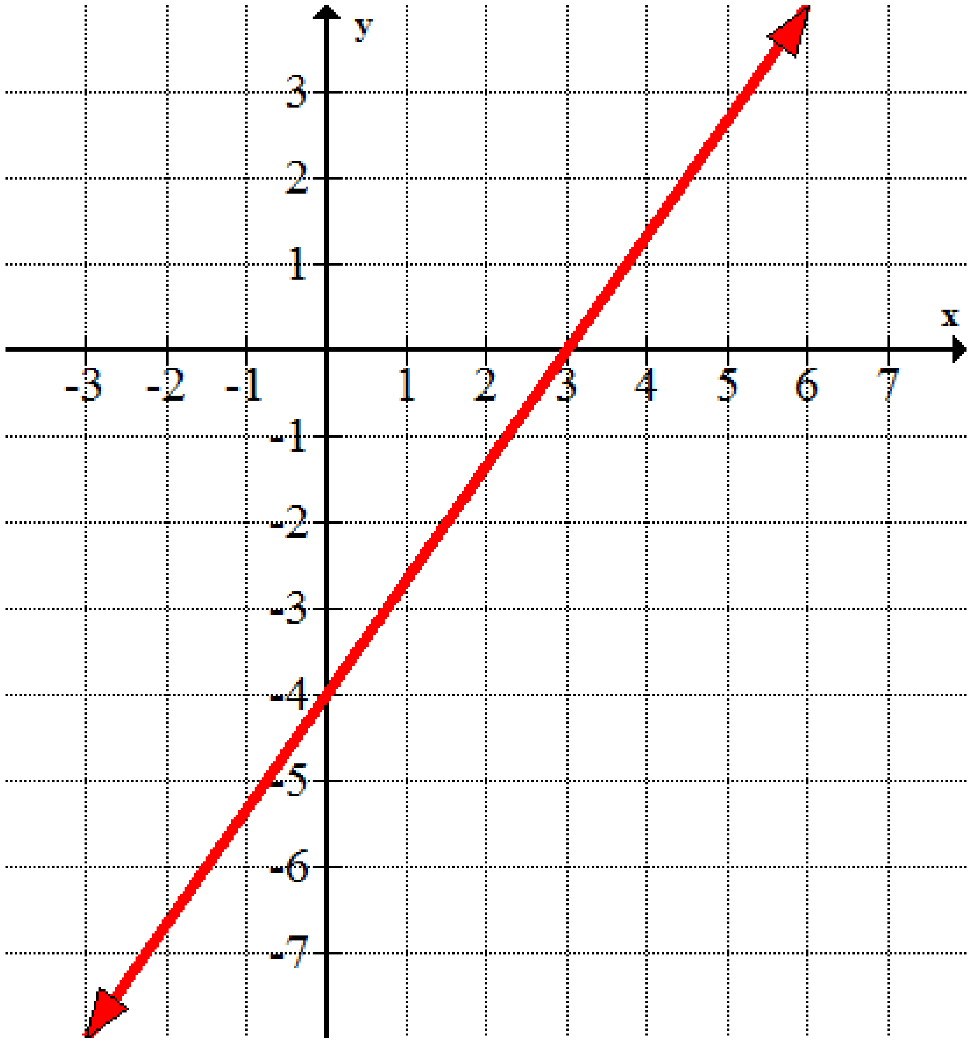
2. $5x+2y=-10$

Let $y=0$. Solve for x.	Let $x=0$. Solve for y.
$5x+2y=-10$	$5x+2y=-10$
$5x+2(0)=-10$	$5(0)+2y=-10$
$5x+0=-10$	$0+2y=-10$
$\frac{5x}{5}=\frac{-10}{5}$	$\frac{2y}{2}=\frac{-10}{2}$
$x=-2$	$y=-5$
The x-intercept is $(-2,0)$	The y-intercept is $(0,-5)$



3. Determine the $x-$ and y-intercept for each of the equation:

Let $y=0$. Solve for x.	Let $x=0$. Solve for y.
$4x-3y-12=0$	$4x-3y-12=0$
$4x-3y-12+12=0+12$	$4x-3y-12+12=0+12$
$4x-3y=12$	$4x-3y=12$
$4x-3(0)=12$	$4(0)-3y=12$
$4x-0=12$	$0-3y=12$
$4x=12$	$-3y=12$
$\frac{4x}{4}=\frac{12}{4}$	$\frac{-3y}{-3}=\frac{12}{-3}$
$x=3$	$y=-4$
The x-intercept is $(3,0)$	The y-intercept is $(0,-4)$



Practice Problems

For 1-10, complete the following table:

TABLE 3.1:

Function	x-intercept	y-intercept
$7x-3y=21$	1.	2.
$8x-3y+24=0$	3.	4.
$\frac{x}{4}-\frac{y}{2}=3$	5.	6.
$7x+2y-14=0$	7.	8.
$\frac{2}{3}x-\frac{1}{4}y=-2$	9.	10.

Use the intercept method to graph each of the linear equations in questions 11 - 15.

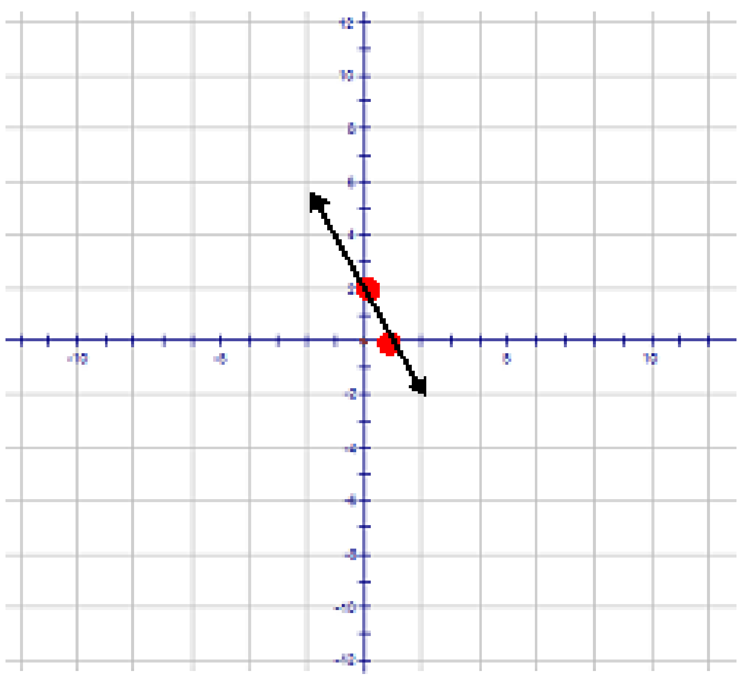
11. $7x-3y=21$
12. $8x-3y+24=0$
13. $\frac{x}{4}-\frac{y}{2}=3$
14. $7x+2y-14=0$

15. $\frac{2}{3}x - \frac{1}{4}y = -2$

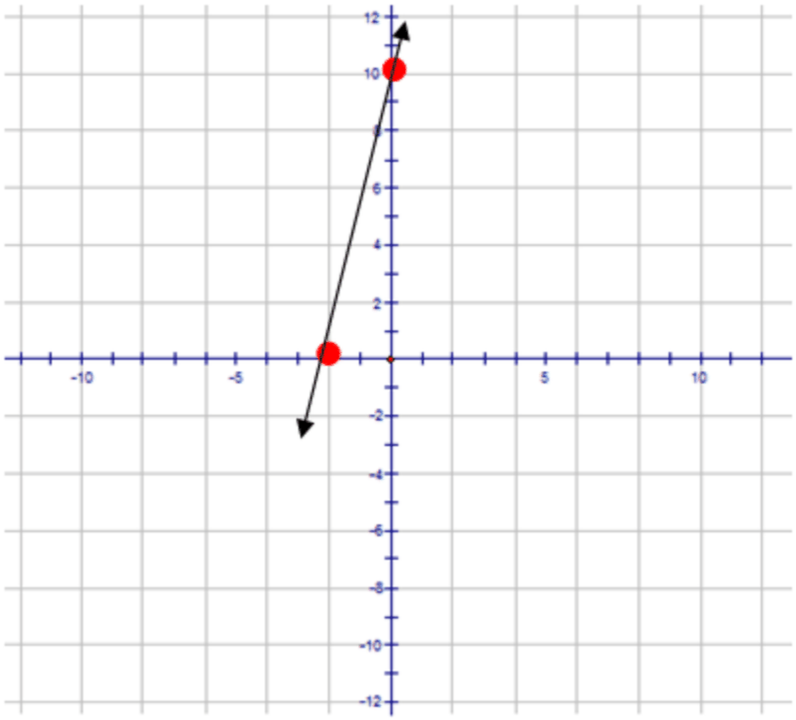
Use the $x-$ and y-intercepts to match each graph to its equation in questions 16 - 19

a. $7x + 5y - 35 = 0$
b. $y = 5x + 10$
c. $2x + 4y + 8 = 0$
d. $2x + y = 2$

16.



17.



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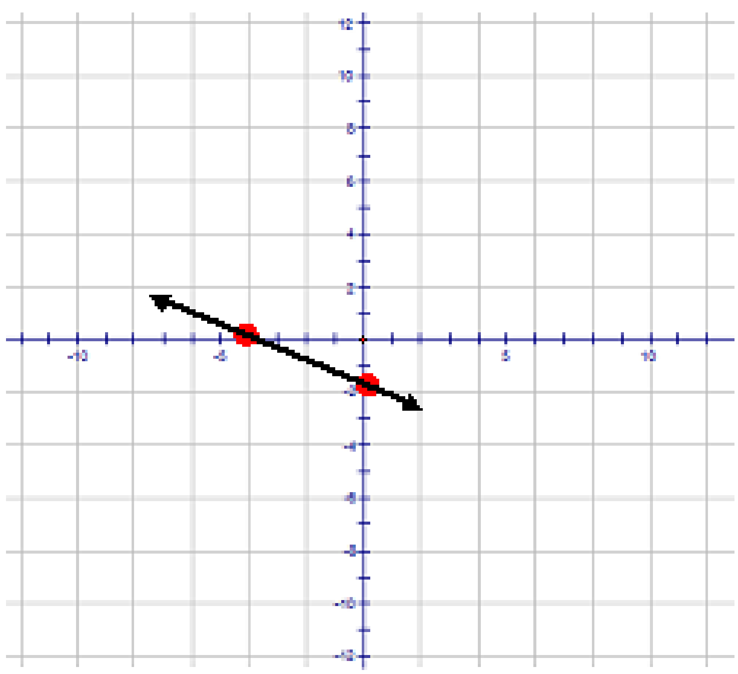
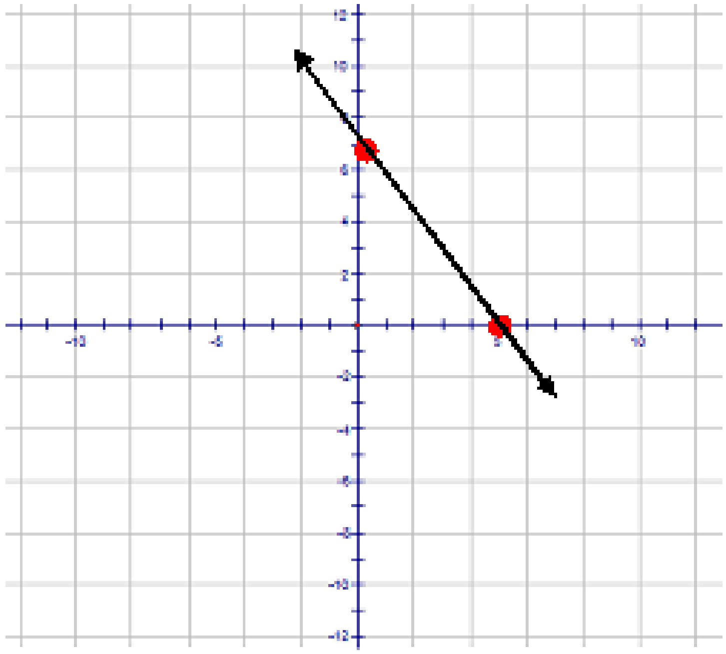


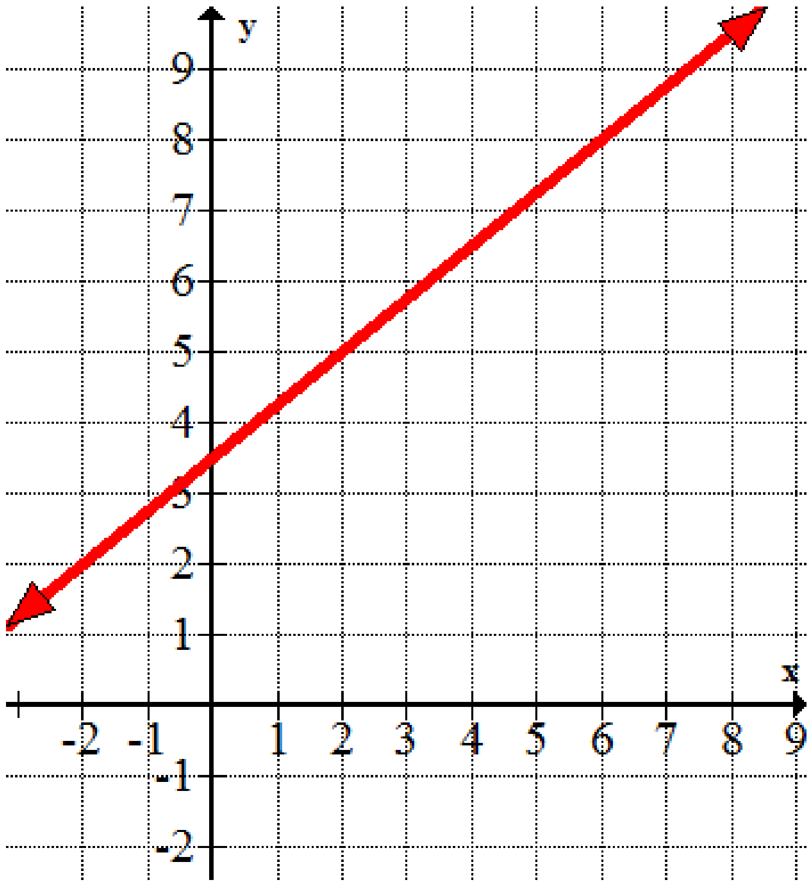
19.



3.6 Equations of Lines from Graphs

Concept Problem

Write the equation, in standard form, of the following graph:



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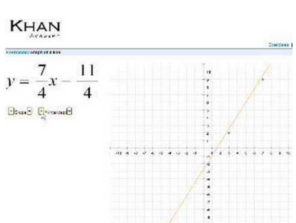
Khan Academy Slope and y-Intercept Intuition



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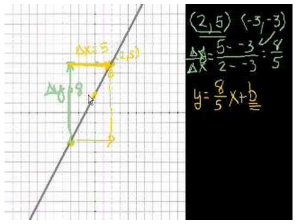
Khan Academy Slope 2



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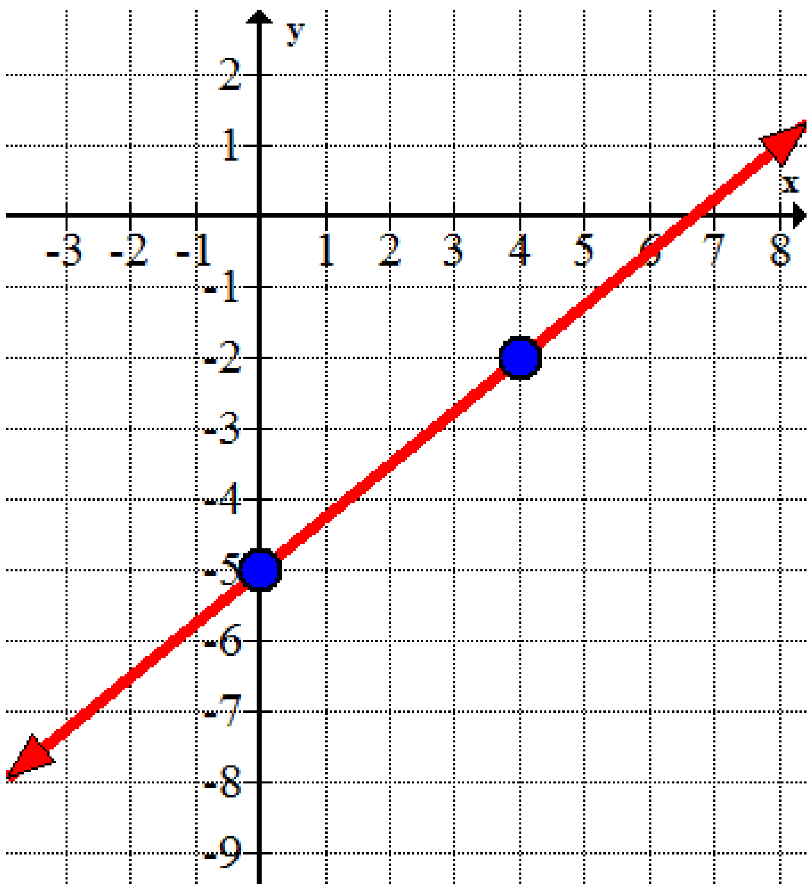
Guidance

You can determine the equation of a line from its graph by identifying points on the line. Any two points will work as long as they are clearly identifiable. The x- and y-intercepts are good points to use, when they are integers. Once you have the points, you can find the equation in slope intercept form as we have done before..



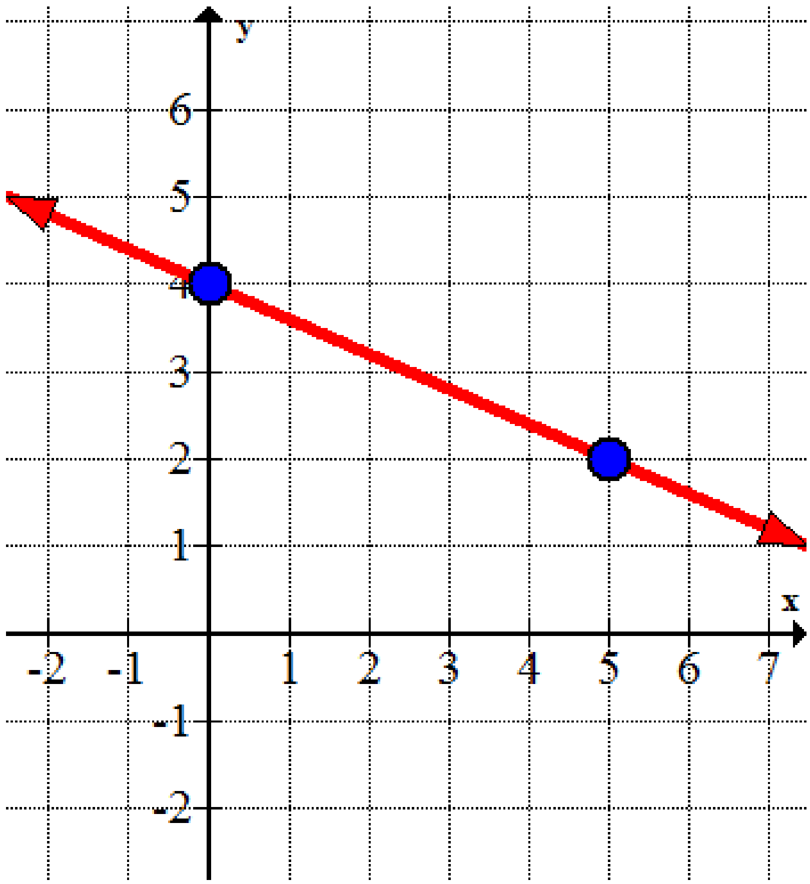
The y-intercept of the graph is (0, -5). To get from one point to the other, we go up 3 and over 4, so the slope of the line is $\frac{3}{4}$. The equation of the line in slope-intercept form is

$$\boxed{y = \frac{3}{4}x - 5}$$

.

Example A

Determine the equation of the following graph. Write the equation in slope-intercept form.

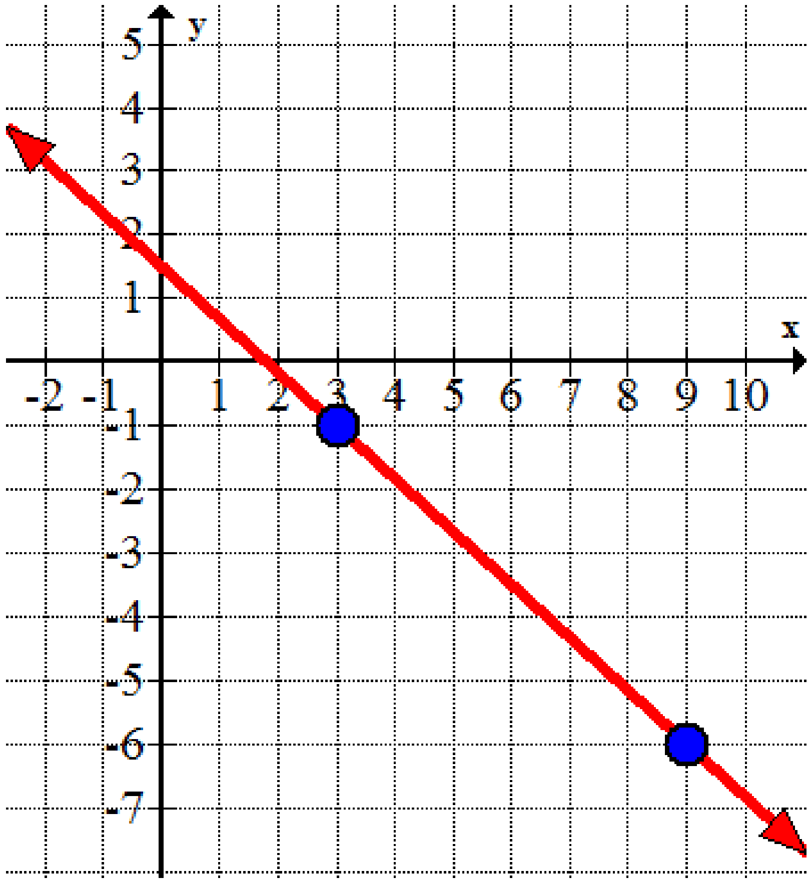


Solution:

The y-intercept is (0, 4) so $b = 4$. The slope has a run of five units to the right and a rise of 2 units downward. The slope of the line is $-\frac{2}{5}$. The equation of the line in slope-intercept form is $y = mx + b$ so $y = -\frac{2}{5}x + 4$.

Example B

Determine the equation in slope-intercept form of the line shown on the following graph:



Solution:

The y-intercept is not an exact point on this graph. The value of fractions on a Cartesian grid can only be estimated. Therefore, the points (3, -1) and (9, -6) will be used to determine the slope of the line. To get from the top point to the bottom one, we go down 5 units and to the right 6 units, so the slope is $-\frac{5}{6}$. We can use this and the coordinates of one of the points in the point-slope form:

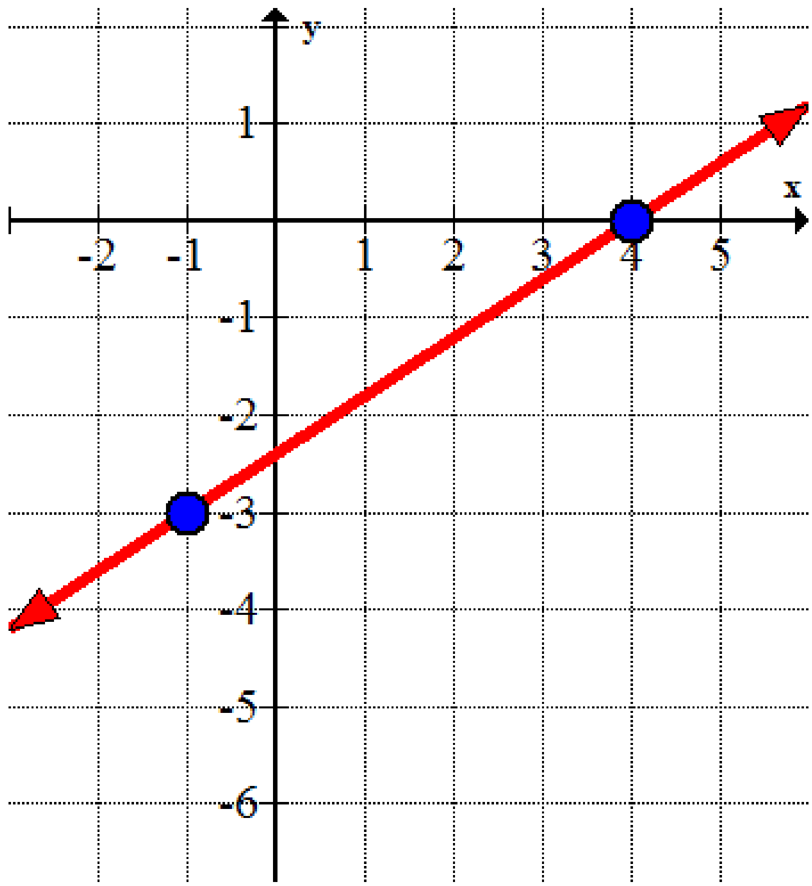
$$\begin{aligned} y &= m(x - x_1) + y_1 \\ y &= -\frac{5}{6}(x - 3) + (-1) && \text{Plug in} \\ y &= -\frac{5}{6}(x - 3) - 1 && \text{Simplify} \\ y &= -\frac{5}{6}x + \frac{5}{2} - 1 && \text{Distribute} \\ y &= -\frac{5}{6}x + \frac{3}{2} && \text{Add} \end{aligned}$$

The equation in slope-intercept form is

$$\boxed{y = -\frac{5}{6}x + \frac{3}{2}}$$

Example C

Determine the equation, in standard form, for the line on the following graph:



Solution:

The y-intercept is not an exact point on this graph. Therefore, the points (4, 0) and (-1, -3) will be used to determine the slope of the line. To go from the left point to the right point we go up 3 and to the right 5, so the slope is $\frac{3}{5}$. The slope and one of the points will be used to algebraically calculate the equation of the line in standard form.

$y-y_1=m(x-x_1)$	Use this formula to determine the equation in standard form.
$y-0=\frac{3}{5}(x-4)$	Fill in the value for m of $\frac{3}{5}$ and $\begin{pmatrix} x_1, & y_1 \\ 4, & 0 \end{pmatrix}$
$y=\frac{3}{5}x-\frac{12}{5}$	
$5(y)=5\left(\frac{3}{5}x\right)-5\left(\frac{12}{5}\right)$	Multiply every term by 5.
$5(y)=\not{5}\left(\frac{3}{\not{5}}x\right)-\not{5}\left(\frac{12}{\not{5}}\right)$	Simplify and set the equation equal to zero.

$$5y=3x-12$$
$$5y-3x=3x-3x-12$$
$$5y-3x=-12$$
$$-3x+5y=-12$$

Coefficient on x needs to be positive, so divide both sides by -1

$$3x-5y=12$$

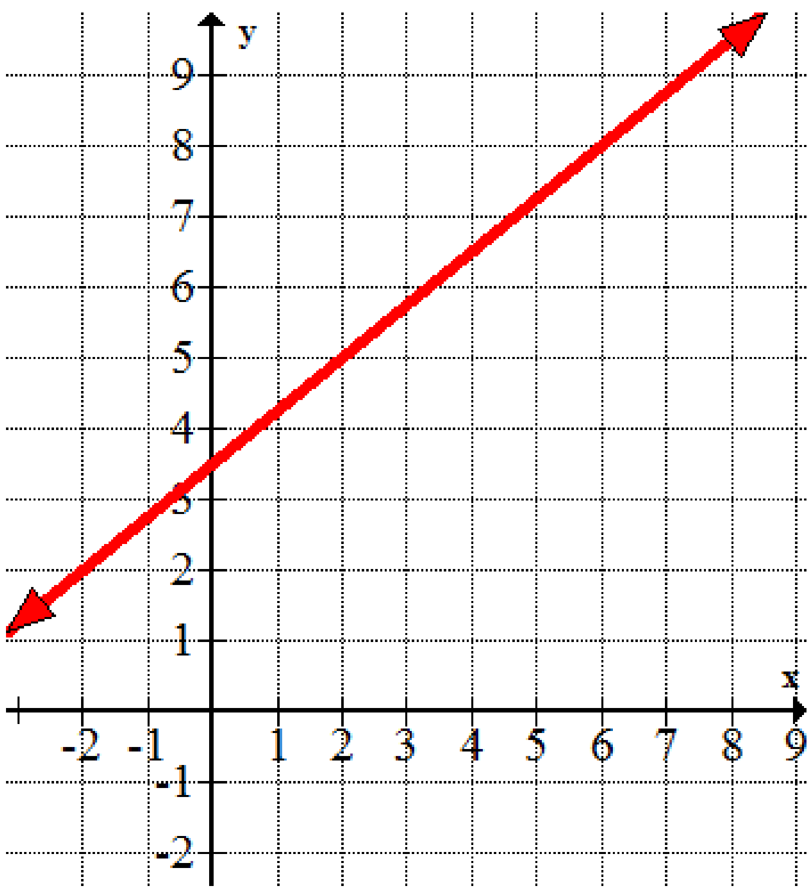
The equation of the line in standard form is

$$\boxed{3x - 5y = 12}$$

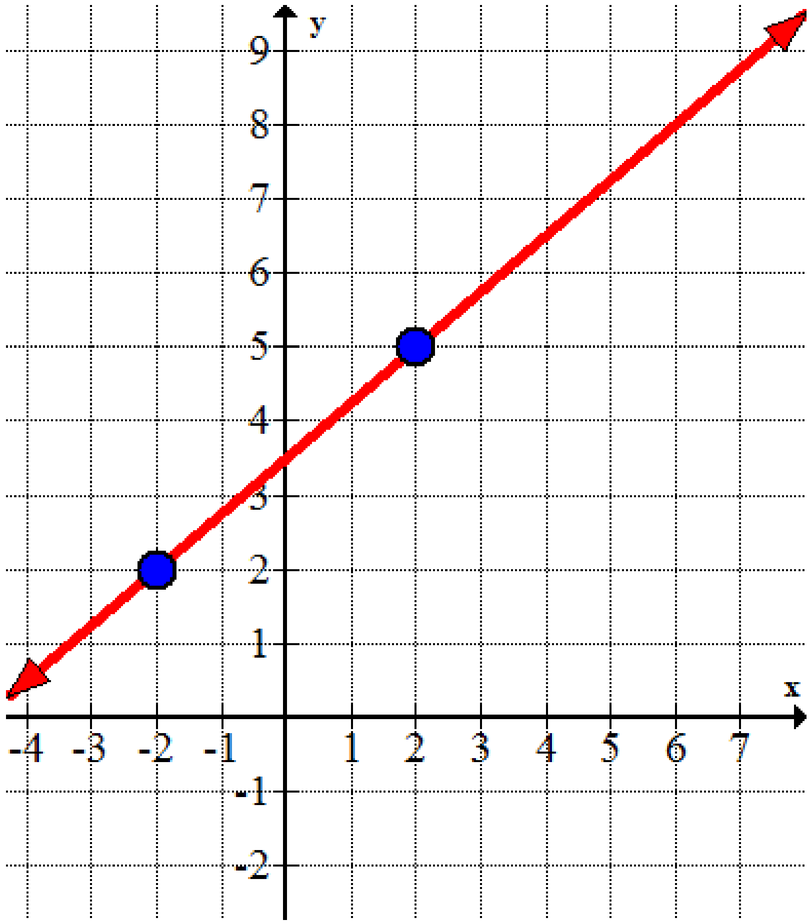
.

Concept Problem Revisited

Write the equation, in standard form, of the following graph:



The first step is to determine the slope of the line.



To get fro the left point to the right, we go up 3 and to the right by 4, so the slope of the line is $\frac{3}{4}$. The coordinates of one point on the line are (2, 5).

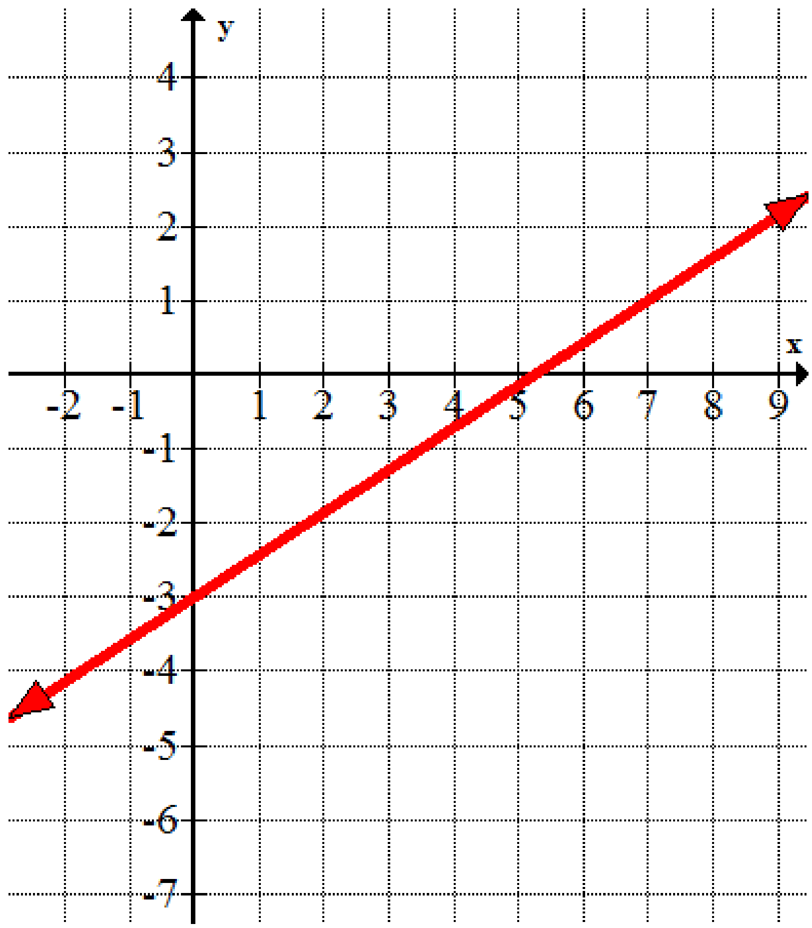
$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 5 &= \frac{3}{4}(x - 2) \\
y - 5 &= \frac{3}{4}x - \frac{6}{4} \\
4(y) - 4(5) &= 4\left(\frac{3}{4}\right)x - 4\left(\frac{6}{4}\right) && \text{Multiply by 4 to clear fractions} \\
4(y) - 4(5) &= \cancel{4}\left(\frac{3}{\cancel{4}}\right)x - \cancel{4}\left(\frac{6}{\cancel{4}}\right) \\
4y - 20 &= 3x - 6 \\
-3x + 4y - 20 &= 3x - 3x - 6 && \text{Move 3x to the left side} \\
-3x + 4y - 20 &= -6 \\
-3x + 4y - 20 + 20 &= -6 + 20 && \text{Add 20 to both sides} \\
-3x + 4y &= 14 \\
3x - 4y &= -14 && \text{Divide both sides by -1 to get positive coefficient on x}
\end{aligned}
$$

The equation of the line in standard form is

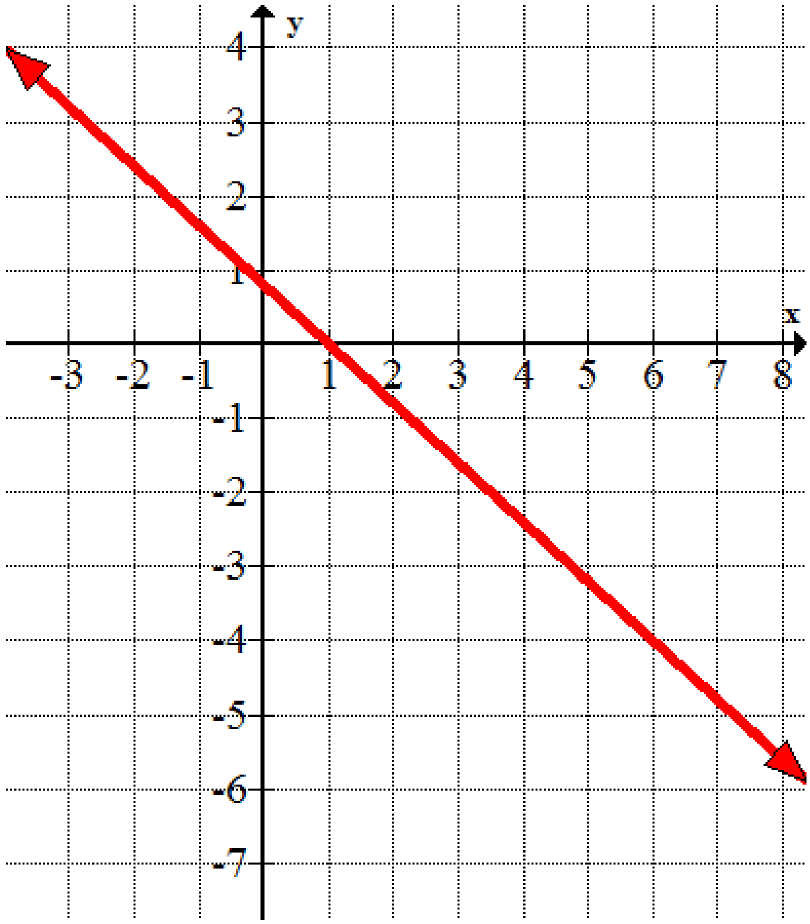
$$\boxed{3x - 4y = -14}$$

Guided Practice

1. Write the equation, in slope-intercept form, of the following graph:



2. Write the equation, in slope-intercept form, of the following graph:



3. Rewrite the equation of the line from #2 in standard form.

Answers:

1. The first step is to determine the coordinates of the y-intercept. The y-intercept is (0, -3) so $b = -3$. The second step is to count to determine the value of the slope. Another point on the line is $(7, 1)$ so the slope is $\frac{4}{7}$. The equation of the line in slope-intercept form is

$$\boxed{y = \frac{4}{7}x - 3}$$

2. Looking at the graph, the coordinates of two points on the line are (1, 0) and (6, -4). The slope is is $-\frac{4}{5}$. Now, use the point-slope form:

$$\begin{aligned}
y &= m(x - x_1) + y_1 \\
y &= -\frac{4}{5}(x - 1) + 0 \\
y &= -\frac{4}{5}x + \frac{4}{5} + 0 \\
y &= -\frac{4}{5}x + \frac{4}{5}
\end{aligned}$$

The equation of the line in slope-intercept form is

$$\boxed{y = -\frac{4}{5}x + \frac{4}{5}}$$

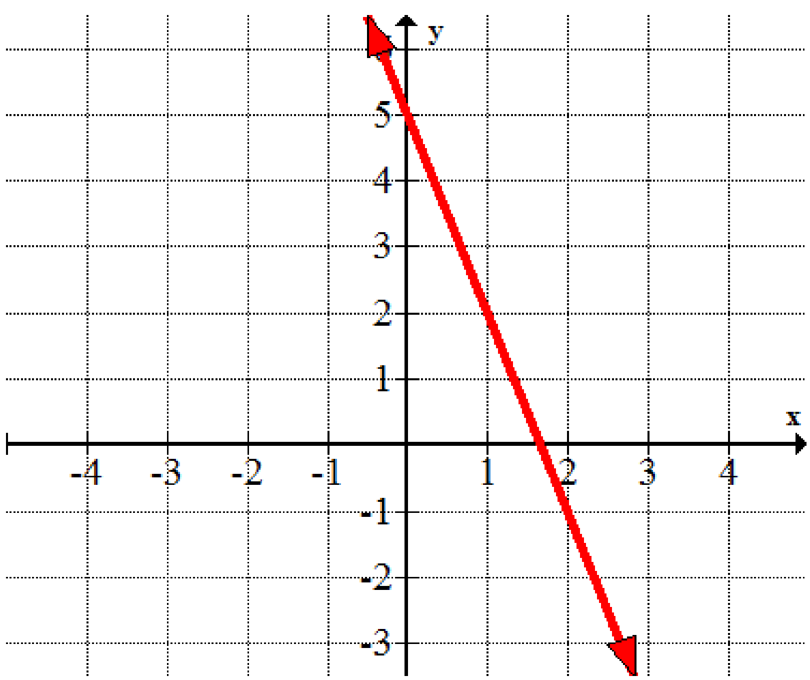
3. To rewrite the equation in standard form, first multiply the equation by 5 to get rid of the fractions. Then, set the equation equal to 0.

$$\begin{aligned} y &= -\frac{4}{5}x + \frac{4}{5} \\ 5y &= -4x + 4 \\ 4x + 5y &= 4 \end{aligned}$$

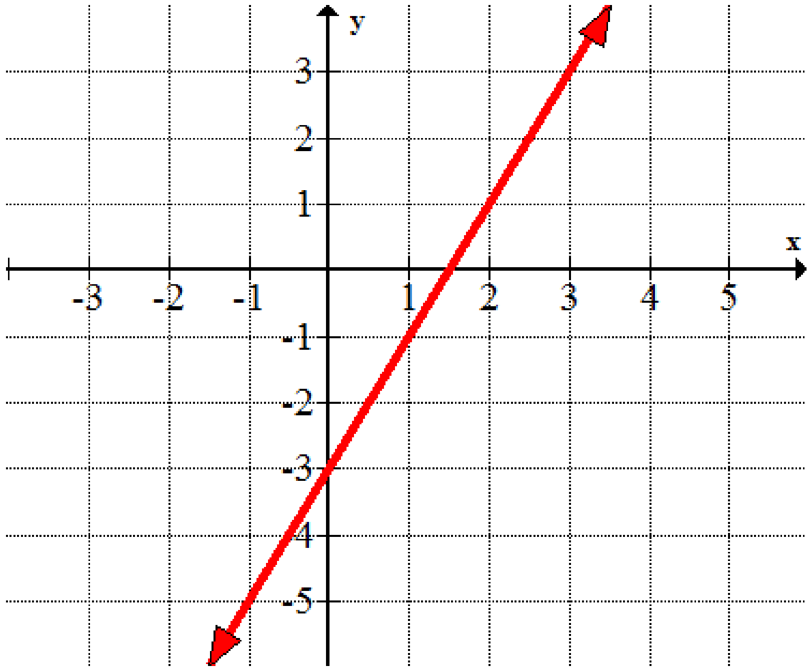
Practice Problems

For each of the following graphs, write the equation in slope-intercept form:

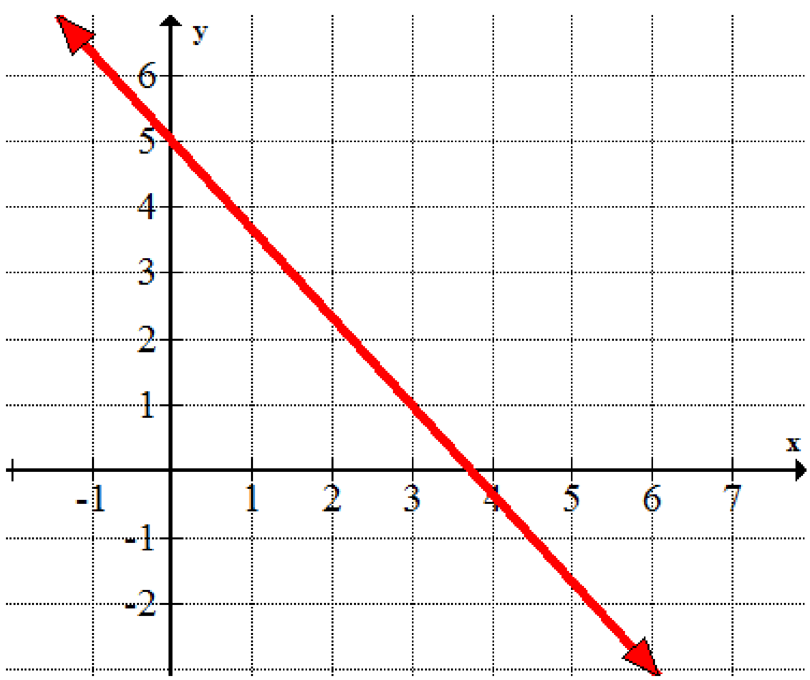
1.



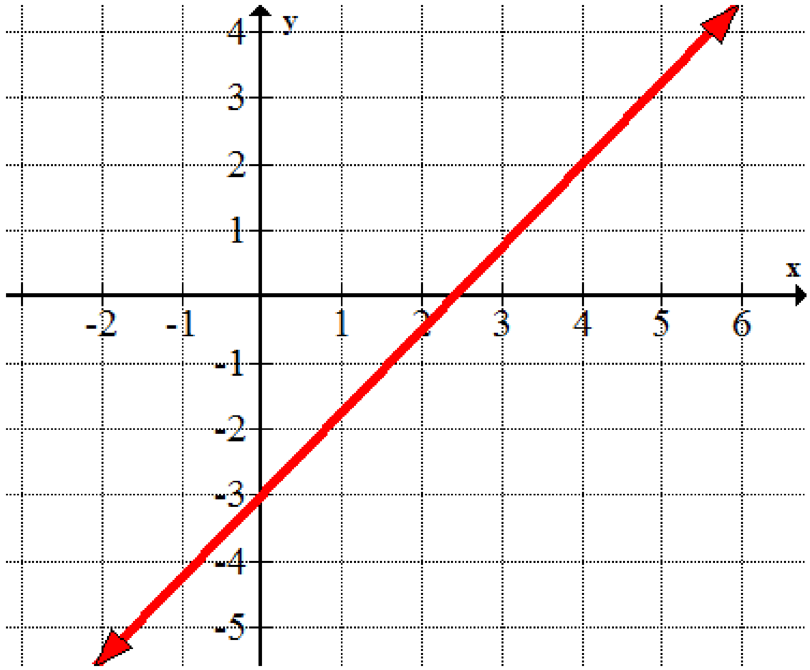
2.



3.

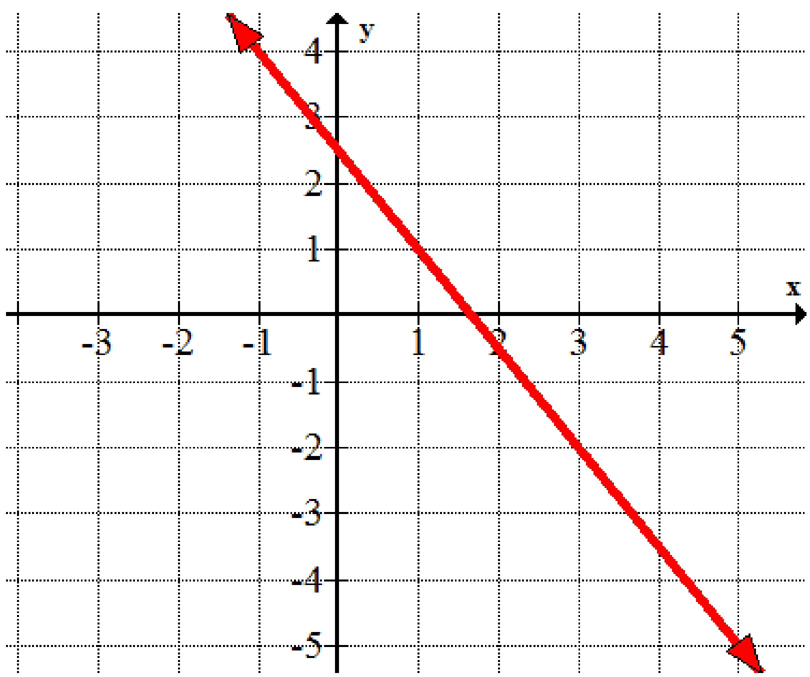


4.

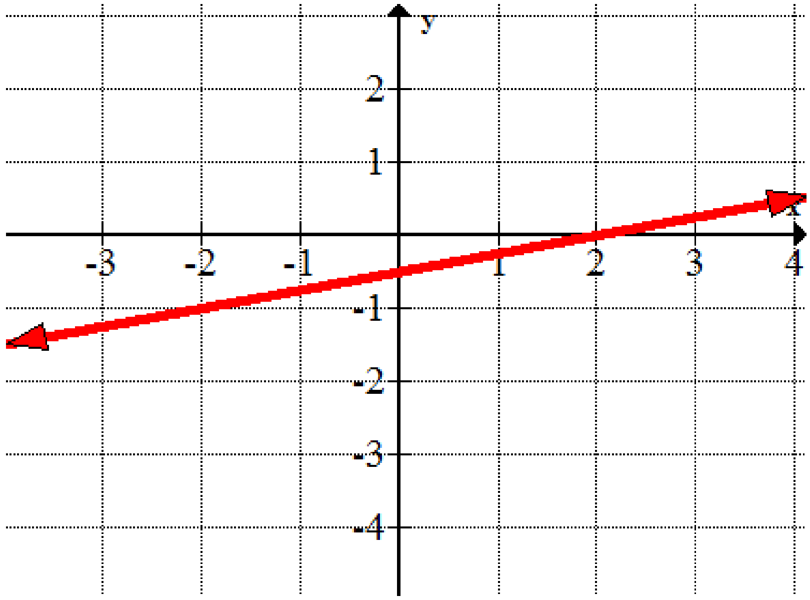


For each of the following graphs, write the equation in slope-intercept form:

5.



6.



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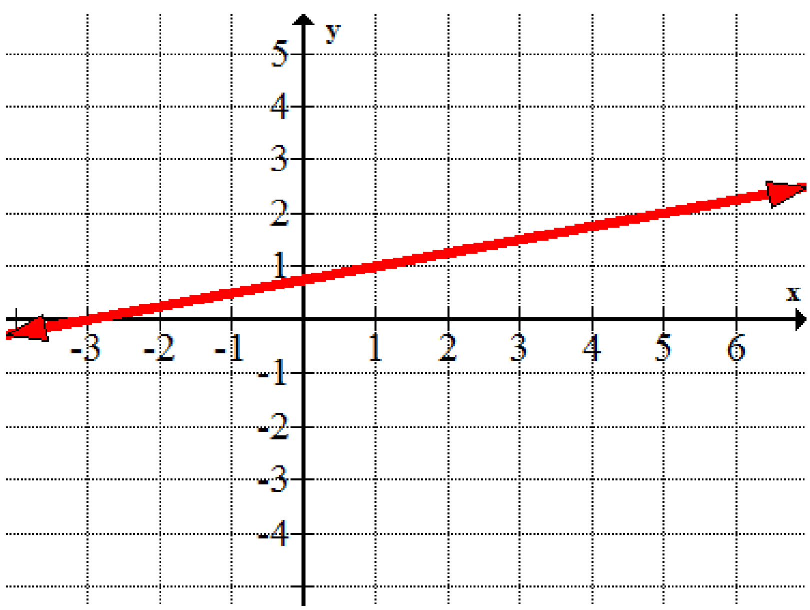
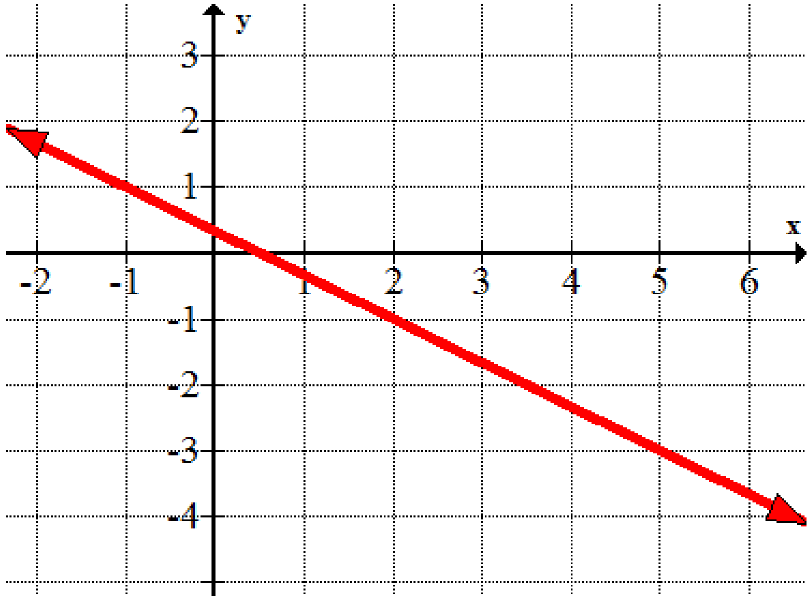
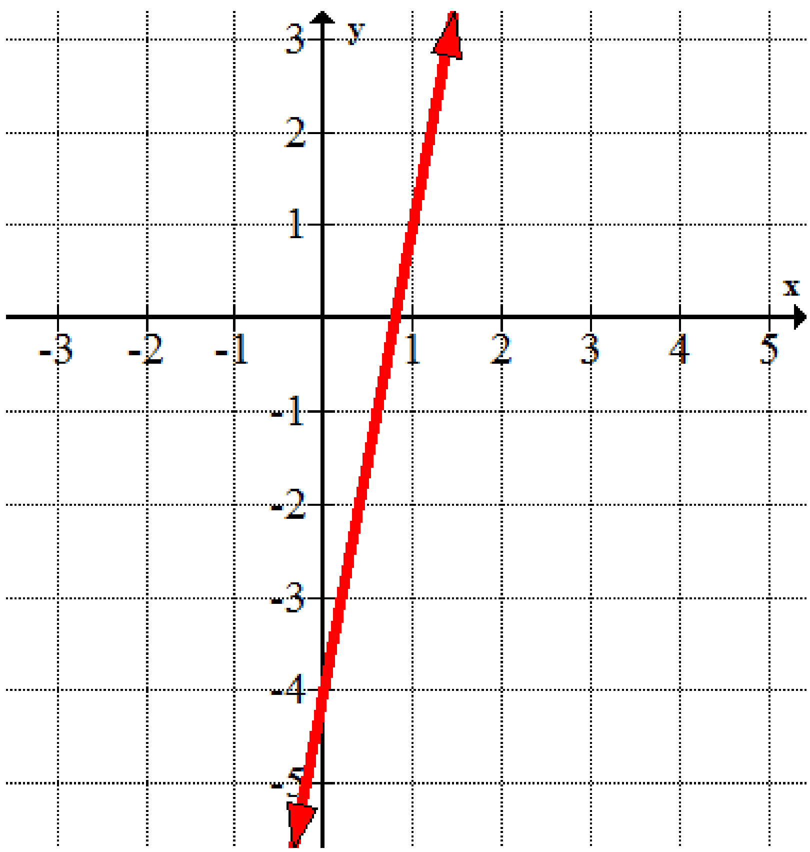


8.

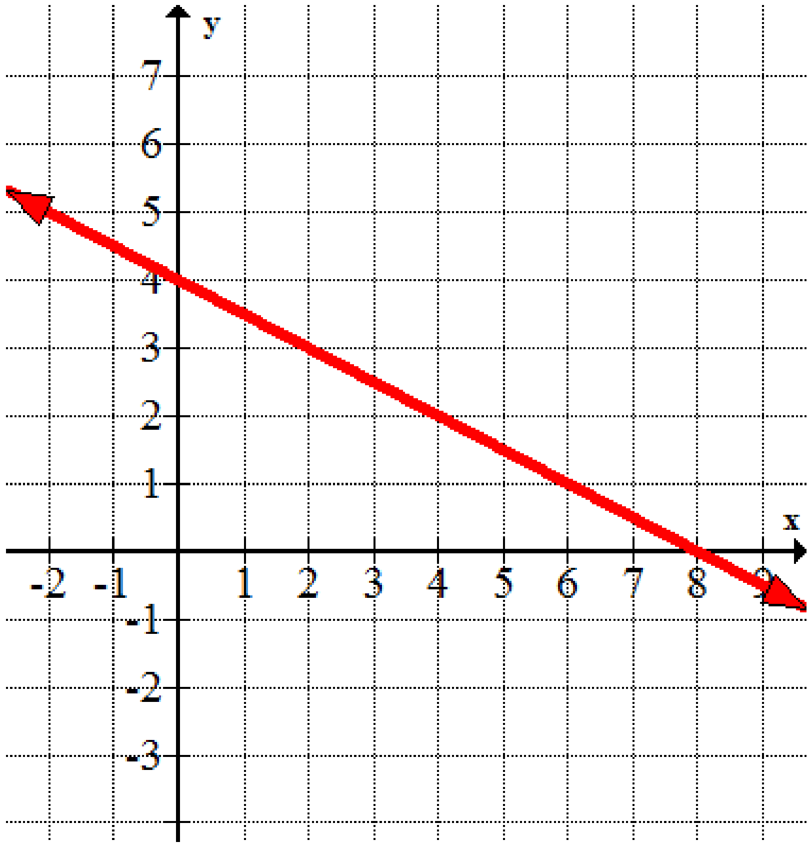


For each of the following graphs, write the equation standard form:

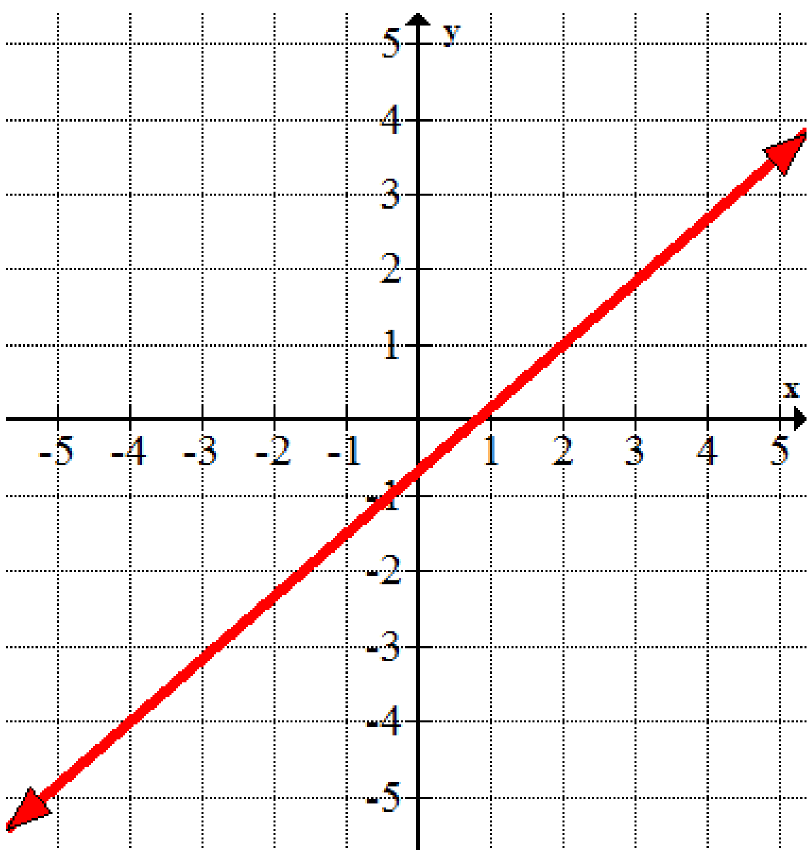
9.



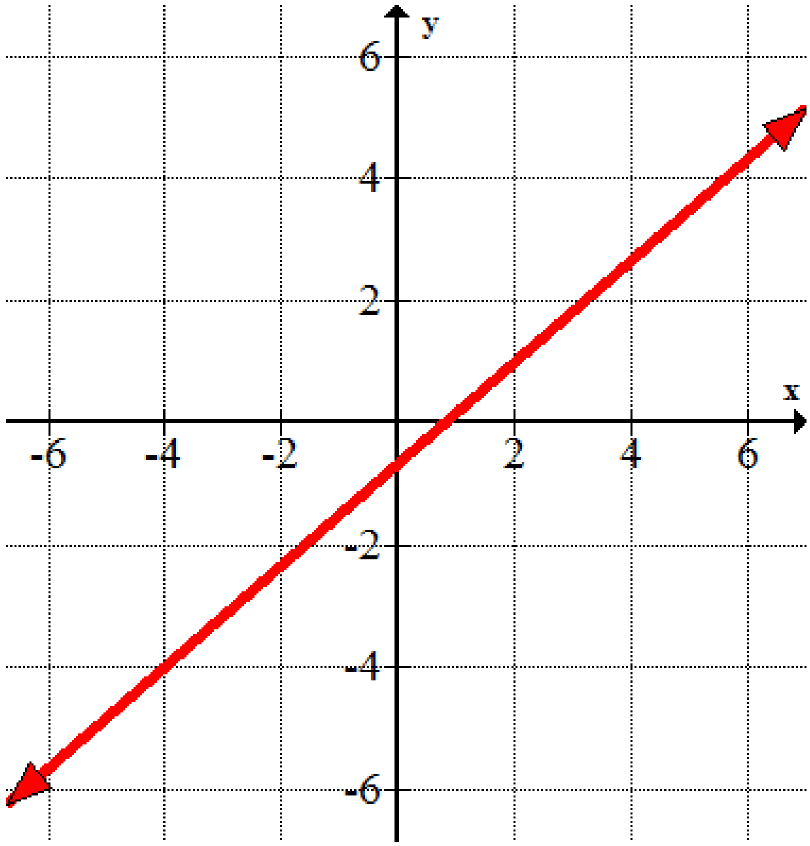
10.



11.



12.



13. Can you always find the equation of a line from its graph?
14. How do you find the equation of a vertical line? What about a horizontal line?
15. Rewrite the equation $y = \frac{1}{4}x - 5$ in standard form.
16. Rewrite the equation $y = \frac{2}{3}x + 1$ in standard form.
17. Rewrite the equation $y = \frac{1}{3}x - \frac{3}{7}$ in standard form.

3.7 Equations of Parallel and Perpendicular Lines

Concept Problem

Can you write the equation for the line that passes through the point (-2, -3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

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Khan Academy Parallel Lines

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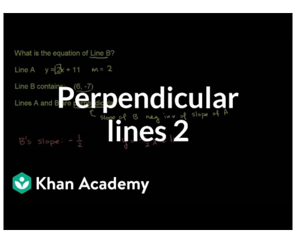
Khan Academy Perpendicular Lines



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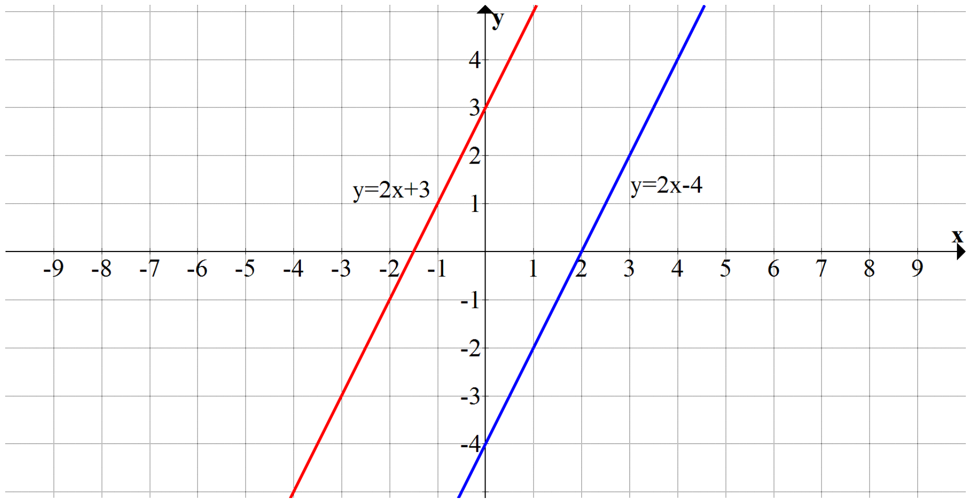
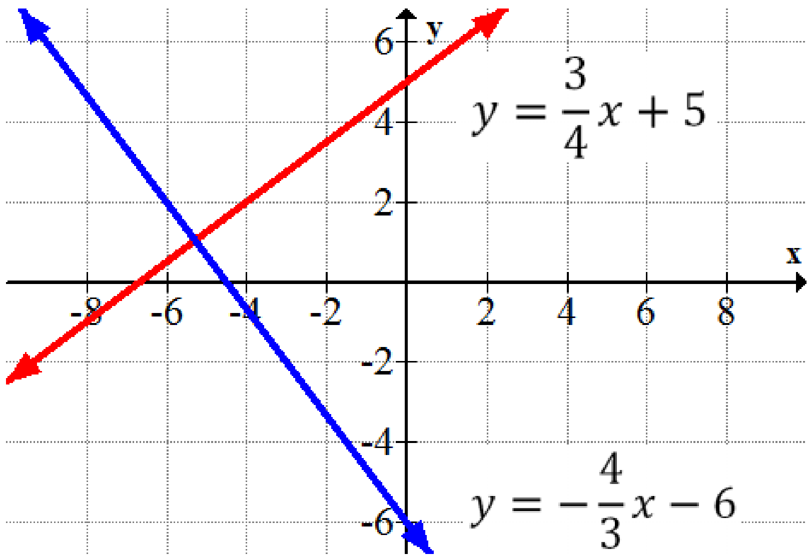
Guidance

Parallel lines are lines that never intersect. Parallel lines maintain the **same slope**, and the same distance from each other. The following graph shows two lines with the same slope. The slope of each line is 2. Notice that the lines are the same distance apart for the entire length of the lines. The lines will never intersect.



Two lines that intersect at a right angle are **perpendicular**. Perpendicular lines have **slopes that are opposite reciprocals**. The following graph shows two lines with slopes that are opposite reciprocals. The slope of one line is $\frac{3}{4}$ and the slope of the other line is $-\frac{4}{3}$. Notice that the lines intersect at a right angle. These lines are perpendicular:



You can use the relationship between the slopes of parallel lines and the slopes of perpendicular lines to write the equations of other lines.

Example A

Given the slopes of two lines, tell whether the lines are parallel, perpendicular or neither.

i) $m_1 = 4, m_2 = \frac{1}{4}$

ii) $m_1 = -3, m_2 = \frac{1}{3}$

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$

iv) $m_1 = -1, m_2 = 1$

v) $m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$

Solutions:

i) $m_1 = 4, m_2 = \frac{1}{4}$ The slopes are reciprocals but **not** opposite reciprocals. The lines are neither parallel nor perpendicular.

ii) $m_1 = -3, m_2 = \frac{1}{3}$ The slopes are reciprocals and are also opposites. The lines are perpendicular.

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$ The slopes are the same. The fractions are equivalent. The lines are parallel.

iv) $m_1 = -1, m_2 = 1$ The slopes are reciprocals and are also opposites. The lines are perpendicular.

v)$m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$ The slopes are not the same. The lines are neither parallel nor perpendicular.

Example B

Determine the equation of the line passing through the point (-4, 6) and parallel to the graph of $3x + 2y - 7 = 0$. Write the equation in standard form.

Solution:

If the equation of the line you are looking for is parallel to the given line, then the two lines have the same slope. Begin by expressing $3x + 2y - 7 = 0$ in slope-intercept form in order to find its slope.

$$\begin{aligned}
3x + 2y - 7 &= 0 \\
3x - 3x + 2y - 7 &= 0 - 3x \\
2y - 7 &= -3x \\
2y - 7 + 7 &= -3x + 7 \\
2y &= -3x + 7 \\
\frac{\cancel{2}y}{\cancel{2}} &= -\frac{3x}{2} + \frac{7}{2} \\
y &= -\frac{3}{2}x + \frac{7}{2} \\
&\updownarrow \\
y &= mx + b
\end{aligned}$$

The slope of the line is $-\frac{3}{2}$. The line passes through the point $(-4, 6)$.

Substitute the values into this equation.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 6 &= \frac{-3}{2}(x - -4) \\
y - 6 &= \frac{-3}{2}(x + 4) \\
y - 6 &= \frac{-3x}{2} - \frac{12}{2} \\
2(y) - 2(6) &= \cancel{2}\left(\frac{-3x}{\cancel{2}}\right) - \cancel{2}\left(\frac{12}{\cancel{2}}\right) \\
2y - 12 &= -3x - 12 \\
2y - 12 + 12 &= -3x - 12 + 12 \\
2y &= -3x \\
3x + 2y &= -3x + 3x \\
3x + 2y &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{3x + 2y = 0}$$

Example C

Determine the equation of the line that passes through the point (6, -2) and is perpendicular to the graph of $3x = 2y - 4$. Write the equation in standard form.

Solution: Begin by writing the equation $3x = 2y - 4$ in slope-intercept form.

$$\begin{aligned}
3x &= 2y-4\\
2y-4 &= 3x\\
2y-4+4 &= 3x+4\\
2y &= 3x+4\\
\frac{\cancel{2}y}{\cancel{2}} &= \frac{3x}{2}+\frac{4}{2}\\
y &= \frac{3}{2}x+2\\
&\updownarrow\\
y &= mx+b
\end{aligned}$$

The slope of the given line is $\frac{3}{2}$. The slope of the perpendicular line is

$$\boxed{-\frac{2}{3}}$$

. The line passes through the point (6, -2).

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y--2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2x}{3}+\frac{12}{3}\\
3(y)+3(2) &= 3\left(-\frac{2x}{3}\right)+3\left(\frac{12}{3}\right)\\
3(y)+3(2) &= \cancel{3}\left(-\frac{2x}{\cancel{3}}\right)+\cancel{3}\left(\frac{12}{\cancel{3}}\right)\\
3y+6 &= -2x+12\\
3y+6-6 &= -2x+12-6\\
3y &= -2x+6\\
2x+3y &= -2x+6+2x\\
2x+3y &= 6
\end{aligned}$$

The equation of the line is

$$\boxed{2x+3y=6}$$

.

Concept Problem Revisited

Can you write the equation for the line that passes through the point (-2, -3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the parallel line is the same as the slope of the given line.

$$\begin{aligned}
y+2x &= 8 \\
y+2x-2x &= -2x+8 \\
y &= -2x+8
\end{aligned}$$

The slope of the given line is -2. The slope of the parallel line is also

$$\boxed{-2}$$

.

$$\begin{aligned}
y-y_1 &= m(x-x_1) \\
y--3 &= -2(x--2) \\
y+3 &= -2(x+2) \\
y+3 &= -2x-4 \\
y+3 &= -2x-4 \\
2x+y+3 &= -2x+2x-4 \\
2x+y+3 &= -4 \\
2x+y+3+4 &= -4+4 \\
2x+y+7 &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{2x+y+7=0}$$

Guided Practice

Determine whether the lines that pass through the two pairs of points are parallel, perpendicular or neither parallel nor perpendicular.

1. (-2, 8), (3, 7) and (4, 3), (9, 2)

2. (2, 5), (8, 7) and (-3, 1), (-2, -2)

3. (4, 6), (-3, -1) and (6, -3), (4, 5)

4. Write the equation for the line that passes through the point (-3, 6) and is perpendicular to the graph of $3x=5y+6$. Write the equation of the line in slope-intercept form.

Answers:

1. (-2, 8), (3, 7) and (4, 3), (9, 2)

$m = \frac{y_2 - y_1}{x_2 - x_1}$	$m = \frac{y_2 - y_1}{x_2 - x_1}$	Use the formula to calculate the slopes
$m = \frac{7-8}{3--2}$	$m = \frac{2-3}{9-4}$	Calculate the slopes for each pair of points
$m = \frac{7-8}{3+2}$	$\boxed{m = \frac{-1}{5}}$	
$\boxed{m = \frac{-1}{5}}$		

The slopes of the lines are the same. The lines are parallel.

2. (2, 5), (8, 7) and (-3, 1), (-2, -2)

$m = \frac{y_2 - y_1}{x_2 - x_1}$	$m = \frac{y_2 - y_1}{x_2 - x_1}$	Use the formula to calculate the slopes
$m = \frac{7-5}{8-2}$	$m = \frac{-2-1}{-2--3}$	Calculate the slopes for each pair of points
$m = \frac{2}{6}$	$m = \frac{-2-1}{-2+3}$	
$\boxed{m = \frac{1}{3}}$	$\boxed{m = \frac{-3}{1}}$	

The slopes of the lines are opposite reciprocals. The lines are perpendicular.

3. (4, 6), (-3, -1) and (6, -3), (4, 5)

$m = \frac{y_2 - y_1}{x_2 - x_1}$	$m = \frac{y_2 - y_1}{x_2 - x_1}$	Use the formula to calculate the slopes
$m = \frac{-1-6}{-3-4}$	$m = \frac{5--3}{4-6}$	Calculate the slopes for each pair of points
$m = \frac{-7}{-7}$	$m = \frac{5+3}{4-6}$	
$\boxed{m = 1}$	$m = \frac{8}{-2}$	
	$\boxed{m = -4}$	

The lines are neither parallel nor perpendicular.

4. Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the perpendicular line is the opposite reciprocal.

$$3x = 5y + 6$$
$$5y + 6 = 3x$$
$$5y + 6 - 6 = 3x - 6$$
$$5y = 3x - 6$$
$$\frac{5y}{5} = \frac{3x}{5} - \frac{6}{5}$$
$$\frac{\cancel{5}y}{\cancel{5}} = \frac{3x}{5} - \frac{6}{5}$$
$$y = \frac{3}{5}x - \frac{6}{5}$$

The slope of the given line is $\frac{3}{5}$. The slope of the perpendicular line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the perpendicular line that passes through the point (-3, 6) is:

$$y = m(x - x_1) + y_1$$
$$y = -\frac{5}{3}(x - (-3)) + 6$$
$$y = -\frac{5}{3}(x + 3) + 6$$
$$y = -\frac{5}{3}x - 5 + 6$$
$$y = -\frac{5}{3}x + 1$$

The y-intercept is (0, 1) and the slope of the line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the line is

$$\boxed{y = -\frac{5}{3}x + 1}$$

.

Practice Problems

For each pair of given equations, determine if the lines are parallel, perpendicular or neither.

1. $y = 2x - 5$ and $y = 2x + 3$
2. $y = \frac{1}{3}x + 5$ and $y = -3x - 5$
3. $x = 8$ and $x = -2$

4. $y=4x+7$ and $y=-4x-7$
5. $y=-x-3$ and $y=x+6$
6. $3y=9x+8$ and $y=3x-4$

Determine the equation of the line satisfying the following conditions:

7. through the point (5, -6) and parallel to the line $y=5x+4$
8. through the point (-1, 7) and perpendicular to the line $y=-4x+5$
9. containing the point (-1, -5) and parallel to $3x+2y=9$
10. containing the point (0, -6) and perpendicular to $6x-3y+8=0$
11. through the point (2, 4) and perpendicular to the line $y=-\frac{1}{2}x+3$
12. containing the point (-1, 5) and parallel to $x+5y=3$
13. through the point (0, 4) and perpendicular to the line $2x-5y+1=0$

If $D(4,-1), E(-4,5)$ and $F(3,6)$ are the vertices of ΔDEF determine:

14. the equation of the line through D and parallel to EF.
15. the equation of the line containing the altitude from D to EF (the line perpendicular to EF that contains D).

3.8 Applications of Linear Functions

Learning Objectives

Here you will learn how to use what you know about the equations and graphs of lines to help you to solve real-life problems.

Concept Problem

Joe's Warehouse has banquet facilities to accommodate a maximum of 250 people. When the manager quotes a price for a banquet she is including the cost of renting the room plus the cost of the meal. A banquet for 70 people costs $1300. For 120 people, the price is $2200.

(a) Plot a graph of cost versus the number of people.

(b) From the graph, estimate the cost of a banquet for 150 people.

(c) Determine the slope of the line. What quantity does the slope of the line represent?

(d) Write an equation to model this real-life situation.

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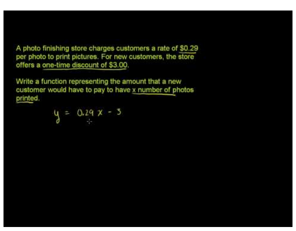
Khan Academy Basic Linear Function



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Guidance

Linear relationships are often used to model real-life situations. In order to create an equation and graph to model the real-life situation, you need at least two data values related to the real-life situation. When the data values have been represented graphically and the equation of the line has been determined, questions relating to the real-life situation can be presented and answered.

Example A

Stan's Industrial Clean-o-Mart laundromat charges a base price for each load of laundry plus an additional charge per pound of laundry in the load. One customer did 28 pounds of laundry and was charged $65. Another customer did 42 pounds of laundry and was charged $82.50. What is the base price that Stan's charges, and what is the charge per pound?

Solution: First, define the variables that matter in the problem:

$$x = \text{\# pounds of laundry}$$
$$y = \text{cost of laundry load}$$

Now, translate the information given in the problem into points: $(28, 65)$ and $(42, 82.5)$.

Since the price charged per pound of laundry is constant, we know that a linear function will work in this situation. We can use the slope formula on the two points to find the slope:

$$m = \frac{82.5 - 65}{42 - 28} = \frac{17.5}{14} = 1.25$$

Now, use the slope and one of the points in the point-slope form:

$$\begin{aligned} y &= m(x - x_1) + y_1 \\ y &= 1.25(x - 28) + 65 && \text{Plug in} \\ y &= 1.25x - 35 + 65 && \text{Distribute} \\ y &= 1.25x + 30 && \text{Add} \end{aligned}$$

The slope tells us that there is a charge of $1.25 per pound of laundry. The y-intercept is 30, which indicates that the base charge (the charge for a 0-pound load of laundry) is $30.

Example B

When a weight is hung from a spring, the spring stretches. In fact, the relationship between the size of the weight and the length of stretching is linear. When a 40 gram mass was suspended from a coil spring, the length of the spring was 24 inches. When an 80 gram mass was suspended from the same coil spring, the length of the spring was 36 inches.

(a) Plot a graph of length versus mass.

(b) From the graph, estimate the length of the spring for a mass of 70 grams.

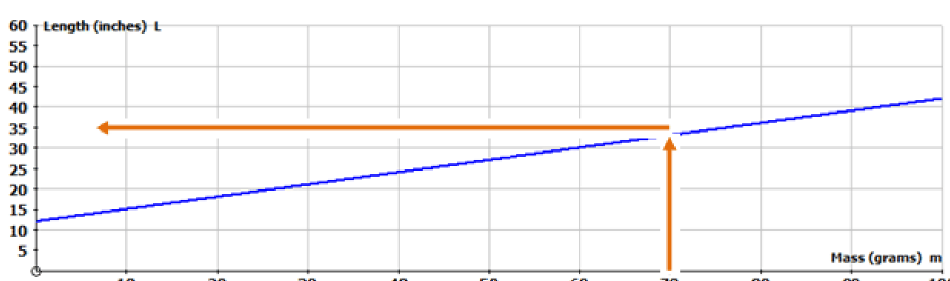
(c) Determine an equation that models this situation. Write the equation in slope-intercept form.

(d) Use the equation to determine the length of the spring for a mass of 60 grams.

(e) What is the y-intercept? What does the y-intercept represent?

Solution:

(a) On the x-axis is the mass in grams and on the y-axis is the length of the spring in inches.



(b) The length of the coil spring for a mass of 70 grams is approximately 33 inches.

(c) The equation of the line can be determined by using the two data values (40, 24) and (80, 36).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{36 - 24}{80 - 40}$$
$$m = \frac{12}{40}$$
$$m = \frac{3}{10}$$

$$y = m(x - x_1) + y_1$$
$$y = \frac{3}{10}(x - 40) + 24$$
$$y = \frac{3}{10}x - 12 + 24$$
$$y = \frac{3}{10}x + 12$$

The y-intercept is (0, 12). The equation that models the situation is

$y = \frac{3}{10}x + 12$

$$\boxed{l = \frac{3}{10}m + 12}$$

where 'l' is the length of the spring in inches and 'm' is the mass in grams.

(d)

$$l = \frac{3}{10}m + 12 \qquad \text{Use the equation and substitute 60 in for } m.$$
$$l = \frac{3}{10}(60) + 12$$
$$l = \frac{3}{\cancel{10}}\left(\overset{6}{\cancel{60}}\right) + 12$$
$$l = 18 + 12$$
$$\boxed{l = 30 \text{ } inches}$$

(e) The y-intercept is (0, 12). The y-intercept represents the length of the coil spring before a mass was suspended from it. The length of the coil spring was 12 inches.

Concept Problem Revisited

Joe's Warehouse has banquet facilities to accommodate a maximum of 250 people. When the manager quotes a price for a banquet she is including the cost of renting the room plus the cost of the meal. A banquet for 70 people costs \$1300. For 120 people, the price is \$2200.

(a) Plot a graph of cost versus the number of people.

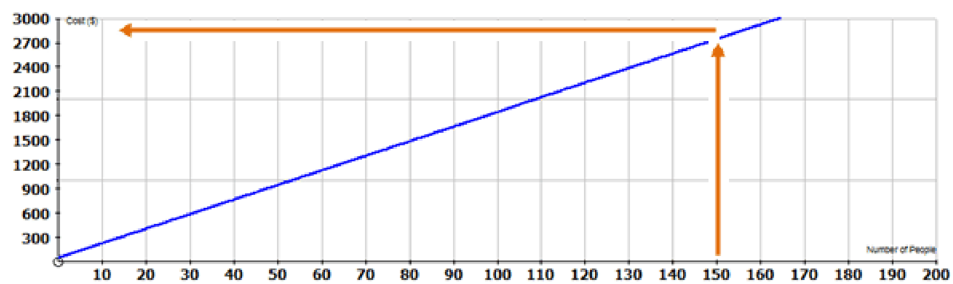
(b) From the graph, estimate the cost of a banquet for 150 people.

(c) Determine the slope of the line. What quantity does the slope of the line represent?

(d) Write an equation to model this real-life situation.

Solution:

(a) On the x-axis is the number of people and on the y-axis is the cost of the banquet.



(b) The approximate cost of a banquet for 150 people is \$2700.

(c) The two data points (70, 1300) and (120, 2200) will be used to calculate the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{2200 - 1300}{120 - 70}$$
$$m = \frac{900}{50}$$
$$m = \frac{18}{1}$$

The slope represents the cost of the banquet for each person. The cost is \$18 per person.

When a linear function is used to model the real life situation, the equation can be written in the form or in the form $y = mx + b$ or in the form $Ax + By + C = 0$.

(d)

$$y = m(x - x_1) + y_1$$
$$y = 18(x - 70) + 1300$$
$$y = 18x - 1260 + 1300$$
$$y = 18x + 40$$

The y-intercept is (0, 40)

The equation to model the real-life situation is $y = 18x + 40$. The variables should be changed to match the labels on the axes. The equation that best models the situation is $c = 18n + 40$ where 'c' represents the cost and 'n' represents the number of people.

Guided Practice

1. Some college students who plan on becoming math teachers decide to set up a tutoring service for high school math students. One student was charged \$25 for 3 hours of tutoring. Another student was charged \$55 for 7 hours

of tutoring. The relationship between the cost and time is linear.

(a) What is the independent variable?
(b) What is the dependent variable?
(c) What are two data values for this relationship?
(d) Draw a graph of cost versus time.
(e) Determine an equation to model the situation.
(f) What is the significance of the slope?
(g) What is the cost-intercept? What does the cost-intercept represent?

2. A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for $1950 and 15 in another for $2850. The number of computers and cost forms a linear relationship

(a) State the dependent and independent variables.
(b) Sketch a graph.
(c) Find an equation expressing cost in terms of the number of computers.
(d) State the slope of the line and tell what the slope means to the problem.
(e) State the cost-intercept and tell what it means to this problem.
(f) Using your equation, calculate the number of computers you could get for $6000.

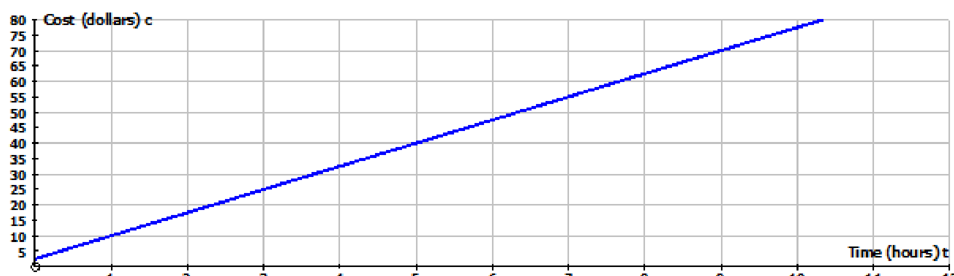
3. Handy Andy sells one quart can of paint thinner for $7.65 and a two quart can for $13.95. Assume there is a linear relationship between the volume of paint thinner and the price.

(a) What is the independent variable?
(b) What is the dependent variable?
(c) Write two data values for this relationship.
(d) Draw a graph to represent this relationship.
(e) What is the slope of the line?
(f) What does the slope represent in this problem?
(g) Write an equation to model this problem.
(h) What is the cost-intercept?
(i) What does the cost-intercept represent in this problem?
(j) How much would you pay for 6 quarts of paint thinner?

Answers:

1. (a) The cost for tutoring depends upon the amount of time. The independent variable is the time.

(b) The dependent variable is the cost.
(c) Two data values for this relationship are (3, 25) and (7, 55).
(d) On the x-axis is the time in hours and on the y-axis is the cost in dollars.



(e) Use the two data values (3, 25) and (7, 55) to calculate the slope of the line. $m = \frac{15}{2}$. Determine the y-intercept of the graph.

$$\begin{aligned} y &= mx + b \\ 25 &= \frac{15}{2}(3) + b \qquad \text{Use the slope and one of the data values to determine the value of } b. \\ 25 &= \frac{45}{2} + b \\ 25 - \frac{45}{2} &= \frac{45}{2} - \frac{45}{2} + b \\ \frac{50}{2} - \frac{45}{2} &= b \\ \frac{5}{2} &= b \end{aligned}$$

The equation to model the relationship is $y = \frac{15}{2}x + \frac{5}{2}$. To match the variables of the equation with the graph the equation is

$$\boxed{c = \frac{15}{2}t + \frac{5}{2}}$$

. The relationship is cost in dollars versus time in hours. The equation could also be written as

$$\boxed{c = 7.50t + 2.50}$$

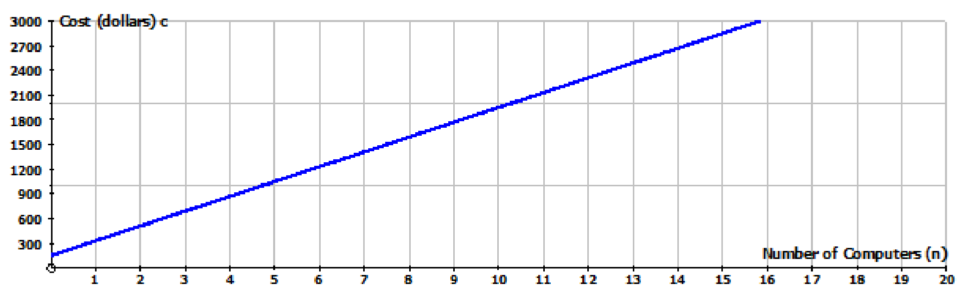
.

(f) The slope of $\frac{15}{2}$ means that it costs \$15.00 for 2 hours of tutoring. If the slope is expressed as a decimal, it means that it costs \$7.50 for 1 hour of tutoring.

(g) The cost-intercept is the y-intercept. The y-intercept is (0, 2.50). This value could represent the cost of having a scheduled time or the cost that must be paid for cancelling the appointment. In a problem like this, the y-intercept must represent a meaningful quantity for the problem.

2. (a) The number of dollars in sales from the computers depends upon the number of computers sold. The dependent variable is the dollars in sales and the independent variable is the number of computers sold.

(b) On the x-axis is the number of computers and on the y-axis is the cost of the computers.



(c) Use the data values (10, 1950) and (15, 2850) to calculate the slope of the line. $m = 180$. Next determine the y-intercept of the graph.

$$\begin{aligned}
y &= mx+b\\
1950 &= 180(10)+b\\
1950 &= 1800+b\\
1950-1800 &= 1800-1800+b\\
150 &= b
\end{aligned}$$

The equation of the line that models the relationship is

$$\boxed{y = 180x+150}$$

. To make the equation match the variables of the graph the equation is

$$\boxed{c = 180n+150}$$

.

(d) The slope is $\frac{180}{1}$. This means that the cost of one computer is \$180.00.

(e) The cost intercept is the y-intercept. The y-intercept is (0, 150). This could represent the cost of renting the location where the sales are being made or perhaps the salary for the sales person.

(f)

$$\begin{aligned}
c &= 180n+150\\
6000 &= 180n+150\\
6000-150 &= 180n+150-150\\
5850 &= 180n\\
\frac{5850}{180} &= \frac{180n}{180}\\
\frac{5850}{180} &= \frac{\cancel{180}n}{\cancel{180}}\\
32.5 &= n
\end{aligned}$$

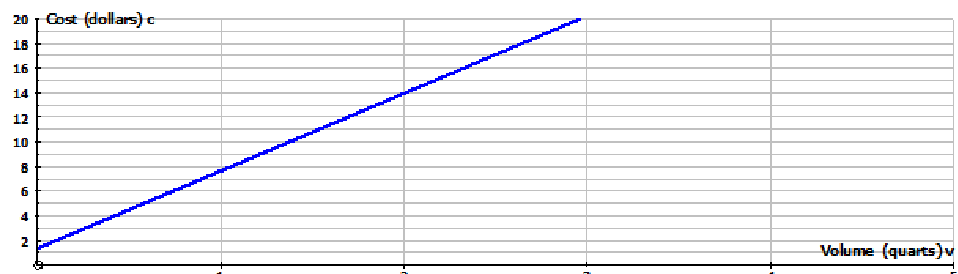
With \$6000 you could get **32** computers.

3. (a) The independent variable is the volume of paint thinner.

(b) The dependent variable is the cost of the paint thinner.

(c) Two data values are (1, 7.65) and (2, 13.95).

(d) On the x-axis is the volume in quarts and on the y-axis is the cost in dollars.



(e) Use the two data values (1, 7.65) and (2, 13.95) to calculate the slope of the line. The slope is $m = 6.30$.

(f) The slope represents the cost of one quart of paint thinner. The cost is \$6.30.

(g)

$$\begin{aligned}
y &= mx + b \\
7.65 &= 6.30(1) + b \\
7.65 &= 6.30 + b \\
7.65 - 6.30 &= 6.30 - 6.30 + b \\
1.35 &= b
\end{aligned}$$

The equation to model the relationship is $y = 6.30x + 1.35$. The equation that matches the variables of the graph is

$$\boxed{c = 6.30v + 1.35}$$

.

(h) The cost-intercept is (0, 1.35).
(i) This could represent the cost of the can that holds the paint thinner.
(j)

$$\begin{aligned}
c &= 6.30v + 1.35 \\
c &= 6.30(6) + 1.35 \\
c &= 37.80 + 1.35 \\
c &= \$39.15
\end{aligned}$$

The cost of 6 quarts of paint thinner is $39.15.

Practice Problems

Players on the school soccer team are selling candles to raise money for an upcoming trip. Each player has 24 candles to sell. If a player sells 4 candles a profit of $30 is made. If he sells 12 candles a profit of $70 is made. The profit and the number of candles sold form a linear relation.

1. State the dependent and the independent variables.
2. What are the two data values for this relation?
3. Draw a graph and label the axis.
4. Determine an equation to model this situation.
5. What is the slope and what does it mean in this problem?
6. Find the profit-intercept and explain what it represents.
7. Calculate the maximum profit that a player can make.
8. Write a suitable domain and range.
9. If a player makes a profit of $90, how many candles did he sell?
10. Is this data continuous, discrete, or neither? Justify your answer.

Jacob leaves his summer cottage and drives home. After driving for 5 hours, he is 112 km from home, and after 7 hours, he is 15 km from home. Assume that the distance from home and the number of hours driving form a linear relationship.

11. State the dependent and the independent variables.
12. What are the two data values for this relationship?
13. Represent this linear relationship graphically.
14. Determine the equation to model this situation.

15. What is the slope and what does it represent?
16. Find the distance-intercept and its real-life meaning in this problem.
17. How long did it take Jacob to drive from his summer cottage to home?
18. Write a suitable domain and range.
19. How far was Jacob from home after driving 4 hours?
20. How long had Jacob been driving when he was 209 km from home?

3.9 Equations with Fractions

Learning Objectives

Here you will learn to solve equations that contain fractions.

Concept Problem

In this year's student election for president, there were two candidates. The winner received 9 votes less than $\frac{3}{5}$ of the total number of votes. If there were 360 ballots cast for the winner, how many total votes were there?

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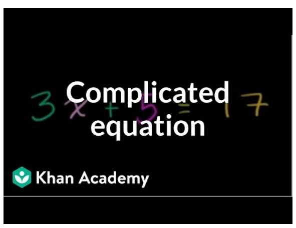
Khan Academy Slightly More Complicated Equations



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Guidance

Fractions are an unavoidable part of life. They often appear when applying linear equations to the real world. Working with fractions can be unwieldy, but there is a way to make solving equations that have fractions easier using the least common denominator (LCD).

When solving an equation involving fractions, you can clear out the fractions by multiplying both sides of the LCD. This will leave you with an equation that is easier to solve. Often, you can do this as the first step of the problem.

Here are the steps:

1. Find the LCD of **all** fractions in the equation
2. Multiply both sides of the equation by the LCD
3. Simplify and solve

Example A

Solve: $\frac{1}{3}t+5=-1$.

Solution:

1. In this case $\frac{1}{3}$ is the only fraction, so the LCD is 3.

2. Multiply both sides by 3.

$$3 \cdot \left(\frac{1}{3}t + 5\right) = 3 \cdot (-1)$$
$$t + 15 = -3$$

3. Solve for t:

$$t + 15 = -3$$
$$t + 15 - 15 = -3 - 15$$
$$t = -18$$

Therefore $t = -18$.

Check:
$$\frac{1}{3}t + 5 = -1$$
$$\frac{1}{3}(-18) + 5 = -1$$
$$-6 + 5 = -1$$
$$-1 = -1$$

Example B

Solve: $\frac{3}{4}x - 3 = 2$.

Solution:

1. In this problem the only value in a denominator is 4, so the LCD is 4.

2. Multiply both sides of the equation by 4:

$$4 \cdot \left(\frac{3}{4}x - 3\right) = 4 \cdot 2$$
$$3x - 12 = 8$$

3. Solve for x:

$$3x - 12 = 8$$
$$3x - 12 + 12 = 8 + 12$$
$$3x = 20$$
$$\frac{3x}{3} = \frac{20}{3}$$
$$x = \frac{20}{3}$$

Therefore $x = \frac{20}{3}$.

$$\begin{aligned} \text{Check:}& \\ \frac{3}{4}x-3&=2 \\ \frac{3}{4}\left(\frac{20}{3}\right)-3&=2 \\ \frac{20}{4}-3&=2 \\ 5-3&=2 \\ 2&=2 \end{aligned}$$

Example C

Solve: $\frac{2}{5}x-4=-\frac{3}{4}x+8$.

Solution:

1. In this problem the numbers 4 and 5 are our denominators, so the LCD is 20.

2. Multiply both sides of the equation by 20:

$x=\frac{240}{23}$

3. Now solve for x:

$$\begin{aligned} 8x-80&=-15x+160 \\ 8x-80+80&=-15x+160+80 \\ 8x&=-15x+240 \\ 8x+15x&=-15x+240+15x \\ 23x&=240 \\ \frac{23x}{23}&=\frac{240}{23} \\ x&=\frac{240}{23} \end{aligned}$$

Therefore $x=\frac{240}{23}$.

Concept Problem Revisited

We are asked to find the total number of votes in the election, so define a variable accordingly: $x=$ total votes cast.

We know that the winner received 360 votes and that this represents 9 less than $\frac{3}{5}$ of the total vote. We can translate this information into an equation:

$$360=\frac{3}{5}x-9$$

1. The LCD is this case is 5.

2. Multiply both sides by 5 to clear the fraction:

$$5 \cdot 360 = 5 \cdot \left(\frac{3}{5}x - 9\right)$$
$$1800 = 3x - 45$$

3. Now solve for x:

$$\begin{aligned}
1800 &= 3x - 45 \\
1800 + 45 &= 3x - 45 + 45 \\
1845 &= 3x \\
\frac{1845}{3} &= \frac{3x}{3} \\
x &= 615
\end{aligned}$$

So there were a total of 615 votes cast in the election.

Vocabulary

Least Common Denominator

The ***least common denominator*** or lowest common denominator is the smallest number that all of the denominators (or the bottom numbers) can be divided into evenly. For example with the fractions $\frac{1}{2}$ and $\frac{1}{3}$, the smallest number that both 2 and 3 will divide into evenly is 6. Therefore the least common denominator is 6.

Guided Practice

1. Solve for x: $\frac{2}{3}x = 12$.

2. Solve for x: $\frac{3}{4}x - 5 = 19$.

3. Solve for w: $\frac{1}{4}w - 3 = \frac{2}{3}w$.

Answers:

1.

$$\begin{aligned}
\frac{2}{3}x &= 12 \\
(\cancel{3})\frac{2}{\cancel{3}}x &= 12(3) && \text{(Multiply both sides by the denominator (3) in the fraction)} \\
2x &= 36 && \text{(Simplify)} \\
\frac{\cancel{2}x}{\cancel{2}} &= \frac{36}{2} && \text{(Divide both sides by the numerator (2) in the fraction)} \\
x &= 18 && \text{(Simplify)}
\end{aligned}$$

Therefore $x = 18$.

Check:

$$\begin{aligned}\frac{2}{3}x &= 12\\ \frac{2}{3}(18) &= 12\\ \frac{36}{3} &= 12\\ 12 &= 12\end{aligned}$$

2.

$$\begin{aligned}\frac{3}{4}x-5 &= 19\\ 4\cdot\left(\frac{3}{4}x-5\right) &= 4\cdot 19 && \text{(Multiply both sides by the LCD 4)}\\ 3x-20 &= 76 && \text{(Simplify)}\\ 3x-20+20 &= 76+20 && \text{(Add 20 to both sides)}\\ 3x &= 96 && \text{(Simplify)}\\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{96}{3} && \text{(Divide both sides by numerator (3) in the fraction)}\\ x &= 32 && \text{(Simplify)}\end{aligned}$$

Therefore $x = 32$.

Check:

$$\begin{aligned}\frac{3}{4}x-5 &= 19\\ \frac{3}{4}(32)-5 &= 19\\ \frac{96}{4}-5 &= 19\\ 24-5 &= 19\\ 19 &= 19\end{aligned}$$

3.

$$\begin{aligned}
\frac{1}{4}w-3 &= \frac{2}{3}w \\
12\cdot\left(\frac{1}{4}w-3\right) &= 12\cdot\left(\frac{2}{3}w\right) && \text{(Multiply both sides by the LCD 12)} \\
3w-36 &= 8w && \text{(Simplify)} \\
3w-36-3w &= 8w-3w && \text{(Subtract 3w from both sides)} \\
-36 &= 5w && \text{(Simplify)} \\
\frac{-36}{5} &= \frac{5w}{5} && \text{(Divide both sides by 5)} \\
\frac{-36}{5} &= w \\
w &= -\frac{36}{5}
\end{aligned}$$

Therefore $w=-\frac{36}{5}$.

$$\begin{aligned}
&\text{Check:} \\
\frac{1}{4}w-3 &= \frac{2}{3}w \\
\frac{1}{4}\left(\frac{-36}{5}\right)-3 &= \frac{2}{3}\left(\frac{-36}{5}\right) \\
\frac{-36}{20}-3 &= \frac{-72}{15} \\
\frac{-108}{60}-\frac{180}{60} &= \frac{-288}{60} \\
\frac{-288}{60} &= \frac{-288}{60}
\end{aligned}$$

Practice

Solve for the variable in each of the following equations.

1. $\frac{1}{3}p=5$
2. $\frac{3}{7}j=8$
3. $\frac{2}{5}b+4=6$
4. $\frac{2}{3}x-2=1$
5. $\frac{1}{3}x+3=-3$

6. $\frac{1}{8}k+\frac{2}{3}=5$
7. $\frac{1}{6}c+\frac{1}{3}=-2$
8. $\frac{4}{5}x+3=\frac{2}{3}$
9. $\frac{3}{4}x-\frac{2}{5}=\frac{1}{2}$
10. $\frac{1}{4}t+\frac{2}{3}=\frac{1}{2}$

11. $\frac{1}{3}x+\frac{1}{4}x=1$
12. $\frac{1}{5}d+\frac{2}{3}d=\frac{5}{3}$
13. $\frac{1}{2}x-1=\frac{1}{3}x$

14. $\frac{1}{3}x - \frac{1}{2} = \frac{3}{4}x$
15. $\frac{2}{3}j - \frac{1}{2} = \frac{3}{4}j + \frac{1}{3}$

Summary

You learned that slope is a measure of the steepness of a line. You learned how to find the slope of a line given two points on the line or the graph of the line. You also learned how to write the equation of a line in two forms: slope-intercept form, point-slope form and standard form. You learned how to graph a line from its equation.

You also learned about parallel and perpendicular lines. You learned parallel lines always have the same slope while perpendicular lines always have slopes that are opposite reciprocals. Finally, you learned how to model real-world problems with linear functions.

CHAPTER 4 The Empty Chapter

Chapter Outline

This chapter holds many secrets, but the greatest is this: I am a fake chapter

4.1 The Empty Section

This chapter only exists so that the chapter numbers in this book match up with the unit numbers of the course, but since you are reading this, here are some interesting math facts:

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...=1$

$e^{i\pi}+1=0$

$.99999999...=1$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}-...=.693147...$
but
$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}+...=.346573...$

And in the end, the love you take is equal to the love you make.

CHAPTER 5 Unit 5 - Systems of Equations

Chapter Outline

In the last chapter, we learned things we can do with a single line. What if you have two lines? How might they relate to each other? Do their graphs intersect? If so, where? These are ideas that we will investigate.

5.1 Introduction to Linear Systems and Their Graphs

Concept Problem

Suppose that you own a taco truck. You sell tacos for \$3 apiece and burritos for \$6 apiece. One particular day, you take in \$450 from 110 total sales. As the owner, you would like to know how many of each item you sold. Is it possible to determine from the given information?

We can get partway to the answer by translating the information given above into mathematical expressions. First, notice that there are two separate quantities we are interested in (tacos and burritos), so we will need to define two variables:

$$\begin{aligned} x &= \text{\# tacos sold} \\ y &= \text{\# burritos sold} \end{aligned}$$

We can use x and y to construct equations. For instance, we know there were a total of 110 sales, so we can say:

$$\underbrace{x}_{\text{\# tacos}} + \underbrace{y}_{\text{\# burritos}} = \underbrace{110}_{\text{total sales}}$$

Similarly, since we know we took in a total of \$490 we can say:

$$\underbrace{3x}_{\text{taco revenue}} + \underbrace{6y}_{\text{burrito revenue}} = \underbrace{450}_{\text{total revenue}}$$

From our two pieces of information we were able to create two equations... now what?

Linear Systems

A **system of equations** is a collection of two or more equations. These often arise in situations where you are given two separate pieces of information about a situation (like the sales and revenue in the problem above). We want to find an 'answer' or 'solution' to these equations, but what does that mean, precisely?

A **solution** to a system is an ordered pair that that satisfies *all* of the equations in the system.

Example A

A system is given below:

$$\begin{cases} y = 4x - 1 \\ y = 2x + 3 \end{cases}$$

Are any of the points (1, 3), (0, 2), and (2, 7) a solution to the system?

A solution must satisfy all of the equations in the system. To determine if a particular ordered pair is a solution, we can plug the coordinates in for the variables x and y in each equation and check.

Check (1, 3) : $\begin{cases} 3 = 4(1) - 1;\ 3 = 3. \text{ Yes, this ordered pair checks.} \\ 3 = 2(1) + 3;\ 3 = 5. \text{ No, this ordered pair does not check.} \end{cases}$

Check (0, 2) : $\begin{cases} 2 = 4(0) - 1;\ 2 = \text{-}1. \text{ No, this ordered pair does not check.} \\ 2 = 2(0) + 3;\ 2 = 3. \text{ No, this ordered pair does not check.} \end{cases}$

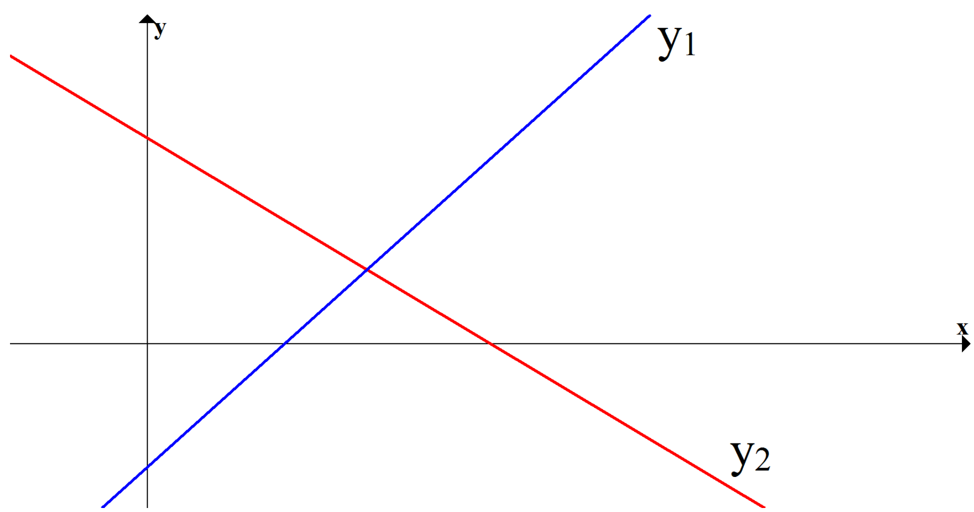
Check (2, 7) : $\begin{cases} 7 = 4(2) - 1;\ 7 = 7. \text{ Yes, this ordered pair checks.} \\ 7 = 2(2) + 3;\ 7 = 7. \text{ Yes, this ordered pair checks.} \end{cases}$

Because the coordinate (2, 7) works in both equations simultaneously, it is a solution to the system.

Graphs and Systems

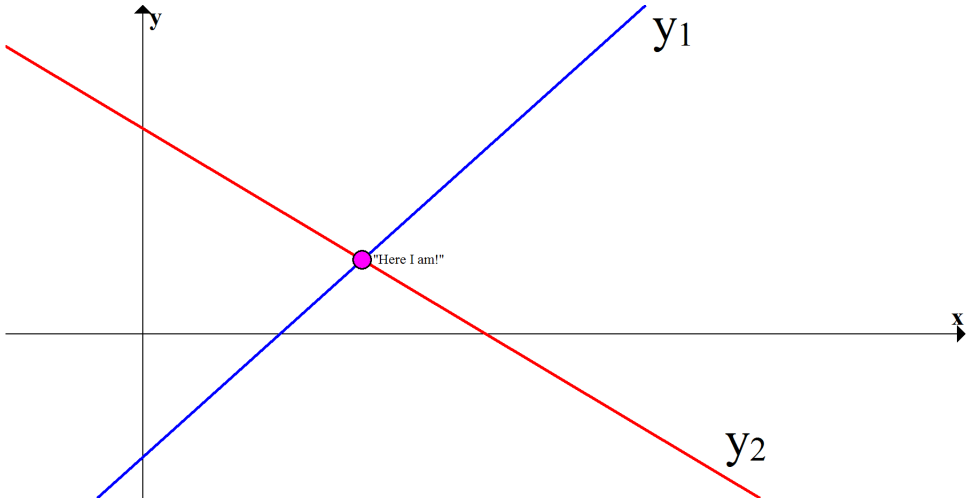
What does the graph of a system of equations look like? What does a solution look like in a graph?

Suppose that y_1 and $y2$ are linear equations in a system. If we graph them, the picture will have two lines and might look like this:



Where is the solution to the system in this picture? Is it on one of the lines? Is it at one of the x- or y-intercepts?

The solution is an ordered pair (i.e. a point) that satisfies y_1 so it must lie somewhere on the blue line. After all, the blue line is the set of **all** the points that satisfy y_1. Similarly, the solution also satisfies y_2, so it must also lie on the red line. Are there any points that lie on both the red line *and* the blue line...?



Yes! The point of intersection lies on both the red line and the blue line; it is a solution!

Graphically, a solution to a system corresponds to a point of intersection of the graphs of the system.

The solution can be written two ways:

- As an ordered pair, such as (2,7)
- By writing the value of each variable, such as $x = 2, \ y = 7$.

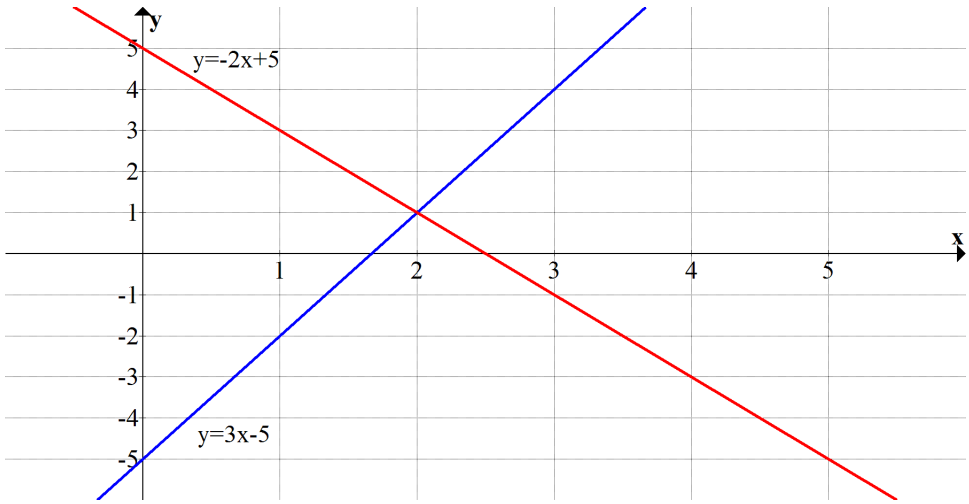
Example B

Find the solution of the system by graphing:

$$\begin{cases} y = 3x - 5 \\ y = -2x + 5 \end{cases}$$

By graphing each equation and finding the point of intersection, you find the solution to the system.

Each equation is written in slope-intercept form and can be graphed accordingly.



The lines appear to intersect at the ordered pair (2,1). Is this the solution to the system?

$$\text{Check (2,1):} \begin{cases} 1 = 3(2) - 5; & 1 = 1 \\ 1 = -2(2) + 5; & 1 = 1 \end{cases}$$

The coordinates check in both sentences. Therefore, (2, 1) is a solution to the system.

The greatest strength of the graphing method is that it offers a very visual representation of a system of equations and its solution. You can see, however, that determining a solution from a graph would require very careful graphing of the lines and is really practical only when you are certain that the solution gives integer values for x and y. In most cases, this method can offer only approximate solutions to systems of equations. For exact solutions, other methods are necessary.

Concept Problem Revisited

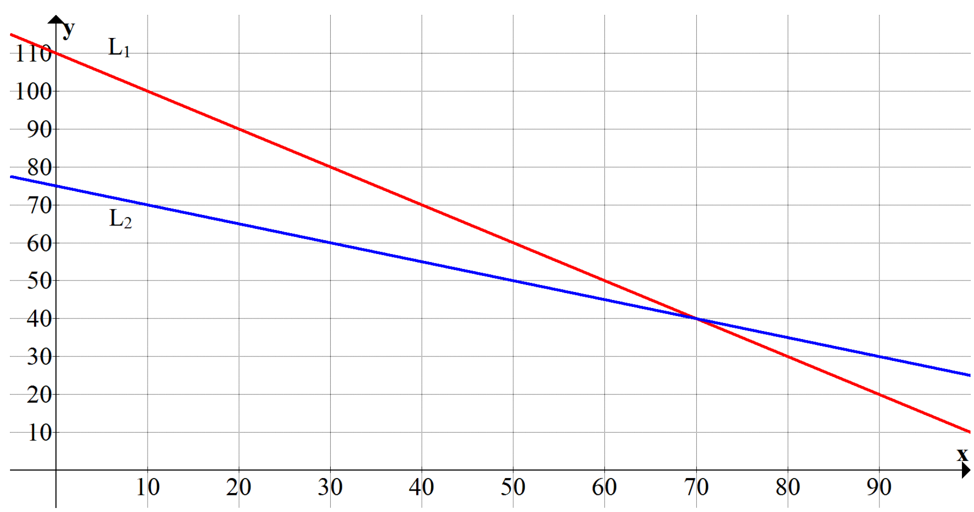
The sales and revenue information led us to this system of equations:

$$\begin{aligned} x + y &= 110 & (L_1) \\ 3x + 6y &= 450 & (L_2) \end{aligned}$$

We can solve this system by graphing, but first we have to get the two equations into slope-intercept form:

$$\begin{aligned} y &= -x + 110 & (L_1) \\ y &= -\frac{1}{2}x + 75 & (L_2) \end{aligned}$$

Now, we make the graph:



It looks like the lines intersect at the point (70,40). Let's check those values in the system:

$$\begin{aligned} 70 + 40 &= 110 & (L_1)\checkmark \\ 3 \cdot (70) + 6 \cdot (40) &= 450 & (L_2)\checkmark \end{aligned}$$

So we conclude that our taco truck sold 70 tacos and 40 burritos on the day in question.

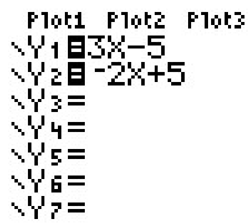
Solving Systems Using a Graphing Calculator

A graphing calculator can be used to find or check solutions to a system of equations. To solve a system graphically, you must graph the two lines on the same coordinate axes and find the point of intersection. You can use a graphing calculator to graph the lines as an alternative to graphing the equations by hand.

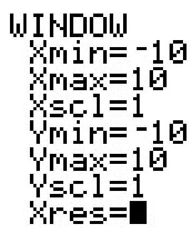
Let's use a graphing calculator to solve the problem:

Using the system from the first example, $\begin{cases} y = 3x - 5 \\ y = -2x + 5 \end{cases}$ use a graphing calculator to find the approximate solutions to the system.

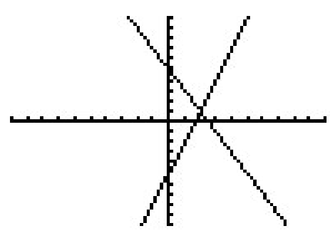
Begin by entering the equations into the $Y =$ menu of the calculator.



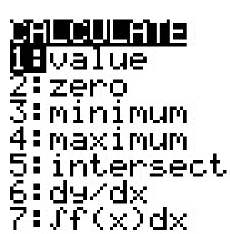
You already know the solution to the system is (2,1). The window needs to be adjusted so an accurate picture is seen. Change your window to the **default window**.



See the graphs by pressing the **GRAPH** button.



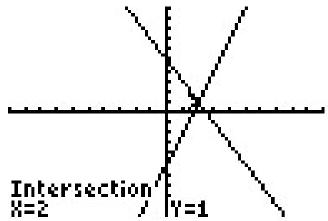
The solution to a system is the intersection of the graphs of the equations. To find the intersection using a graphing calculator, locate the **Calculate** menu by pressing 2^{nd} and **TRACE**. Choose option #5 - **intersect**.



The calculator will ask you **"First Curve?"** Check that the equation of the first line appears at the top of the screen and hit **ENTER**.

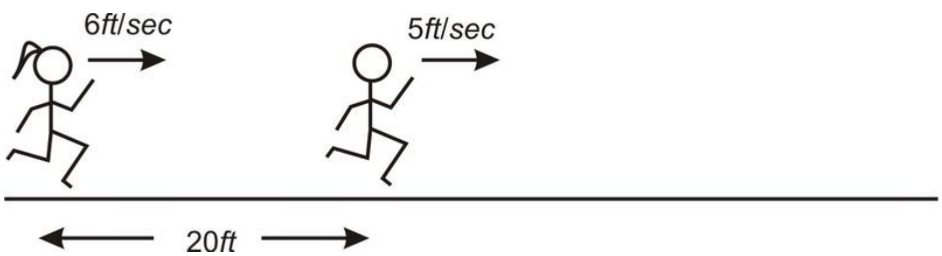
The calculator will automatically jump to the other curve and ask you **"Second Curve?"** Check that the equation of the second line appears at the top of the screen and hit **ENTER**.

Lastly, the calculator will ask, **"Guess?".** The calculator is asking for a starting point that is near the point of intersection. Use the arrow keys to position the cursor near the intersection point (it doesn't have to be right on it) and hit **ENTER**. The intersection will appear at the bottom of the screen.



Example C

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?



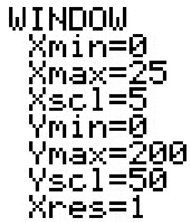
The basic formula in this problem is distance = rate × time. We can modify it to model Peter and Nadia's positions.

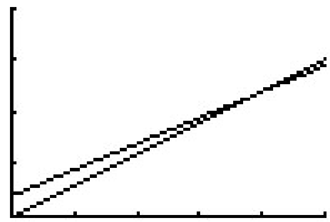
Peter: $d = \underbrace{5t}_{r \cdot t} \underbrace{+20}_{\text{head start}}$

Nadia: $d = \underbrace{6t}_{r \cdot t}$

The question asks when Nadia catches Peter. Nadia runs faster than Peter, so we know it will happen eventually. The solution is the time when the distances are equal, which is at the point of intersection of the two lines. We need to graph the equations and find the intersection, but to do this we first must set the viewing window on the calculator. How do we figure what values to use? There is a bit of an art to this, but here is one way to approach it:

First, notice that the race starts when the time is 0 and when Nadia's distance is 0. This suggests that Xmin=0 (time) and Ymin=0 (distance) would be reasonable choices. Now, what is a reasonable race length? Let's somewhat arbitrarily choose 25 seconds, so Xmax = 25. (If this doesn't work, we can then try a larger value.) What about Ymax? After 25 seconds, Nadia's distance will be $6 \cdot 25 = 150$ feet. If we choose a value larger than this for Ymax it will guarantee that we can see the graph of Nadia's equation, so let's give ourselves some room and choose Ymax=200.





The two lines cross at the coordinate $t = 20, \ d = 120$. This means after 20 seconds Nadia will catch Peter. At this time, they will be at a distance of 120 feet.

Example D

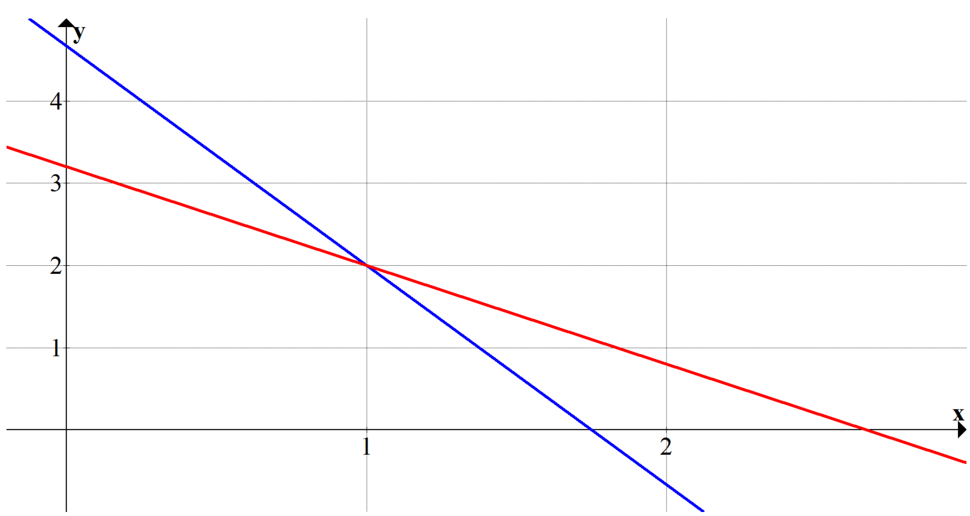
Suppose that Julia and Jason both took a test. The test had multiple choice and fill-in-the-blank questions. Julia answered 8 multiple choice questions and 3 fill-in-the-blank questions correctly while Jason answered 6 multiple choice questions and 5 fill-in-the-blank questions correctly. If Julia got a total of 14 points and Jason got 16 points, how many points is each type of question worth?

Let x be the point value of the multiple choice questions and y be the point value of fill-in-the-blank questions. The system of equations that represents this situation is:

$$\begin{cases} 8x+3y=14 \\ 6x+5y=16 \end{cases}$$

First, put the equations into slope-intercept form: $y = -\frac{8}{3}x + \frac{14}{3} y = -\frac{6}{5}x + \frac{16}{5}$

Next, graph the equations:



The intersection looks like the point (1, 2). Let's check the answer:

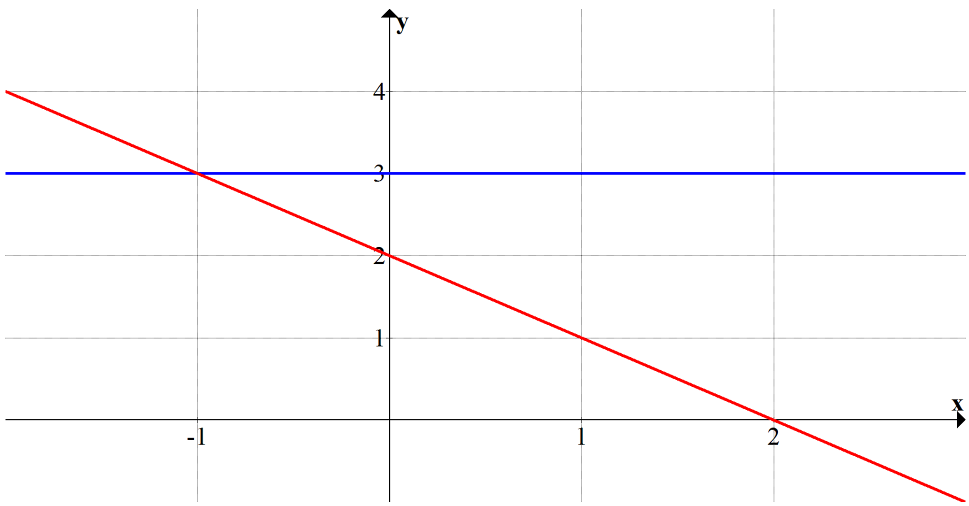
$$\text{Check (1,2):} \begin{cases} 8(1)+3(2)=14; & 8+6=14; 14=14 \\ 6(1)+5(2)=16; & 6+10=16; 16=16 \end{cases}$$

The multiple choice questions are worth 1 point and the fill-in-the-blank questions are worth 2 points.

Example E

Solve the system $\begin{cases} x+y=2 \\ \quad\ y=3 \end{cases}$

The first equation is written in standard form. To graph it, we can convert to slope-intercept form: $y=-x+2$.



The second equation is a horizontal line three units up from the origin.

The lines appear to intersect at (-1, 3).

$$\text{Check (-1,3):} \begin{cases} -1+3=2; & 2=2 \\ & 3=3 \end{cases}$$

The coordinates are a solution to each equation, and so they are a solution to the system.

Practice Problems

1. Define a system of equations.
2. What is the solution to a system of equations?
3. Explain the process of solving a system by graphing.
4. What is one problem with using a graph to solve a system?
5. What are the two main ways to write the solution to a system of equations?
6. Suppose Horatio says the solution to a system is (4, -6). What does this mean visually about the graph of the system?
7. Where is the **"intersect"** command located on your graphing calculator? What does it do?

8. In the Example 1, who is farther from the starting line at 19.99 seconds? At 20.002 seconds?

Determine which of the given ordered pairs satisfies the system of linear equations.

9. $\begin{cases} y = 3x - 2 \\ y = -x \end{cases}$ $(1,4),\ (2,9),\ \left(\frac{1}{2},\ -\frac{1}{2}\right)$
10. $\begin{cases} y = 2x - 3 \\ y = x + 5 \end{cases}$ $(8,13),\ (-7,6),\ (0,4)$
11. $\begin{cases} 2x + y = 8 \\ 5x + 2y = 10 \end{cases}$ $(-9,1),\ (-6,20),\ (14,2)$
12. $\begin{cases} 3x + 2y = 6 \\ y = \frac{x}{2} - 3 \end{cases}$ $\left(3, -\frac{3}{2}\right),\ (-4,3),\ \left(\frac{1}{2},4\right)$

In 13 - 22, solve the following systems by graphing.

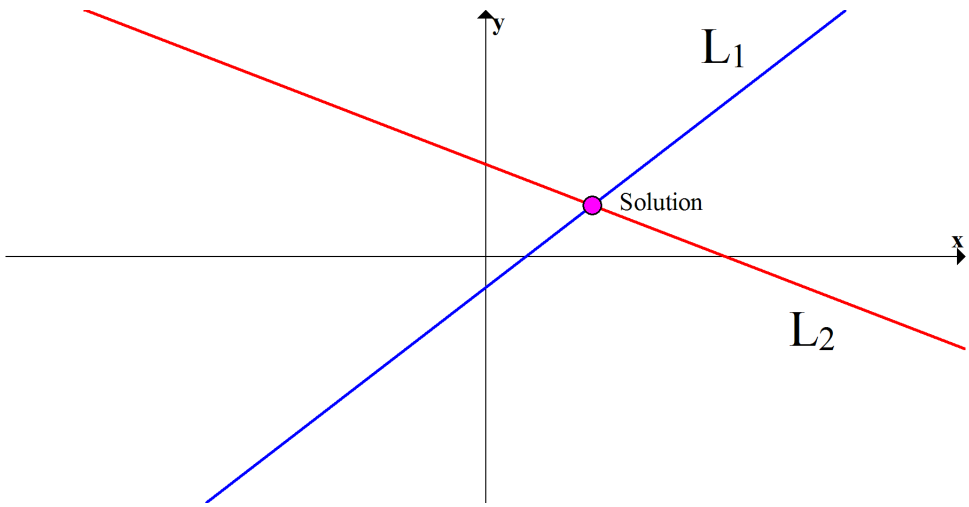
13. $\begin{cases} y = x + 3 \\ y = -x + 3 \end{cases}$
14. $\begin{cases} y = 3x - 6 \\ y = -x + 6 \end{cases}$
15. $\begin{cases} 2x = 4 \\ y = -3 \end{cases}$
16. $\begin{cases} y = \text{-}x + 5 \\ \text{-}x + y = 1 \end{cases}$
17. $\begin{cases} x + 2y = 8 \\ 5x + 2y = 0 \end{cases}$
18. $\begin{cases} 3x + 2y = 12 \\ 4x - y = 5 \end{cases}$
19. $\begin{cases} 5x + 2y = -4 \\ x - y = 2 \end{cases}$
20. $\begin{cases} 2x + 4 = 3y \\ x - 2y + 4 = 0 \end{cases}$
21. $\begin{cases} y = \frac{x}{2} - 3 \\ 2x - 5y = 5 \end{cases}$
22. $\begin{cases} y = 4 \\ x = 8 - 3y \end{cases}$
23. Mary's car is 10 years old and has a problem. The repair man indicates that it will cost her $1200.00 to repair her car. She can purchase a different, more efficient car for $4,500.00. Her present car averages about $2,000.00 per year for gas while the new car would average about $1,500.00 per year. Find the number of years for which the total cost of repairs plus gas would equal the total cost of replacement plus gas.
24. Juan is considering two cell phone plans. The first company charges $120.00 for the phone and $30 per month for the calling plan that Juan wants. The second company charges $40.00 for the same phone, but charges $45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
25. A tortoise and hare decide to race 30 feet. The hare, being much faster, decided to give the tortoise a head start of 20 feet. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long will it be until the hare catches the tortoise?

5.2 Types of Linear Systems

Now that we know what systems of equations are, it is natural to wonder: do all systems of equations have a solution? How many solutions can a system of linear equations have? These questions may sound philosophical, but they have actual answers, as we shall see in this section.

Types of Linear Systems

In the last section, we found that a system of linear equations corresponds to a graph of two lines. The examples looked more or less like this:



This picture represents a 'nice' case, because there is a single solution, and the lines L_1 and L_2 are distinct from one another.

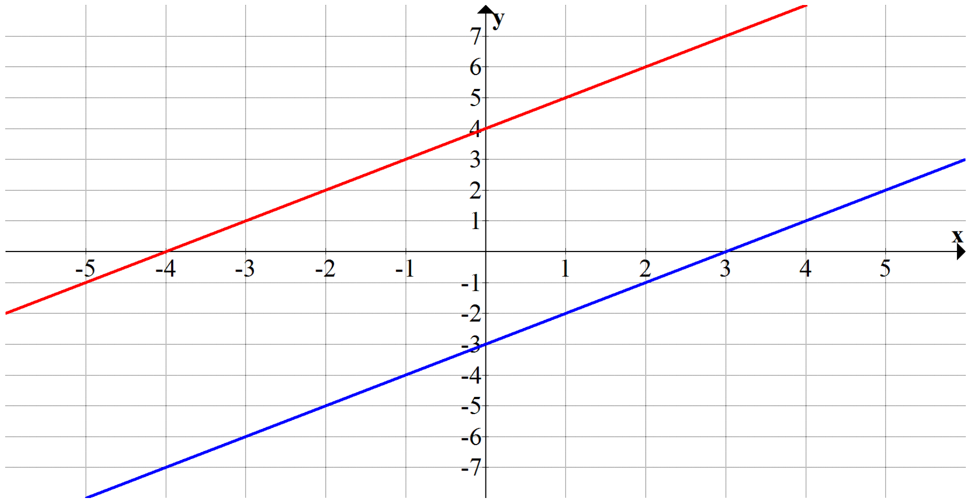
This is not the only possibility, however.

Inconsistent Systems

Here is a system of equations:

$$y = x - 3$$
$$y = x + 4$$

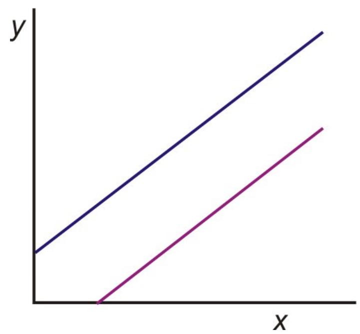
The equations are in slope-intercept form, so we can easily graph them:



Does this system have a solution? A solution would correspond to an intersection point of the lines, but since the lines are parallel, they never intersect. This system has *no solutions*.

A system with no solutions is called **inconsistent**.

Remember that parallel lines have the same slope. When graphed, the lines will run in the same direction but have different $y-$intercepts. Parallel lines will never intersect, so they have no solution.



Example A

$$\begin{cases} 4y = 5 - 3x \\ 6x + 8y = 7 \end{cases}$$

We can get both equations into slope-intercept form and then draw conclusions about their graphs.

$$\begin{cases} 4y = 5 - 3x \\ 6x + 8y = 7 \end{cases} \quad \rightarrow \quad \begin{cases} y = -\frac{3}{4}x + \frac{5}{4} \\ y = -\frac{3}{4}x + \frac{7}{8} \end{cases}$$

Notice that the slopes of the two equations are the same, but their y-intercepts are different. These lines are parallel, therefore the system will have no solution. This is an inconsistent system.

In practice, inconsistent systems arise when you are given two pieces of contradictory information.

Example B

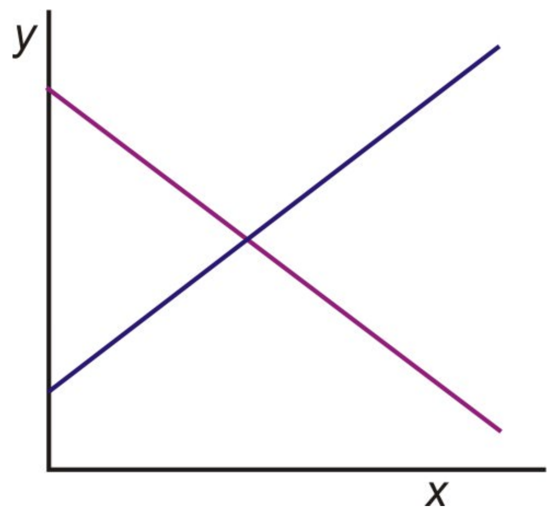
You are the owner of a business that sells xylophones (x) and yams (y). One report indicates a total of 25 sales for the day, while another indicates 35 total sales. It's clear that both cannot be true, and if you translated this information into a system of equations, the system would be inconsistent:

$$\begin{aligned} x+y &= 25 & \rightarrow & & amp; y &= -x+25 \\ x+y &= 35 & \rightarrow & & amp; y &= -x+35 \end{aligned}$$

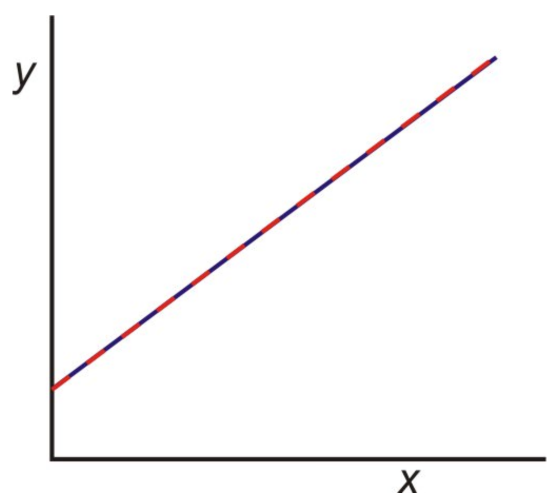
Consistent Systems

A system is **consistent** if it has at least one solution. Graphically, this means there is at least one intersection of the lines. There are two cases for consistent linear systems:

- One intersection - this kind of system is called **consistent and independent**, and is the type of system we see most often.



- Infinitely many intersections



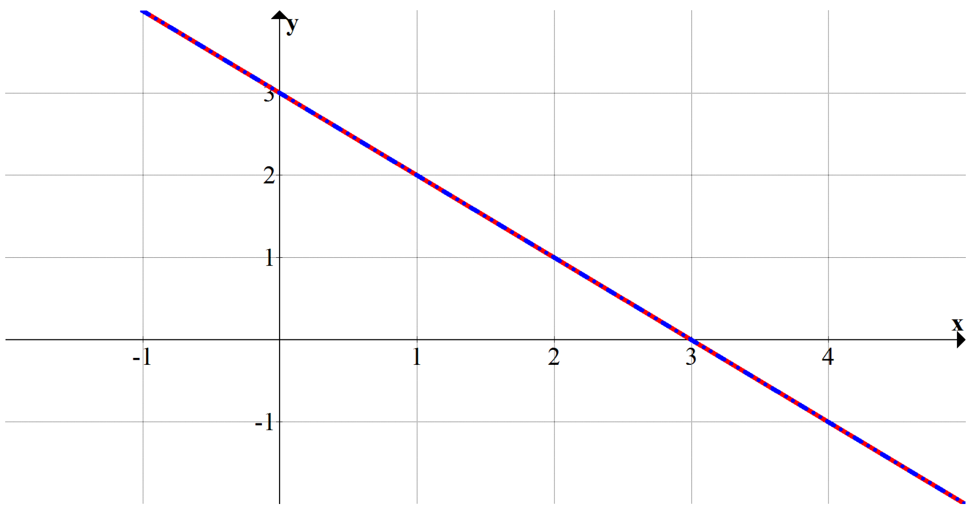
What's going on in this picture? It looks like there is just one line, not two. Let's use a specific example to see.

Example C

Examine the following system: $\begin{cases} x+y=3 \\ 3x+3y=9 \end{cases}$. Let's put both equations into slope-intercept form as we did with Example 1.

$$\begin{aligned} x+y=3 \quad & \rightarrow \quad amp; y=-x+3 \\ 3x+3y=9 \quad & \rightarrow \quad amp; y=-x+3 \end{aligned}$$

That's not a mistake; the equations are identical. What does this mean about their graphs? They lie right on top of each other!



How many times do these lines 'intersect'? An infinite number! Every point on the line is a point of intersection.

Graphs that lie precisely on top of one another are called *coincident*. A system of coincident lines is consistent has an infinite number of solutions. Such a system is called **dependent** (or **consistent-dependent** for clarity)

In practice, consistent-dependent systems appear when you are given two pieces of equivalent information.

Example D

Returning to our xylophone and yam store, the yam salesman may say "We sold 4 times more yams than xylophones today!" while the xylophone salesman may say "We only sold one-fourth the number of xylophones as we did yams today." We can see that these two statements are saying the same thing in two different ways, and they will produce equivalent equations:

$$\begin{aligned} y=4x \quad & \rightarrow \quad amp; y=4x \\ x=\frac{1}{4}y \quad & \rightarrow \quad amp; y=4x \end{aligned}$$

Example E

Determine if the system is consistent, inconsistent, or consistent-dependent

$$\begin{aligned} 3x-2y&=4 \\ 9x-6y&=1 \end{aligned}$$

Fist, put the equations into slope-intercept form:

$$3x - 2y = 4 \quad \rightarrow \quad amp; y = \frac{3}{2}x - 2$$
$$9x - 6y = 1 \quad \rightarrow \quad amp; y = \frac{3}{2}x - \frac{1}{6}$$

We see that these lines have the same slope but different y-intercepts. Therefore:

- These lines are parallel.
- The system has no solution.
- The system is inconsistent.

Summary

There are three possibilities for the solutions to a linear system:

- One solution - Consistent and Independent
- No solutions - Inconsistent
- An infinite number of solutions - Consistent and dependent

Example F

Two movie rental websites are in competition. Blamazon charges an annual membership of \$60 and charges \$3 per movie rental, while Nitflex charges an annual membership of \$40 and charges \$3 per movie rental. After how many movie rentals would Blamazon become the better option?

It should already be clear to see that Blamazon will never become the better option, since its membership is more expensive and it charges the same amount per movie as Nitflex.

Let's see how this works algebraically.

Define the variables: Let $x =$ number of movies rented and $y =$ total rental cost

$$y = 60 + 3x \qquad \text{Blamazon}$$
$$y = 40 + 3x \qquad \text{Nitflex}$$

The lines that describe each option have different $y-$intercepts, namely 60 for Blamazon and 40 for Nitflex. They have the same slope, three dollars per movie. This means that the lines are parallel and the system is inconsistent. The system has no solutions, so there is no number of rentals for which the two websites will have the same cost.

Multimedia

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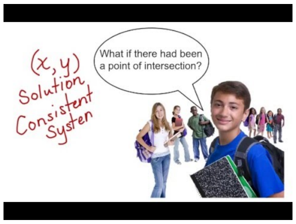

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Examples

Example G

In economics, a line representing supply and a line representing demand are often graphed on the same coordinate plane and that the lines intersect in one point. How many solutions does this system of linear equations have? Is the system consistent, consistent-dependent, or inconsistent?

Since the lines intersect in a single point, the lines have one solution. There is at least one solution so the system is not inconsistent. This means that the system is either consistent or consistent-dependent. It is not consistent-dependent because the lines are not the same and do not have an infinite number of solutions. Therefore, the system is just consistent.

Example H

Determine whether the following system of linear equations has zero, one, or infinitely many solutions:

$$\begin{cases} 2y+6x=20 \\ y=-3x+7 \end{cases}$$

Rewrite the first equation in slope-intercept form.

$$2y+6x=20 \Rightarrow y+3x=10 \Rightarrow y=-3x+10$$

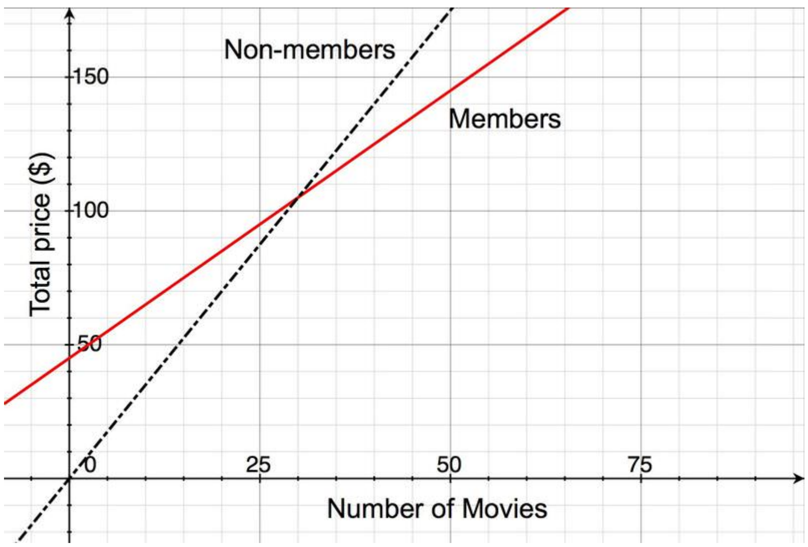
Now the system looks like this:

$$\begin{aligned} y&=-3x+10 \\ y&=-3x+7 \end{aligned}$$

Since the two equations have the same slope, but different y-intercepts, they are parallel lines. It is an inconsistent system, so it has zero solutions.

Practice Problems

1. Define an inconsistent system. What is true about these systems?
2. What are the three types of consistent systems?
3. The graph of a system consists of only one line. What can you conclude?
4. You graph a system and see the lines have an intersection point. What can you conclude?
5. The lines you graphed appear parallel. How can you verify the system will have no solution?
6. You graph a system and obtain the following graph. Is the system consistent or inconsistent? How many solutions does the system have?



In 7 - 24, state whether the system is inconsistent, consistent, or consistent-dependent.

7.
$$3x-4y=13$$
$$y=-3x-7$$

8.
$$4x+y=3$$
$$12x+3y=9$$

9.
$$10x-3y=3$$
$$2x+y=9$$

10.
$$2x-5y=2$$
$$4x+y=5$$

11.
$$\frac{3x}{5}+y=3$$
$$1.2x+2y=6$$

12.

$$3x-4y=13$$
$$y=-3x-7$$

13.

$$3x-3y=3$$
$$x-y=1$$

14.

$$0.5x-y=30$$
$$0.5x-y=-30$$

15.

$$4x-2y=-2$$
$$3x+2y=-12$$

16.

$$3x+2y=4$$
$$-2x+2y=24$$

17.

$$5x-2y=3$$
$$2x-3y=10$$

18.

$$3x-4y=13$$
$$y=-3x-y$$

19.

$$5x-4y=1$$
$$-10x+8y=-30$$

20.

$$4x+5y=0$$
$$3x=6y+4.5$$

21.

$$-2y+4x=8$$
$$y-2x=-4$$

22.

$$x-\frac{y}{2}=\frac{3}{2}$$
$$3x+y=6$$

23.

$$0.05x+0.25y=6$$
$$x+y=24$$

24.

$$x+\frac{2y}{3}=6$$
$$3x+2y=2$$

5.3 Solving Systems Using Substitution

Concept Problem

Suppose that at a bakery, bagels are sold for one price and muffins are sold for another price. 4 bagels and 2 muffins cost \$11, while 3 bagels and 3 muffins cost \$12. How much do individual bagels and muffins cost? We can approach this problem as we have previous problems: create a system and look at its graph. In this case, let $b=$ bagel cost and $m=$ muffin cost. The statements above lead us to these equations:

$$4b+2m=11$$
$$3b+3m=12$$

Solving each equation for m we get:

$$m=-2b+\frac{11}{2}$$
$$m=-b+4$$

Now we can look at the graph:

We can see the solution, but what are its coordinates? Does it happen when b=... 1.5? 1.6? $\sqrt{2}$? We can't eyeball this one easily. We need to solve this using an algebraic method rather than a graphical one.

Substitution Property of Equality

The graphical approach to solving systems is helpful, but it does not always provide exact answers. For that, we will need a symbolic method of solving. One option is the **Substitution Method**, which uses the **Substitution Property of Equality**.

The **Substitution Property of Equality** states that if $y=f(x)$ (an algebraic expression), then $f(x)$ can be substituted for any y in an equation or an inequality.

Example A

Find the solution to the system $\begin{cases} 2x+5y=29 & (1) \\ y=3x-1 & (2) \end{cases}$ using substitution:

Remember, an $=$ symbol is the mathematical way of saying that two things are the same. So equation (2) is saying y 'is the same as' $3x-1$. This means that any place where we have a y we can instead write $3x-1$. How about we do that in equation (1)? We replace (substitute) the y in that equation with the expression $3x-1$.

$$2x+5(3x-1)=29$$

This looks more complicated than it was before, but notice that we now have an equation that only has x in it. We can solve this!

$$\begin{aligned} 2x+5(3x-1)&=29 && \\ 2x+15x-5&=29 && \text{(Distribute 5)} \\ 17x-5&=29 && \text{(Combine like terms)} \\ 17x&=34 && \text{(Add 5 to both sides)} \\ x&=2 && \text{(Divide both sides by 17)} \end{aligned}$$

Are we done yet? No. Remember that the solution to a system is a point; it has *two* coordinates. What we have is the first half of the solution. To find the other half, we can take the value we just found and plug it back in somewhere to find y. Does it matter where? Not really, as long as the equation has both x and y in it. For this problem, let's plug $x=2$ into equation (2):

$$\begin{aligned} y&=3(2)-1 \\ y&=5 \end{aligned}$$

The solution to the system is $x=2,\ y=5$ or the ordered pair (2, 5).

Steps for Solving with Substitution

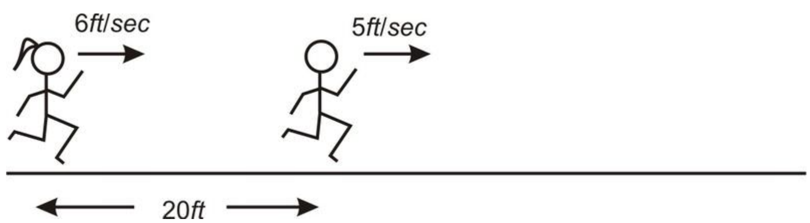
Here are the general steps you can follow to solve using substitution:

1. Solve one equation for one of the variables.
   a. Substitution is particularly useful when one equation in the system is of the form $y=$ algebraic expression or $x=$ algebraic expression since this step will already be done.
2. Substitute the expression you obtained for the variable into the *other* equation.
   a. It is important that you plug your expression into the other equation, not the same one you used in step 1.
3. Solve the resulting equation.
   a. After step 2 you should have an equation with only one variable in it. Solve for this variable and you will have the first half of the solution.
4. Plug the answer from step 3 in and solve for the other variable
   a. To find the second half of the solution, plug the result from step 3 into one of the equations, and you will be able to solve for the other variable.

Example B

Let's revisit a problem we saw in section 4.1. We solved it graphically in that section, but now we can solve it algebraically.

1. Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?



The two racers' information can be translated into two equations.

$$\text{Peter: } d = 5t + 20 \qquad (1)$$
$$\text{Nadia: } d = 6t \qquad (2)$$

We want to know when the two racers will be the same distance from the start. Notice that d is already isolated in both equations, so we can skip to step 2 and plug in the equation (1) expression in for d in equation (2).

$$d = 6t$$
$$\downarrow$$
$$5t + 20 = 6t$$

Now solve for t.

$$5t - 5t + 20 = 6t - 5t$$
$$20 = t$$

After 20 seconds, Nadia will catch Peter.

Now we need to determine how far from the start the two runners are. We already know $t = 20$, so we will substitute to determine the distance. Using either equation, substitute the known value for t and find d.

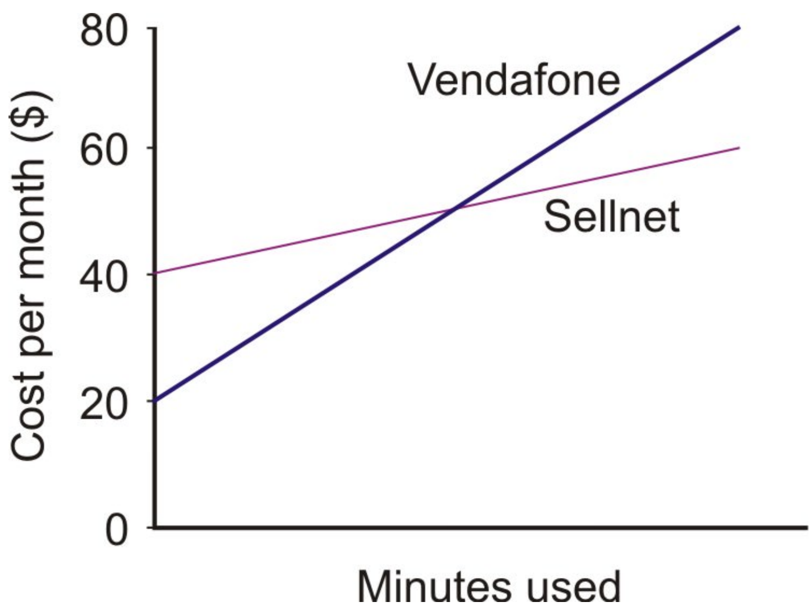
$$d = 5(20) + 20 \rightarrow 120$$

When Nadia catches Peter, the runners are 120 feet from the starting line.

2. Anne is trying to choose between two phone plans. Vendaphone's plan costs \$20 per month, with calls costing an additional 24 cents per minute. Sellnet's plan charges \$40 per month, but calls cost only 8 cents per minute. Which should she choose?

Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations (one for each company's plan) for the cost in dollars in terms of the minutes used. Since the number of minutes is the independent variable, it will be our x. Cost is dependent on minutes. The cost per month is the dependent variable and will be assigned y.

For Vendafone	$y = 0.24x + 20$	(1)
For Sellnet	$y = 0.08x + 40$	(2)



By graphing two equations, we can see that at some point the two plans will charge the same amount, represented by the intersection of the two lines. Before this point, Sellnet's plan is more expensive. After the intersection, Sellnet's plan is cheaper.

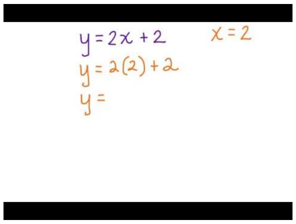
Let's use substitution to find the point that the two plans are the same: Plug the expression for y in equation (1) into equation (2) to obtain:

$$\underbrace{.24x + 20}_{y} = .08x + 40$$

Now solve for x:

$$\begin{aligned} 0.25x + 20 &= 0.08x + 40 && \text{Subtract 20 from both sides.} \\ 0.25x &= 0.08x + 20 && \text{Subtract } 0.08x \text{ from both sides.} \\ 0.16x &= 20 && \text{Divide both sides by 0.16.} \\ x &= 125 \text{ minutes} \end{aligned}$$

We can now use our sketch, plus this information, to provide an answer. If Anne will use 125 minutes or fewer every month, she should choose Vendafone. If she plans on using 126 or more minutes, she should choose Sellnet.



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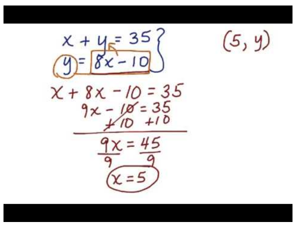

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In the beginning of this section, we were told that 4 bagels and 2 muffins cost \$11 and 3 bagels and 3 muffins cost \$12. If bagels and muffins cost different amounts, how much do individual bagels and muffins cost? From this information we produced the following system:

$$\begin{aligned} 4b+2m &= 11 && (1)\\ 3b+3m &= 12 && (2) \end{aligned}$$

To solve this with substitution, we need to solve one of the equations for one of the variables. Let's solve for the b in equation (2).

$$\begin{aligned} 3b+3m &= 12\\ 3b &= 12-3m\\ b &= 4-m \end{aligned}$$

Now, we can substitute the expression in for b into equation (1).

$$\begin{aligned} 4b+2m &= 11\\ 4(4-m)+2m &= 11\\ 16-4m+2m &= 11\\ 16-2m &= 11\\ -2m &= -5\\ m &= 2.5 \end{aligned}$$

Finally, we need to substitute the value for m into the equation that is solved for b.

$$\begin{aligned}b &= 4 - m\\ b &= 4 - 2.5\\ b &= 1.5\end{aligned}$$

The solution to the system is $(1.5, 2.5)$, so a bagel costs $1.50 and a muffin costs $2.50.

Example C

Solve the system $\begin{cases} x + y = 2 \\ y = 3 \end{cases}$

The second equation is solved for the variable y. Therefore, we can substitute the value "3" for any y in the system.

$$x + y = 2 \rightarrow x + 3 = 2$$

Now solve the equation for x :

$$\begin{aligned}x + 3 - 3 &= 2 - 3\\ x &= -1\end{aligned}$$

The x-coordinate of the intersection of these two equations is -1. Since we were given the y-value of 3 in the question, we have our answer: $x = -1,\ y = 3$.

The solution to the system is (-1, 3), which you can verify by graphing the two lines.

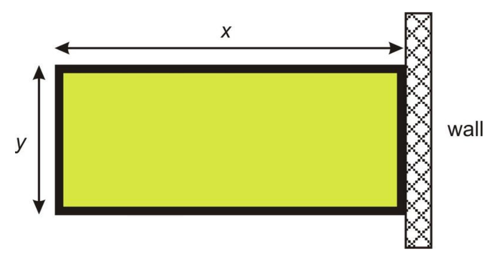
Practice Problems

1. Explain the process of solving a system using the Substitution Property.
2. Which systems are easier to solve using substitution?

Solve the following systems. Remember to find the value for both variables!

3. $\begin{cases} y = \text{-}3 \\ 6x - 2y = 0 \end{cases}$
4. $\begin{cases} \text{-}3 - 3y = 6 \\ 3x + y = 4 \end{cases}$
5. $\begin{cases} y = 3x + 16 \\ x - y = -8 \end{cases}$
6. $\begin{cases} y + 3 = -6x \\ y = 3 \end{cases}$

7. $\begin{cases} 2x+y=5 \\ y=-1-8x \end{cases}$
8. $\begin{cases} y=6+x \\ y=-2x-15 \end{cases}$
9. $\begin{cases} y=-2 \\ y=5x-17 \end{cases}$
10. $\begin{cases} x+y=5 \\ 3x+y=15 \end{cases}$
11. $\begin{cases} 12y-3x=-1 \\ x-4y=1 \end{cases}$
12. $\begin{cases} x+2y=9 \\ 3x+5y=20 \end{cases}$
13. $\begin{cases} x-3y=10 \\ 2x+y=13 \end{cases}$
14. Solve the system $\begin{cases} y=\frac{1}{4}x-14 \\ y=\frac{19}{8}x+7 \end{cases}$ by both graphing and substitution. Which method do you prefer? Why?
15. Admission to Tarantula Ranch is $7 for adults and $4 for children. One afternoon, the Ranch sells 200 tickets and earns $1160 in revenue. How many of each type of ticket did they sell?
16. The sum of two numbers is 70. They differ by 11. What are the numbers?
17. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



18. A steam boat operating on the Mississippi River can travel upstream 18 miles upstream in 4 hours. It can make the return trip downstream in 2.4 hours. What would the speed of the boat be in still water? What is the speed of the river current?

5.4 Solving Systems Using Elimination

Concept Problem

Suppose two families went to an amusement park, where adult tickets and children's tickets have different prices. If the first family had 2 adults and 3 children and paid a total of $48, and the second family had 6 adults and 12 children and paid $168, how much does each type of ticket cost?

This problem is similar to one we saw in the last section. First, we need to define our variables: $a =$ # of adults $c =$ # of children. Using these we can construct two equations:

$$\underbrace{2a}_{\text{2 adult revenue}} + \underbrace{3c}_{\text{3 child revenue}} = \underbrace{48}_{\text{1st family revenue}}$$

$$\underbrace{6a}_{\text{6 adult revenue}} + \underbrace{12c}_{\text{12 child revenue}} = \underbrace{168}_{\text{2nd family revenue}}$$

We could solve this system using substitution; however, there is not a variable that would be obviously easy to isolate, and the fractions may get unwieldy. Is there another algebraic method we might use?

Solving with Elimination

In the last section, we saw that solving a system algebraically will give you the most accurate answer and in some cases, it is easier than graphing. However, we found some problems that took some work to rewrite one equation before you could use the Substitution Property. There is another method used to solve systems algebraically: the **Elimination Method**.

In the Elimination method, the goal is to cancel, or eliminate, a variable by either adding or subtracting the two equations. While the Substitution Method works with one equation at a time (first solve one equation, then plug into the other), Elimination works with both equations simultaneously. This method works well if both equations are in standard form.

Example A

Solve the system:

$$\begin{aligned} 3x+2y &= 11 \qquad (1) \\ 5x-2y &= 13 \qquad (2) \end{aligned}$$

We could solve this with substitution, but these equations would require a significant amount of work to rewrite in slope-intercept form. Is there anything else we could do?

Looking at the $y-$variable, you can see the coefficients are 2 and -2. By adding these together, you get zero. Add these two equations and see what happens.

$$\begin{array}{rr} & 3x+2y=11 \quad (1) \\ + & (5x-2y)=13 \quad (2) \\ \hline & 8x+0=24 \quad (3) \end{array}$$

The y's are gone! The resulting equation (3) is $8x = 24$. Solving for x, you get $x = 3$. To find the $y-$coordinate, choose either equation, and substitute the number 3 for the variable x. In this case we will plug into equation (1):

$$\begin{aligned} 3(3)+2y &= 11 \\ 9+2y &= 11 \\ 2y &= 2 \\ y &= 1 \end{aligned}$$

The point of intersection of these two equations is (3, 1).

Example B

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, calculate, in miles per hour, the speed of the river and the speed Andrew would travel in calm water.

We have two unknowns to solve for, so we will call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew's speed is boosted by the river current, so his total speed is the canoe speed plus the speed of the river $(x+y)$. Upstream, his speed is hindered by the speed of the river. His speed upstream is $(x-y)$.

$$\begin{array}{rl} \text{Downstream Equation} & x+y=7 \\ \text{Upstream Equation} & x-y=1.5 \end{array}$$

Notice y and $-y$ are additive opposites. If you add them together, their sum equals zero. Therefore, by adding the two equations together, the variable y will cancel, leaving you to solve for the remaining variable, x.

$$\begin{array}{r} x+y=7 \\ +\ (x-y)=1.5 \\ \hline 2x+0y=8.5 \\ 2x=8.5 \end{array}$$

Therefore, $x = 4.25$; Andrew is paddling 4.25 $miles/hour$. To find the speed of the river, substitute your known value into either equation and solve.

$$\begin{aligned} 4.25-y &= 1.5 \\ -y &= -2.75 \\ y &= 2.75 \end{aligned}$$

The stream's current is moving at a rate of 2.75 $miles/hour$.

Steps for Solving with Elimination

The two examples above are 'nice' cases, because the equations were already set up for the elimination step. Unfortunately, this is not always the case. Here are steps you can follow to solve with elimination:

1. Manipulate the equations so that one variable has opposite coefficients
   a. The main tool we have for this is multiplication. We can multiply both sides of an equation by a constant to change its coefficients
   b. 'Opposite' in this case means opposite signs. For instance, if you had $10x$ in one equation you would want $-10x$ in the other
   c. Usually you will only have to multiply one equation, but sometimes you will need to multiply both equations to get what you want
2. Add the two equations
   a. The this step, the variable with opposite coefficients should cancel off, leaving you with an equation with just one variable
3. Solve the resulting equation
   a. This gives you the first half of your solution
4. Plug the value you just found to find the value of the other variable
   a. This is the same as the last step of the substitution method

Example C

Solve the system:

$$2x-3y=-11 \qquad (1)$$
$$4x+5y=55 \qquad (2)$$

1. Neither x nor y has opposite coefficients, so we must multiply to make it happen. Notice the coefficients on x: The 4 in equation (2) is a multiple of the 2 in equation (1). We can change the $2x$ into $-4x$ by multiplying equation (1) by -2:

$$-2\cdot(2x-3y)=(-11)\cdot -2 \qquad \text{Multiply on both sides!}$$
$$-4x+6y=22$$

2. Now that we have $-4x$ on equation (1) and $4x$ on equation (2), we can add them together:

$$\begin{array}{rr} -4x+6y=22 & (1) \\ +\quad 4x+5y=55 & (2) \\ \hline 0+11y=77 & (3) \end{array}$$

3. We just created equation (3) that we can solve for y:

$$\begin{aligned} 11y &= 77 \\ y &= 7 \end{aligned}$$

4. Now that we have the first half of our solution, let's plug it into equation (2) and solve for x:

$$\begin{aligned} 2x-3(7) &= -11 \\ 2x-21 &= -11 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

So the solution is (5,7).

Concept Problem Revisited

One family going to an amusement park bought 2 adult tickets and 3 children's tickets and paid $48 while another family bought 6 adult tickets and 12 children's tickets and paid $168. How much does each type of ticket cost?

Let a represent the cost of adult tickets and c represent the cost of children's tickets. The system of equations that represents this situation is:

$$\begin{aligned} 2a+3c &= 48(1) \\ 6a+12c &= 168(2) \end{aligned}$$

We can change the $2a$ in equation (1) into $-6a$ by multiplying equation (1) by -3:

$$\begin{aligned} -3 \cdot (2a+3c) &= (24) \cdot -3 \\ -6a-9c &= -72 \end{aligned}$$

Now, add the equations (1) and (2) and solve for c:

$$\begin{aligned} -6a-9c &= -144 && (1) \\ +\quad 6a+12c &= 168 && (2) \\ \hline 0+3c &= 24 && (3) \\ 3c &= 24 \\ c &= 8 \end{aligned}$$

Now, substitute the value for c in to solve for a:

$$\begin{aligned} 5a+8(8) &= 124 \\ 5a+64 &= 124 \\ 5a &= 60 \\ a &= 12 \end{aligned}$$

The adult tickets cost $12 and the children's tickets cost $8.

Example D

Solve the system, and express your answer as a point in the form (s,t).

$$\begin{aligned} 10s - 3t &= 34 && (1) \\ 4s + 5t &= -5 && (2) \end{aligned}$$

In this case, we won't be able to get away with only multiplying one of the equations, since 10 is not a multiple of 4 and 5 is not a multiple of 3; we will need to multiply both equations. Looking at the s coefficients 10 and 4, we can see that 20 is a common multiple of both. We need to change the $10s$ and $4s$ terms into $20s$ and $-20s$. We can do this by multiplying equation (1) by 2 and equation (2) by -5:

$$\begin{aligned} 2 \cdot (10s - 3t) &= (34) \cdot 2 && (1) \\ 20s - 6t &= 68 && (1) \\ \\ -5 \cdot (4s + 5t) &= (-5) \cdot -5 && (2) \\ -20s - 25t &= 25 && (2) \end{aligned}$$

Now that we have opposite coefficients, we can add the equations together:

$$\begin{aligned} & 20s - 6t = 68 && (1) \\ + \quad & -20s - 25t = 25 && (2) \\ \hline & 0 - 31t = 93 && (3) \end{aligned}$$

Next, solve the new equation (3) for t:

$$\begin{aligned} -31t &= 93 \\ t &= -3 \end{aligned}$$

Last, plug $t = -3$ into equation (1) and solve for s:

$$\begin{aligned} 10s - 3(-3) &= 34 \\ 10s + 9 &= 34 \\ 10s &= 25 \\ s &= \tfrac{25}{10} = \tfrac{5}{2} \end{aligned}$$

So the solution is $\left(\frac{5}{2}, -3\right)$.

Practice Problems

1. What is the purpose of the elimination method to solve a system? When is this method appropriate?

In 2 - 10, solve each system using elimination.

2.

$$\begin{cases} 2x+y=-17 \\ 8x-3y=-19 \end{cases}$$

3.

$$\begin{cases} x+4y=-9 \\ -2x-5y=12 \end{cases}$$

4.

$$\begin{cases} -2x-5y=-10 \\ x+4y=8 \end{cases}$$

5.

$$\begin{cases} x-3y=-10 \\ -8x+5y=-15 \end{cases}$$

6.

$$\begin{cases} -x-6y=-18 \\ x-6y=-6 \end{cases}$$

7.

$$\begin{cases} 5x-3y=-14 \\ x-3y=2 \end{cases}$$

8.

$$\begin{aligned} 3x+4y&=2.5 \\ 5x-4y&=25.5 \end{aligned}$$

9.

$$\begin{aligned} 5x+7y&=-31 \\ 5x-9y&=17 \end{aligned}$$

10.

$$3y - 4x = -33$$
$$5x - 3y = 40.5$$

11. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for \$2.84. Peter also buys three candy bars, but he can afford only one fruit roll-up. His purchase costs \$1.79. What is the cost of each candy bar and each fruit roll-up?
12. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane), and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another identical plane moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
13. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs \$14.29 and a 17-mile journey costs \$19.91, calculate:
    a. the pick-up fee
    b. the per-mile rate
    c. the cost of a seven-mile trip
14. Calls from a call-box are charged per minute at one rate for the first five minutes, and then at a different rate for each additional minute. If a seven-minute call costs \$4.25 and a 12-minute call costs \$5.50, find each rate.
15. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns \$35 per hour, and the builder earns \$28 per hour. Together they were paid \$330.75, but the plumber earned \$106.75 more than the builder. How many hours did each work?
16. Paul has a part-time job selling computers at a local electronics store. He earns a fixed hourly wage, but he can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned \$220. In his second week, he managed to sell 13 warranties and earned \$280. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

Mixed Review

17. Baxter the golden retriever is standing in the sun and casts a shadow of 3 feet. The doghouse he is next to is 3 feet tall and casts an 8-foot shadow. What is Baxter's height?
18. A botanist watched the growth of a lily. At 3 weeks, the lily was 4 inches tall. Four weeks later, the lily was 21 inches tall. Assuming this relationship is linear:
    a. Write an equation to show the growth pattern of this plant.
    b. How tall was the lily at the 5.5-week mark?
    c. Is there a restriction on how high the plant will grow? Does your equation show this?

5.5 Applications of Systems

Concept Problem

What if you had two types of grape drink, one with 5% real fruit juice and another with 10% real fruit juice? Suppose you wanted a gallon of grape drink with 6% real fruit juice. How much of the 5% drink and how much of the 10% drink should you mix together to produce it?

Summary of Methods of Solving Systems

The previous sections have provided three methods to solve systems: graphing, substitution, and elimination. Each of these methods have strengths and weaknesses. Below is a summary.

Graphing

✓ A good technique to visualize the equations and when both equations are in slope-intercept form.

✓ Solving a system by graphing is often imprecise and will not provide exact solutions.

Substitution

✓ Works well when an equation is given in slope-intercept form or can easily be put into slope-intercept form.

✓ Gives exact answers.

✓ Can be difficult to use substitution when both equations are in standard form.

Elimination by Addition or Subtraction

✓ Works well when both equations are in standard form

✓ Gives exact answers.

✓ Can be difficult to use if one equation is in standard form and the other is in slope-intercept form.

Applications

Systems of equations arise in chemistry when mixing chemicals in solutions and can even be seen in things like mixing nuts and raisins or examining the change in your pocket!

Here are several examples of the kinds of problems in the real world that can be solved with systems.

Concept Problem Revisited

You have two types of grape drink, one with 5% real fruit juice and one with 10% real fruit juice. Suppose you wanted a gallon of a grape drink with 6% real fruit juice. How much of the 5% drink and how much of the 10% drink should you mix together to produce it?

Let x be the amount of the 5% juice and y be the amount of 10% juice. We want one gallon of the grape drink so one of our equations is $x+y=1$. Our second equation is $.05x+.10y=.06$ since we have x gallons of the 5% juice, y gallons of the 10% juice, and we want 1 gallon of the 6% juice (since 6% of 1 is $.06\cdot(1)=.06$).

The system is:

$$\begin{cases} x+y=1 \\ .05x+.1y=.06 \end{cases}$$

We can isolate x in the first equation, then use substitution to solve: $x=1-y$

$$\begin{aligned} .05(1-y)+.1y &= .06 \\ .05-.05y+.1y &= .06 \\ .05+.05y &= .06 \\ .05y &= .01 \\ y &= .2 \end{aligned}$$

Now, substitute y in to the equation to solve for x.

$$\begin{aligned} x &= 1-y \\ x &= 1-.2 \\ x &= .8 \end{aligned}$$

Our solution is $(.8, .2)$. Let's check it: $.05\cdot(.8)+.10\cdot(.2)=.06\checkmark$

You need .8 gallons of the 5% real fruit juice drink and .2 gallons of the 10% real fruit juice drink to get one gallon of a 6% real fruit juice drink.

Example A

Nadia empties her purse and finds that it contains only nickels and dimes. If she has a total of 7 coins and they have a combined value of 55 cents, how many of each coin does she have?

Begin by choosing appropriate variables for the unknown quantities. Let $n=$ the number of nickels and $d=$ the number of dimes.

There are seven coins in Nadia's purse: $n+d=7$.

The total is 55 cents: $0.05n+0.10d=0.55$.

The system is:

$$\begin{cases} n+d=7 \\ 0.05n+0.10d=0.55 \end{cases}$$

We can quickly rearrange the first equation to isolate d, the number of dimes: $d=7-n$.

Using the Substitution Property, every d can be replaced with the expression $7-n$.

$$\begin{aligned} & 0.05n+0.10(7-n)=0.55 & \\ \text{Now solve for } n: \quad & 0.05n+0.70-0.10n=0.55 & \text{Distributive Property} \\ & -0.05n+0.70=0.55 & \text{Add like terms.} \\ & -0.05n=-0.15 & \text{Subtract 0.70.} \\ & n=3 & \text{Divide by } -0.05. \end{aligned}$$

Nadia has 3 nickels. There are seven coins in the purse; three are nickels, so four must be dimes.Check to make sure this combination is 55 cents: $0.05(3)+0.10(4)=0.15+0.40=0.55$.

Example B

A chemist has two containers, Mixture A and Mixture B. Mixture A has a 60% copper sulfate concentration. Mixture B has a 5% copper sulfate concentration. The chemist needs to have a mixture equaling 500 mL with a 15% concentration. How much of each mixture does the chemist need?

Although not explicitly stated, there are two equations involved in this situation.

- Begin by stating the variables. Let

$$A = \text{\# mL from mix A}$$
$$B = \text{\# mL from mix B}$$

- The total mixture needs to have 500 mL of liquid.

$$A+B=500 \qquad (1)$$

- The total amount of copper sulfate needs to be 15% of the total amount of solution (500 mL). 15% of 500 mL is $0.15 \cdot 500 = 75$ mL

  Since 60% of A and 5% of B is copper sulfate we can create another equation:

$$\underbrace{0.60A}_{\text{sulfate from A}} + \underbrace{0.05B}_{\text{sulfate from B}} = \underbrace{75}_{\text{total sulfate needed}} \qquad (2)$$

  Equations (1) and (2) together are the system we need to solve:

$$\begin{aligned} A+B&=500 && (1)\\ 0.60A+0.05B&=75 && (2) \end{aligned}$$

  By rewriting equation (1), the Substitution Property can be used: $A=500-B$.

  Substitute the expression $500-B$ for the variable A in the second equation.

$$0.60(500 - B) + 0.05B = 75$$

Solve for B.

$$\begin{aligned}
300 - 0.60B + 0.05B &= 75 && \text{Distributive Property}\\
300 - 0.55B &= 75 && \text{Add like terms.}\\
-0.55B &= -225 && \text{Subtract 300.}\\
B = \frac{-225}{-.55} &= \frac{4500}{11}\ \text{mL}
\end{aligned}$$

To find the amount of mixture A use the first equation: $A + \frac{4500}{11} = 500$

$$A = \frac{1000}{11}\ \text{mL}$$

Our solution is $\left(\frac{1000}{11}, \frac{4500}{11}\right)$. Let's check it: $.60 \cdot \frac{1000}{11} + .05 \cdot \frac{4500}{11} = 75 \checkmark$

The chemist needs $\frac{1000}{11} \approx 91$ mL of mixture A and $\frac{4500}{11} \approx 409$ mL of mixture B to get a 500 mL solution with a 15% copper sulfate concentration.

Example C

A coffee company makes a product which is a mixture of two coffees, using a coffee that costs \$10.20 per pound and another coffee that costs \$6.80 per pound. In order to make 20 pounds of a mixture that costs \$8.50 per pound, how much of each type of coffee should it use?

Let m be the amount of the \$10.20 coffee, and let n be the amount needed of the \$6.80 coffee. Since we want 20 pounds of coffee that costs \$8.50 per pound, the total cost for all 20 pounds is $20 \cdot \$8.50 = \$170$. The cost for the 20 pounds of mixture is equal to the cost of each type of coffee added together: $10.20 \cdot m + 6.8 \cdot n = 170$.

Also, the amount of each type of coffee added together equals 20 pounds: $m + n = 20$.

The system is:

$$\begin{aligned}
m + n &= 20 && (1)\\
10.20 \cdot m + 6.8 \cdot n &= 170 && (2)
\end{aligned}$$

We can isolate the m in equation (1), then use substitution to solve: $m = 20 - n$

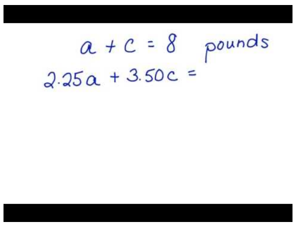
$$\begin{aligned}
& 10.20(20 - n) + 6.8n = 170 &&\\
\text{Now solve for } n: \quad & 204 - 10.20n + 6.8n = 0170 && \text{Distributive Property}\\
& 204 - 3.4n = 170 && \text{Add like terms.}\\
& -3.4n = -34 && \text{Subtract 204.}\\
& n = 10 && \text{Divide by } -3.4.
\end{aligned}$$

Since $n = 10$, we can plug that into $m + n = 20$.

$$m + 10 = 20 \Rightarrow m = 10$$

Our solution is $(10, 10)$. Let's check it: $10.2 \cdot 10 + 6.8 \cdot 10 = 170\checkmark$

The coffee company needs to use 10 pounds of each type of coffee in order to have a 20 pound mixture that costs $8.50 per pound.



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A light green latex paint that is 20% yellow paint is combined with a darker green latex paint that is 45% yellow paint. How many gallons of each paint must be used to create 15 gallons of a green paint that is 25% yellow paint?

Let x be the number of gallons of the 20% yellow paint and let y be the number of gallons of the 40% yellow paint. This means that we want those two numbers to add up to 15:

$$x + y = 15$$

Now if we want 15 gallons of 25% yellow paint, that means we want $0.25 \cdot 15 = 3.75$ gallons of pure yellow pigment. The expression $0.20 \cdot x$ represents the amount of pure yellow pigment in the x gallons of 20% yellow paint. The expression $0.45 \cdot y$ represents the amount of pure yellow pigment in the y gallons of 45% yellow paint. Combing the last two adds up to the 3.75 gallons of pure pigment in the final mixture:

$$0.20x + 0.40y = 3.75$$

The system is:

$$x + y = 15 \quad (1)$$
$$0.20x + 0.45y = 3.75 \quad (2)$$

We can isolate one variable and use substitution to solve the system: $x = 15 - y$

$$
\begin{aligned}
& 0.20(15-y)+0.45y = 3.75 & \\
\text{Now solve for } x: \quad & 3-0.20y+0.45y = 3.75 & \text{Distributive Property} \\
& 3+0.2y = 3.75 & \text{Add like terms.} \\
& 0.25y = 0.75 & \text{Subtract 3.} \\
& y = 3 & \text{Divide by 0.25.}
\end{aligned}
$$

Now we can plug in $y = 3$ into equation (1): $x + y = 15 \Rightarrow x + 3 = 15 \Rightarrow x = 12$.

This means 12 gallons of 20% yellow paint should be mixed with 3 gallons of 45% yellow paint in order to get 15 gallons of 25% yellow paint.

Example E

A cyclist takes a 32-mile round trip ride from Gainesville to Hawthorne and back. On the first leg, he rides against the wind, and the trip takes 2 hours. Coming back, he rides with with wind, and the trip takes him 1.25 hours. What is his cycling speed without the wind? What is the speed of the wind?

First, define variables to represent the quantities we are looking for:

$$x = \text{ cyclist speed}$$
$$y = \text{ wind speed}$$

We are given information about distance, rate, and time, so we can use the $d = r \cdot t$ formula to help us. In this case, there are two separate trips, so we will create two separate equations.

It is pretty clear what will go in for d and t, but what about r? When riding against the wind, the cyclist is being slowed down, so his overall speed (x) is reduced by the speed of the wind (y). When riding with the wind, on the other hand, the cyclist is being sped up, so his overall speed (x) is increased by the speed of the wind (y). Now, we create our equations:

$$\text{1st trip: } \underbrace{16}_{d} = \underbrace{(x-y)}_{r} \cdot \underbrace{2}_{t}$$

$$\text{2nd trip: } \underbrace{16}_{d} = \underbrace{(x+y)}_{r} \cdot \underbrace{1.25}_{t}$$

These equations give us a system to solve:

$$16 = (x-y) \cdot 2 \quad (1)$$
$$16 = (x+y) \cdot 1.25 \quad (2)$$

We could distribute the 2 and 1.25 and then solve, but in this case it will set us up nicely for the elimination method if we divide equation (1) by 2 and equation (2) by 1.25:

$$\frac{16}{2} = \frac{(x-y)\cdot \cancel{2}}{\cancel{2}} \quad \rightarrow \quad 8 = x - y$$

$$\frac{16}{1.25} = \frac{(x+y)\cdot \cancel{1.25}}{\cancel{1.25}} \quad \rightarrow \quad 12.8 = x + y$$

We have opposite coefficients on y so we can add the equations together to cancel it off:

$$\begin{aligned} 8 &= x - y \\ + \quad 12.8 &= x + y \\ \hline 20.8 &= 2x + 0 \end{aligned}$$

Now solve for x:

$$\begin{aligned} 2x &= 20.8 \\ x &= 10.4 \end{aligned}$$

Next, plug this value back into an equation and solve for y:

$$\begin{aligned} (10.4) + y &= 12.8 \\ y &= 2.4 \end{aligned}$$

Our solution is (10.4,2.4). Let's plug these values into equation (1) and check:

$$\begin{aligned} 16 &= (10.4 - 2.4)\cdot 2 \\ 16 &= 8 \cdot 2 \quad \checkmark \end{aligned}$$

So the cyclist rides 10.4 mph on his own, and the wind speed is 2.4 mph.

Example F

Anna, a venture capitalist, is looking to invest in two different companies. Zuntech is a relatively safe investment and promises a 4% interest rate. X-hex, on the other hand, is a riskier investment but predicts a 9% interest rate. Anna has $12,000 to invest and a goal of 5.25% return on her investment. How much should she invest with each company?

She will invest different amounts in each company, so we need two variables:

$$\begin{aligned} z &= \$ \text{ invested in Zuntech} \\ x &= \$ \text{ invested in X-hex} \end{aligned}$$

The total investment will be $12,000, so we can create an equation:

$$z + x = 12,000 \qquad (1)$$

How much interest does Anna need to earn to reach her goal?

$$5.25\% \text{ of } 12,000 = .0525 \cdot 12,000 = 630$$

We can add the interests earned from Zuntech and X-hex to create another equation:

$$\underbrace{.04z}_{\text{Zuntech interest}} + \underbrace{.09x}_{\text{X-hex interest}} = \underbrace{630}_{\text{total interest}} \qquad (2)$$

So the system we need to solve is:

$$\begin{aligned} z + x &= 12000 \qquad (1) \\ .04z + .09x &= 630 \qquad (2) \end{aligned}$$

Substitution will work easily here if we solve equation (1) for x:

$$x = 12,000 - z$$

Plug this into equation (2) and solve:

$$\begin{aligned} .04z + .09(12,000 - z) &= 630 \\ .04z + 1080 - .09z &= 630 \\ -.05z + 1080 &= 630 \\ -.05z &= -450 \\ z &= 9000 \end{aligned}$$

Now, plug the value we just obtained into equation (1) to find x:

$$\begin{aligned} 9000 + x &= 12000 \\ x &= 3000 \end{aligned}$$

Our solution is (9000,3000). Let's plug those values into equation (2) to check:

$$\begin{aligned} .04(9000) + .09(3000) &= 630 \\ 360 + 270 &= 630 \qquad \checkmark \end{aligned}$$

Anna should invest $9000 in Zuntech and $3000 in X-hex.

Practice Problems

1. I have \$15.00 and wish to buy five pounds of mixed nuts for a party. Peanuts cost \$2.20 per pound. Cashews cost \$4.70 per pound. How many pounds of each should I buy?
2. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and 35%. How many liters of each should be mixed to give the acid needed for the experiment?
3. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?
4. Peter buys two apples and three bananas for \$4. Nadia buys four apples and six bananas for \$8 from the same store. How much does one banana and one apple cost?
5. A movie rental website, CineStar, offers customers two choices. Customers can pay a yearly membership of \$45 and then rent each movie for \$2, or they can choose not to pay the membership fee and rent each movie for \$3.50. How many movies would you have to rent before membership becomes the cheaper option?
6. A movie theater charges \$4.50 for children and \$8.00 for adults. On a certain day, 1200 people enter the movie house and \$8,375 is collected. How many children and how many adults attended?
7. Andrew placed two orders with an internet clothing store. The first order was for 13 ties and four pairs of suspenders, and it totaled \$487. The second order was for six ties and two pairs of suspenders, and it totaled \$232. The bill does not list the per-item price but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
8. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane's speed in still air and the jet-stream's speed?
9. Nadia told Peter that she went to the farmer's market, that she bought two apples and one banana, and that it cost her \$2.50. She thought that Peter might like some fruit so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but he told her that he did not like bananas, so he would pay her for only four apples. Nadia told him that the second time she paid \$6.00 for the fruit. Please help Peter figure out how much to pay Nadia for four apples.

In this chapter, you learned about systems of equations. You saw that the solution to a system corresponds to a point of intersection between the graphs of the equations; however, not all lines intersect, so not every system has a solution.

You also learned the algebraic techniques for solving systems: substitution and elimination. Either will work, but sometimes one might be easier to use than another.

Chapter 6 Unit 6 - Polynomials and Factoring

Chapter Outline

Introduction

Here you'll learn all about polynomials. You'll start by learning how to add, subtract, and multiply polynomials. Then you will learn how to factor polynomials, which can be thought of as the opposite of multiplying. Lastly, we will see how factoring can be utilized to solve complicated equations. Factoring and solving are topics that will be vital to the material in the remaining chapters of the book.

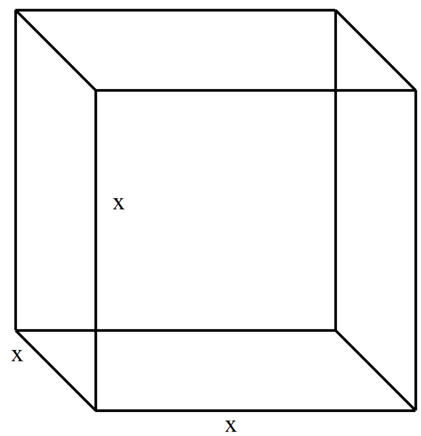
6.1 Addition and Subtraction of Polynomials

Learning Objectives

Here you'll learn how to add and subtract polynomials.

Concept Problem

A cube with a side length of x has a surface area of $6x^2$.



What would the total surface area of three such cubes be? In this section, we will find out.

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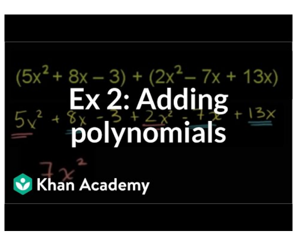
Khan Academy Adding and Subtracting Polynomials 1



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Guidance

We start with the notion of a monomial. A **monomial** is a single algebraic term consisting of a constant multiplied by a nonnegative integer power of one or more variables. Examples:

$$8 \qquad \frac{1}{5}x \qquad -4z^5 \qquad 7x^4y^8$$

Each of these is a monomial. On the other hand, $-2x^{\cdot}5, 3x^{-4}$, and $5x^{\frac{1}{3}}$ are not monomials because their exponents are either negative or not an integer.

Monomials can be added and subtracted (if they are like terms), multiplied and divided (following the rules of exponents).

- Addition: $9x^5 - 3x^5 = 6x^5$
- Subtraction: $6x^4y^5 - 11x^4y^5 = -5x^4y^5$
- Multiplication: $5x^3 \cdot 3x^2 = 15x^5$
- Division: $\frac{6w^8}{18w^3} = \frac{1}{3}w^5$

The word polynomial comes from the Greek word *poly* meaning "many". A **polynomial** is a sum of one or more monomials. Examples:

$$x^2+4x+5 \qquad -4n^5+8n-10 \qquad 4x+1 \qquad 7z^3+11z^2-6z+3 \qquad amp;w^2-16$$

Some common polynomials have special names based on how many terms they have:

- A binomial is a polynomial with two terms. Examples of binomials are $2x+1$, $3x^2-5x$ and $x-5$.
- A trinomial is a polynomial with three terms. An example of a trinomial is $2x^2+3x-4$.

To add and subtract polynomials you will go through two steps.

1. Use the distributive property to remove parentheses. Pay attention to the sign in front. A $-$ sign will need to be distributed as if it was a -1 (because it is)
2. Combine like terms. This means, combine the x^2 terms with the x^2 terms, the x terms with the x terms, etc.

Example A

Find the sum: $(3x^2+2x-7)+(5x^2-3x+3)$.

Solution: First you want to remove the parentheses. Because this is an addition problem, it is like there is a $+1$ in front of each set of parentheses. When you distribute a $+1$, none of the terms will change.

$1(3x^2+2x-7)+1(5x^2-3x+3) = 3x^2+2x-7+5x^2-3x+3$

Next, combine the similar terms. Sometimes it can help to first reorder the expression to put the similar terms next to one another. Remember to keep the signs with the correct terms. For example, in this problem the 7 is negative and the 3x is negative.

$$\begin{aligned} 3x^2+2x-7+5x^2-3x+3 &= 3x^2+5x^2+2x-3x-7+3 \\ &= 8x^2-x-4 \end{aligned}$$

This is your final answer.

Example B

Find the difference: $(5z^2+8z+6)-(4z^2+5z+4)$.

Solution: First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign.

$1(5z^2+8z+6)-1(4z^2+5z+4)=5z^2+8z+6-4z^2-5z-4$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned}5z^2+8z+6-4z^2-5z-4 &= 5z^2-4z^2+8z-5z+6-4\\ &= z^2+3z+2\end{aligned}$$

This is your final answer.

Example C

Find the difference: $(3x^3+6x^2-7x+5)-(4x^2+3x-8)$

Solution: First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign.

$1(3x^3+6x^2-7x+5)-1(4x^2+3x-8)=3x^3+6x^2-7x+5-4x^2-3x+8$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned}3x^3+6x^2-7x+5-4x^2-3x+8 &= 3x^3+6x^2-4x^2-7x-3x+5+8\\ &= 3x^3+2x^2-10x+13\end{aligned}$$

This is your final answer.

Concept Problem Revisited

The cube at the beginning of this section has a surface area of $6x^2$. What would the total surface area of three cubes be?

$$6x^2+6x^2+6x^2=18x^2$$

Vocabulary

Monomial

A **monomial** is a single algebraic term consisting of a constant multiplied by a nonnegative integer power of one or more variables

Polynomial

A **polynomial** is a sum of one or more monomials

Trinomial

A **trinomial** is a polynomial with three terms.

Guided Practice

1. Find the sum: $(2x^2+4x+3)+(x^2-3x-2)$.

2. Find the difference: $(5x^2-9x+7)-(3x^2-5x+6)$.

3. Find the sum: $(8x^3+5x^2-4x+2)+(4x^3+7x-5)$.

Answers:

1. $(2x^2+4x+3)+(x^2-3x-2)=2x^2+4x+3+x^2-3x-2=3x^2+x+1$

2. $(5x^2-9x+7)-(3x^2-5x+6)=5x^2-9x+7-3x^2+5x-6=2x^2-4x+1$

3. $(8x^3+5x^2-4x+2)+(4x^3+7x-5)=8x^3+5x^2-4x+2+4x^3+7x-5=12x^3+5x^2+3x-3$

Practice

For each problem, find the sum or difference.

1. $(x^2+4x+5)+(2x^2+3x+7)$
2. $(2r^2+6r+7)-(3r^2+5r+8)$
3. $(3t^2-2t+4)+(2t^2+5t-3)$
4. $(4s^2-2s-3)-(5s^2+7s-6)$
5. $(5y^2+7y-3)+(-2y^2-5y+6)$
6. $(6x^2+36x+13)-(4x^2+13x+33)$
7. $(12a^2+13a+7)+(9a^2+15a+8)$
8. $(9y^2-17y-12)+(5y^2+12y+4)$
9. $(11b^2+7b-12)-(15b^2-19b-21)$
10. $(25x^2+17x-23)-(-14x^3-14x-11)$
11. $(-3y^2+10y-5)-(5y^2+5y+8)$
12. $(-7x^2-5x+11)+(5x^2+4x-9)$
13. $(9a^3-2a^2+7)+(3a^2+8a-4)$
14. $(3x^2-2x+4)-(x^2+x-6)$
15. $(4s^3+4s^2-5s-2)-(-2s^2-5s+6)$

6.2 Multiplication of Polynomials

Learning Objectives

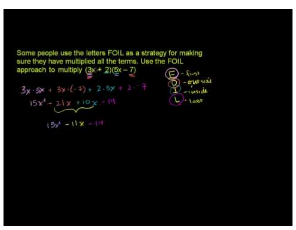
Here you will learn how to multiply polynomials using the distributive property.

Concept Problem

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack notices that the glass is twice as wide as it is tall. Write the expression to determine the area of the picture frame.

Watch This

Khan Academy Multiplying Polynomials



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Guidance

To multiply polynomials you will need to use the distributive property. Recall that the distributive property says:

$$a(b+c) = ab+ac$$

So if you start with an expression like $3(5x+2)$, you can remove the parentheses by multiplying both terms inside by 3 to get a final answer of $15x+6$. Order of terms does not matter with multiplication, so $(5x+2)3 = 15x+6$ as well.

When multiplying polynomials, you will need to use the distributive property more than once for each problem. For example, we can multiply $(x+3)(x+2)$.

First distribute the $(x+3)$ term into the righthand set of parentheses: $(x+3)\cdot x+(x+3)\cdot 2$

Next, distribute the x and 2 into their respective parentheses: $x^2+3x+2x+6$

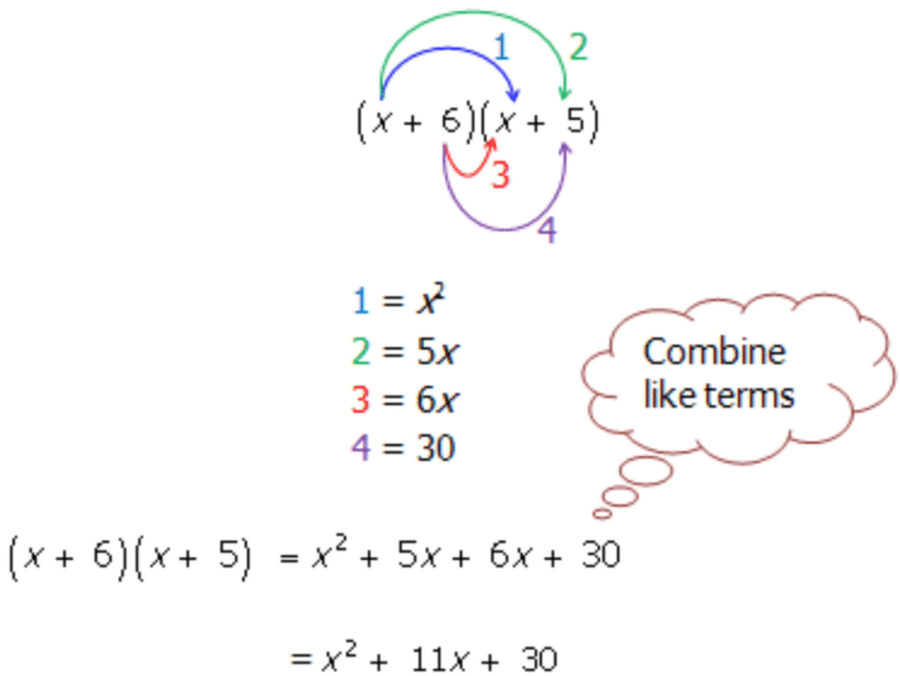
Finally, combine like terms in the middle: x^2+5x+6

You may have heard the acronym FOIL (First, Outer, Inner, Last) before. This is a helpful mnemonic for multiplying binomials, but in general the important thing to remember is that each term in the first set of parentheses gets multiplied by each term in the second set of parentheses.

Example A

Find the product: $(x+6)(x+5)$

Solution: To answer this question you will use the distributive property. The distributive property would tell you to multiply x in the first set of parentheses by everything inside the second set of parentheses , then multiply 6 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



Example B

Find the product: $(2z+5)(z-3)$

Solution: Again, use the distributive property. The distributive property tells you to multiply $2x$ in the first set of parentheses by everything inside the second set of parentheses , then multiply 5 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:

$$\begin{aligned}(2z+5)(z-3)\\ 2z^2+2z+5z-15\\ 2z^2+7z-15\end{aligned}$$

Example C

Find the product: $(4k+3)(2k^2+3k-5)$

Soltuion: Even though at first this question may seem different, you can still use the distributive property to find the product. The distributive property tells you to multiply $4k$ in the first set of parentheses by everything inside the second set of parentheses, then multiply 3 in the first set of parentheses by everything in the second set of parentheses. Here is what that looks like:

$$(4k+3)(2k^2+3k-5)$$
$$\underbrace{8k^3+12k^2-20k}_{\text{Distribute 4k}}+\underbrace{6k^2+9k-15}_{\text{Distribute 3}}$$
$$8k^3+18k^2-11k-15$$

Concept Problem Revisited

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack notices that the glass is twice as wide as it is tall. Write the expression to determine the area of the picture frame.

What is known?

- The width is 5 inches longer than the glass
- The height is 7 inches longer than the glass
- The glass is twice as wide as it is tall

Let x represent the height of the glass. Then 2x is the width (since it is twice as wide as it is tall). Now we can draw the picture:

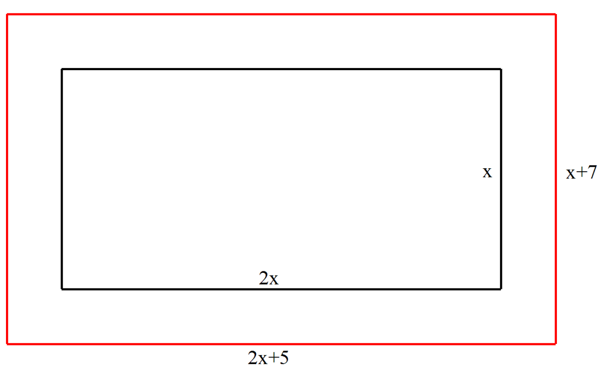


FIGURE 6.1

The equations:

- The height of the picture frame is $x+7$
- The width of the picture frame is $2x+5$

The formula:

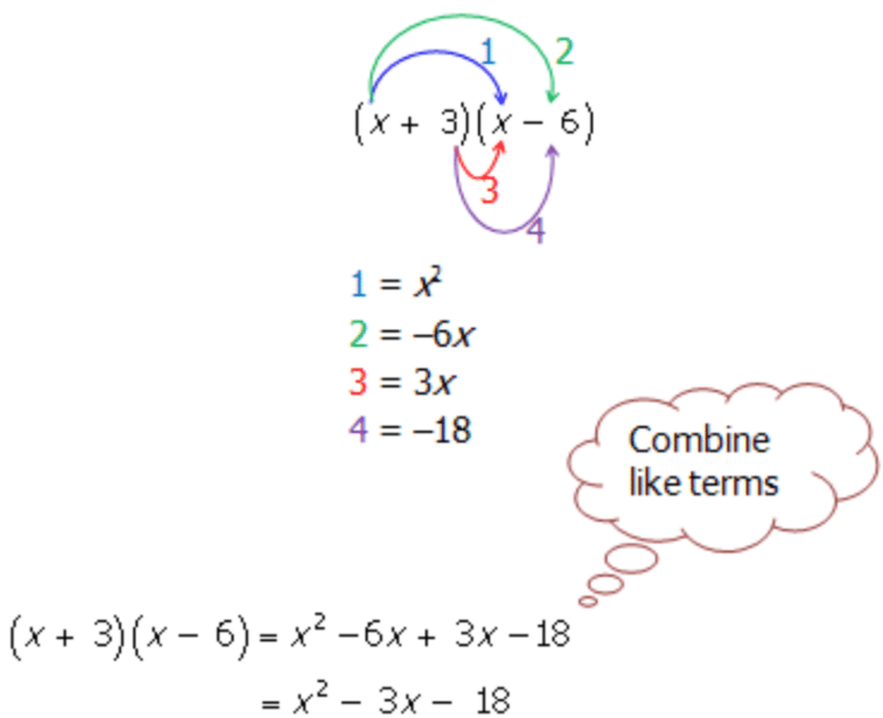
$$\text{Area} = w \times h$$
$$\text{Area} = (2x+5)(x+7)$$
$$\text{Area} = 2x^2+14x+5x+35$$
$$\text{Area} = 2x^2+19x+35$$

Guided Practice

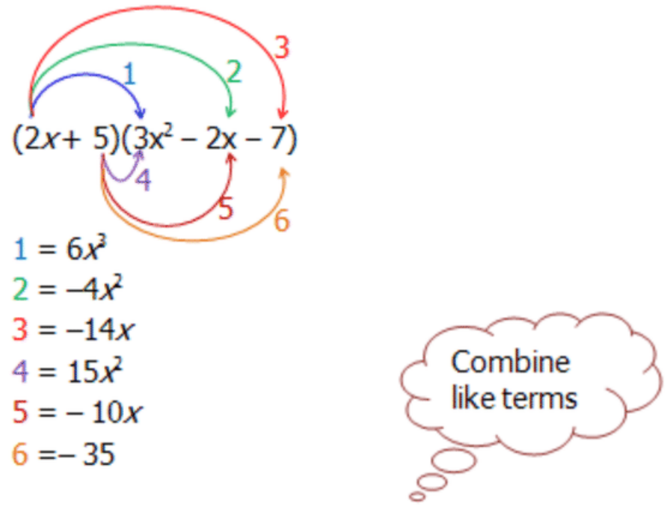
1. Find the product: $(x+3)(x-6)$
2. Find the product: $(2x+5)(3x^2-2x-7)$

Answers:

1. $(x+3)(x-6)$



2. $(2x+5)(3x^2-2x-7)$



$$(4x+3)(2x^2+3x-5) = 6x^3 - 4x^2 - 14x + 15x^2 - 10x - 35$$
$$= 6x^3 + 11x^2 - 24x - 35$$

Practice

Use the distributive property to find the product of each of the following polynomials:

1. $(x+4)(x+6)$
2. $(x+3)(x+5)$
3. $(x+7)(x-8)$
4. $(x-9)(x-5)$
5. $(x-4)(x-7)$
6. $(x+3)(x^2+x+5)$
7. $(x+7)(x^2-3x+6)$
8. $(2x+5)(x^2-8x+3)$
9. $(2x-3)(3x^2+7x+6)$
10. $(5x-4)(4x^2-8x+5)$
11. $9a^2(6a^3+3a+7)$
12. $-4s^2(3s^3+7s^2+11)$
13. $(x+5)(5x^3+2x^2+3x+9)$
14. $(t-3)(6t^3+11t^2+22)$
15. $(2g-5)(3g^3+9g^2+7g+12)$

6.3 Special Products of Polynomials

Learning Objectives

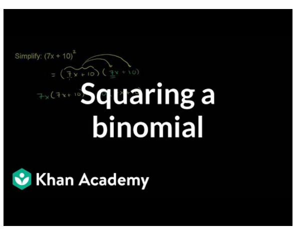
Here you will learn about special cases of binomial multiplication.

Concept Problem

Caroline has a square garden in her back yard that has a side length of 10 feet. She wants to expand it (while keeping the square shape) and would like to know how its area will change. Can she find a formula that will tell her what the new area will be if she increases the side length of the garden by h feet?

Watch This

Khan Academy Special Products of Binomials



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Khan Academy - Difference of Squares

Guidance

If we want to expand the expression $(x+5)^2$ we might be tempted to write x^2+25. Let's evaluate this carefully to see if it is true. To square something means to multiply it by itself, so $(x+5)^2 = (x+5)(x+5)$. Viewed this way, we see that we need to use FOIL to evaluate:

$$\begin{aligned}(x+5)(x+5)\\ x^2+5x+5x+25\\ x^2+10x+25\end{aligned}$$

So, was our intuition correct? No! There is more to squaring a binomial than simply squaring both of its terms. This is a very easy mistake to make.

There are two special cases of multiplying binomials: the Square of a Binomial and Difference of Squares. If you can learn to recognize them, you can multiply these binomials more quickly.

Here are the special products that you should learn to recognize:

Binomial Squared

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$

- Example: $(x+5)^2 = x^2 + 10x + 25$
- Example: $(2x-8)^2 = 4x^2 - 32x + 64$

Special Case 2 (Difference of Perfect Squares)

$$(x+y)(x-y) = x^2 - y^2$$

- Example: $(5x+10)(5x-10) = 25x^2 - 100$
- Example: $(2x-4)(2x+4) = 4x^2 - 16$

Keep in mind that you can always use the distributive property to do the multiplications if you don't notice that the problem is a special case.

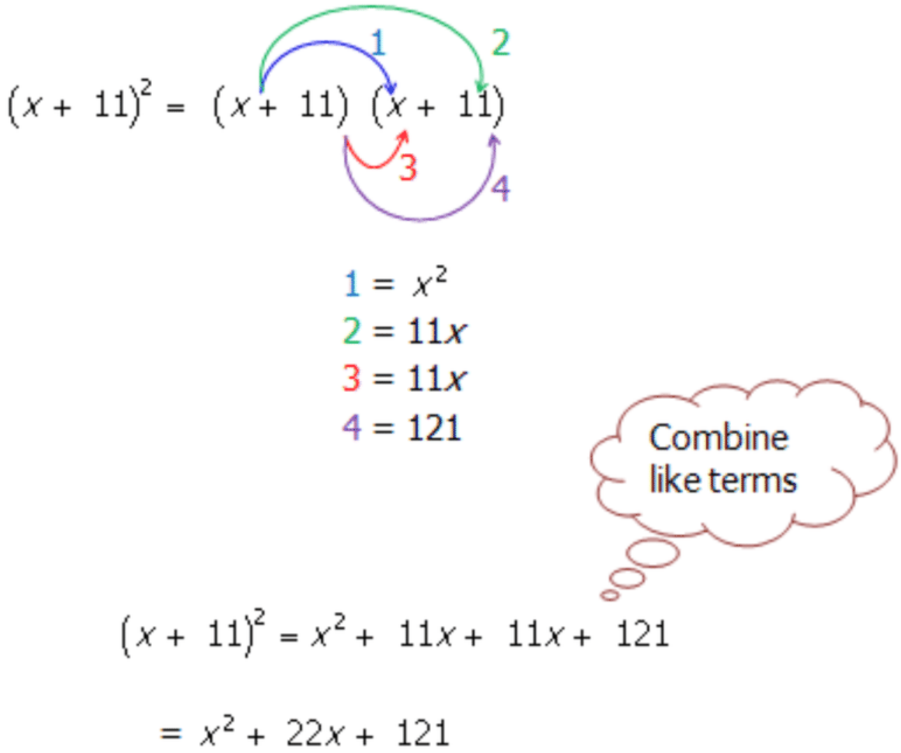
Example A

Find the product: $(x+11)^2$

Solution: These is an example of Special Case 1. You can use that pattern to quickly multiply.

$$\begin{aligned}(x+11)^2 &= x^2 + 2 \cdot x \cdot 11 + 11^2 \\ &= x^2 + 22x + 121\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:



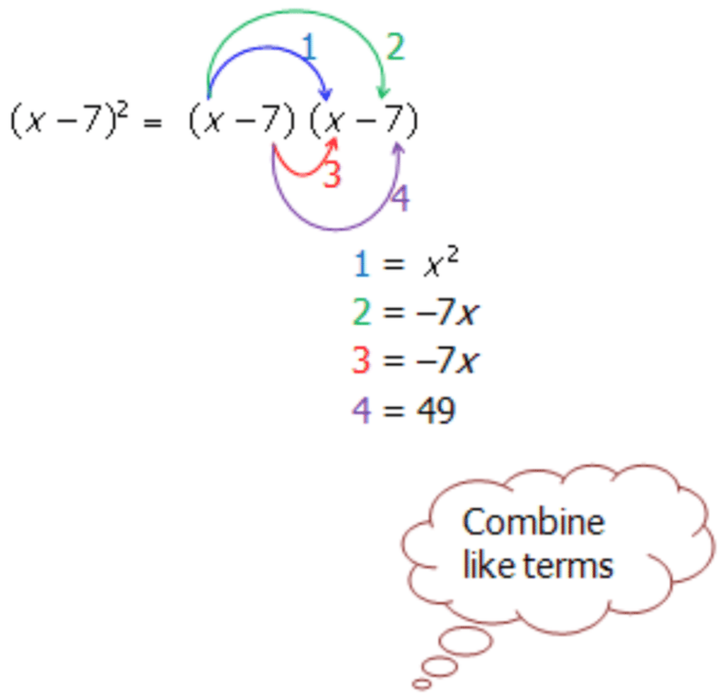
Example B

Find the product: $(x-7)^2$

Solution: These is another example of Special Case 1. You can use that pattern to quickly multiply.

$$\begin{aligned}(x-7)^2 &= x^2 - 2 \cdot x \cdot 7 + 7^2 \\ &= x^2 - 14x + 49\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:



$$(x-7)^2 = x^2 - 7x - 7x + 49$$
$$= x^2 - 14x + 49$$

Example C

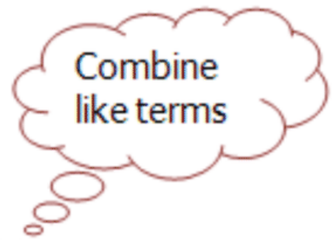
Find the product: $(x+9)(x-9)$

Solution: These is an example of Special Case 2. You can use that pattern to quickly multiply.

$$\begin{aligned}(x+9)(x-9) &= x^2 - 9^2\\ &= x^2 - 81\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:

$1 = x^2$
$2 = -9x$
$3 = 9x$
$4 = -81$



$$(x+9)(x-9) = x^2 + 9x - 9x - 81$$
$$= x^2 - 81$$

Concept Problem Revisited

Caroline has a square garden in her back yard that has a side length of 10 feet. She wants to expand it (while keeping the square shape) and would like to know how its area will change. Can she find a formula that will tell her what the new area will be if she increases the side length of the garden by h feet?

First, make a picture:

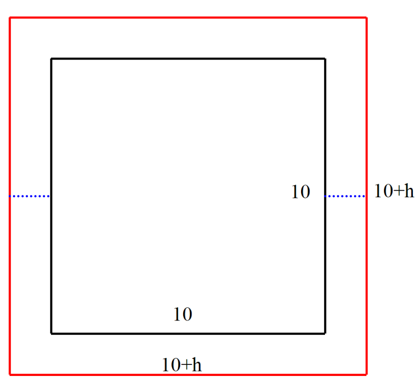


FIGURE 6.2

To express the area A of the expanded garden as a function of h, we multiply its length times its width (which are both $10+h$).

$$A(h) = (10+h)^2$$
$$A(h) = 100+20h+h^2$$

Now, Caroline can calculate the area of her new garden before she builds it. Later on, we will see that she can also use this formula to figure out how much she needs to increase the current size if she want to obtain a specific area (say, 200 square feet).

Guided Practice

1. Expand the following binomial: $(x+4)^2$.

2. Expand the following binomial: $(5x-3)^2$.

Answers:

1. $(x+4)^2 = x^2+2\cdot 4\cdot x+4^2 = x^2+8x+16$.

2.$(5x-3)^2 = (5x)^2-2\cdot 5x\cdot 3+3^2 = 25x^2-30x+9$

Practice

Expand the following binomials:

1. $(t+12)^2$
2. $(w+15)^2$
3. $(2e+7)^2$
4. $(3z+2)^2$
5. $(7m+6)^2$
6. $(g-6)^2$
7. $(d-15)^2$
8. $(4x-3)^2$
9. $(2p-5)^2$
10. $(6t-7)^2$

Find the product of the following binomials:

11. $(x+13)(x-13)$
12. $(x+6)(x-6)$
13. $(2x+5)(2x-5)$
14. $(3x+4)(3x-4)$
15. $(6x+7)(6x-7)$

6.4 Factoring the Greatest Common Factor from a Polynomial

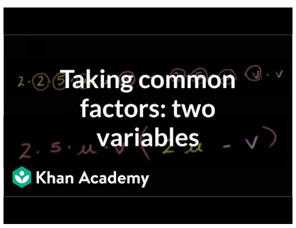
Concept Problem

Can you write the following polynomial as a product of a monomial and a polynomial?

$12x^4 + 6x^3 + 3x^2$

Watch This

Khan Academy Factoring and the Distributive Property



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Guidance

In the past you have studied common factors of two numbers. Consider the numbers 25 and 35. A common factor of 25 and 35 is 5 because 5 goes into both 25 and 35 evenly.

This idea can be extended to polynomials. A common factor of a polynomial is a number and/or variable that are a factor in all terms of the polynomial. The Greatest Common Factor (or GCF) is the largest monomial that is a factor of each of the terms of the polynomial.

To factor a polynomial means to write the polynomial as a product of other polynomials. One way to factor a polynomial is:

1. Look for the greatest common factor.
2. Write the polynomial as a product of the **greatest common factor** and **the polynomial that results when you divide all the terms of the original polynomial by the greatest common factor.**

One way to think about this type of factoring is that you are essentially doing the distributive property in reverse.

Example A

Factor the following binomial: $5a + 15$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5 and 15 can both be divided by 5. The GCF for this binomial is 5.

Step 2: Divide the GCF out of each term of the binomial:

$5a + 15 = 5(a + 3)$

Example B

Factor the following polynomial: $4x^2+8x-2$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 4, 8 and 2 can all be divided by 2. The GCF for this polynomial is 2.

Step 2: Divide the GCF out of each term of the polynomial:

$4x^2+8x-2=2(2x^2+4x-1)$

Example C

Factor the following polynomial: $3x^5-9x^3-6x^2$

Solution: *Step 1:* Identify the GCF. Looking at each of the terms, you can see that 3, 9 and 6 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$3x^5-9x^3-6x^2=3x^2(x^3-3x-2)$

Concept Problem Revisited

To write as a product you want to try to factor the polynomial: $12x^4+6x^3+3x^2$.

Step 1: Identify the GCF of the polynomial. Looking at each of the numbers, you can see that 12, 6, and 3 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$12x^4+6x^3+3x^2=3x^2(4x^2+2x+1)$$

Vocabulary

Greatest Common Factor

The **Greatest Common Factor** (or GCF) of a polynomial is the largest monomial that is a factor of (or divides into evenly) each of its terms.

Guided Practice

1. Factor the following polynomial: $5k^6+15k^4+10k^3+25k^2$
2. Factor the following polynomial: $27x^3y+18x^2y^2+9xy^3$
3. Find the common factors of the following: $a^2(b+7)-6(b+7)$

Answers:

1. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5. Also notice that each of the terms has an k^2 in common. The GCF for this polynomial is $5k^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$5k^6+15k^4+10k^3+25k^2=5k^2(k^4+3k^2+2k+5)$$

2. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 27, 18 and 9 can all be divided by 9. Also notice that each of the terms has an xy in common. The GCF for this polynomial is $9xy$.

Step 2: Divide the GCF out of each term of the polynomial:
$27x^3y+18x^2y^2+9xy^3=9xy(3x^2+2xy+y^2)$

3. *Step 1:* Identify the GCF

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor. The GCF for this expression is $(b+7)$.
Step 2: Divide the GCF out of each term of the expression:

$$a^2(b+7)-6(b+7)=(a^2-6)(b+7)$$

Practice

Factor the following polynomials by looking for a common factor:

1. $7x^2+14$
2. $9c^2+3$
3. $8a^2+4a$
4. $16x^2+24y^2$
5. $2x^2-12x+8$
6. $32w^2x+16xy+8x^2$
7. $12abc+6bcd+24acd$
8. $15x^2y-10x^2y^2+25x^2y$
9. $12a^2b-18ab^2-24a^2b^2$
10. $4s^3t^2-16s^2t^3+12st^2-24st^3$

Find the common factors of the following expressions and then factor:

1. $2x(x-5)+7(x-5)$
2. $4x(x-3)+5(x-3)$
3. $3x^2(e+4)-5(e+4)$
4. $8x^2(c-3)-7(c-3)$
5. $ax(x-b)+c(x-b)$

6.5 Factorization of Quadratic Expressions

Concept Problem

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?

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Khan Academy Factoring trinomials with a leading 1 coefficient

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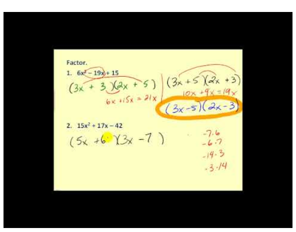
James Sousa: Factoring Trinomials using Trial and Error and Grouping



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Guidance

To factor a polynomial means to write the polynomial as a product of other polynomials. Here, you'll focus on factoring quadratic expressions. Quadratic expressions are polynomials of degree 2, of the form $ax^2 + bx + c$. Consider the steps for finding the product of the following binomials:

$$
\begin{aligned}
(2x+3)(3x-5) &= 6x^2-10x+9x-15 \\
&= 6x^2-x-15
\end{aligned}
$$

When factoring a quadratic expression, your job will be to take an expression like $6x^2-x-15$ and write it as $(2x+3)(3x-5)$. In this sense factoring is the reverse of multiplying. Notice that when factored, the $6x^2$ factors to $2x$ and $3x$. The -15 factors to -5 and 3. You can say then, in general, that with the trinomial ax^2+bx+c, you have to factor both "a" and "c".

-

$$ax^2+bx+c=(dx+e)(fx+g) \text{ where } a=d\times f \text{ and } c=e\times g$$

- The middle term (b) is

$$b=dg+ef$$

There are several techniques that you can use, including trial and error, decomposition, the box method, etc. In this section we will focus on the technique of factoring by grouping, sometimes called the AC method.

Factoring By Grouping

Before getting to factoring quadratics, we must first go over the basics of factoring by grouping. Here are the basic steps:

- **Step 1:** Separate the terms into two groups.
- **Step 2:** Factor out the common terms in each of the two groups.
- **Step 3:** Factor out the common binomial.

Example A

Factor: $x^2+5x+6x+30$

1. Separate the expression into the left and right parts: $\left(x^2+5x\right)+(6x+30)$

2. Factor each part: $x(x+5)+6(x+5)$

Notice that the expression in parentheses is the same for both parts. You cannot go on to the next step if they are different

3. Factor out the common binomial:

$$x\underbrace{(x+5)}_{\text{same}}+6\underbrace{(x+5)}_{\text{same}}=(x+5)(x+6)$$

Since the $(x+5)$ was common to both parts, we were able to factor it out! The $(x+6)$ are the terms that remained after the $(x+5)$ was factored out.

Example B

Factor: $10z^3 + 35z - 6z^2 - 21$.

This looks more complicated, but the same steps will work.

1. Separate the terms into two groups: $(10z^3 + 35z) + (-6z^2 - 21)$
2. Factor each part: $5z(2z^2 + 7) - 3(2z^2 + 7)$
3. Factor out the common binomial: $(2z^2 + 7)(5z - 3)$

Factoring Quadratics Using Grouping

Now, suppose we wanted to factor $2x^2 + 17x + 21$. In this case we don't have left and right halves, so what do we do? We can use a technique called the AC method to split the $17x$ term into two parts that will then allow us to use the grouping method. Here are the steps:

We want to factor something that looks like $ax^2 + bx + c$

Step 1: Multiply a and c together (this is why it is called the ac method)

Step 2: Find factors of ac that add up to b

Step 3: Split bx into two terms according to these factors

Step 4: Factor by grouping

Example C

Factor $2x^2 + 17x + 21$

1. a=2 and c=21, so ac = 42
2. We have ac = 42 and b = 17, so we need two numbers that multiply to 42 and add to 17. The numbers 14 and 3 will do the trick!
3. Go back to the original expression and split up the $17x$: $2x^2 + 14x + 3x + 21$

Note:If we combine like terms we would get our original expression back, so we haven't changed it; we just changed how it looks

4. Factor by grouping:

$$\begin{aligned} 2x^2 + 14x + 3x + 21 \\ 2x(x+7) + 3(x+7) \\ (x+7)(2x+3) \end{aligned}$$

Example D

Factor $12x^2 + 16x - 35$

1. $ac = -420$
2. $ac = -420$ and $b = 16$, so our factors should be 30 and -14
3. Split: $12x^2 + 30x - 14x - 35$
4. Factor by grouping:

$$\begin{aligned} 12x^2+30x-14x-35 \\ 6x(2x+5)-7(2x+5) \\ (2x+5)(6x-7) \end{aligned}$$

Box Method

The AC method described above looks very technical but you should notice that steps 3 and 4 play out in more or less the exact same way every time. The real work is in step 2 when you find the appropriate factors of ac.

The Box Method is another way of doing the AC Method. Some students prefer it because it looks cleaner and less technical than the AC Method. Several examples are worked out below using the Box Method. If you use it, though, keep in mind that it is still the AC Method going on in the background.

Example E

Factor: $2x^2+11x+15$

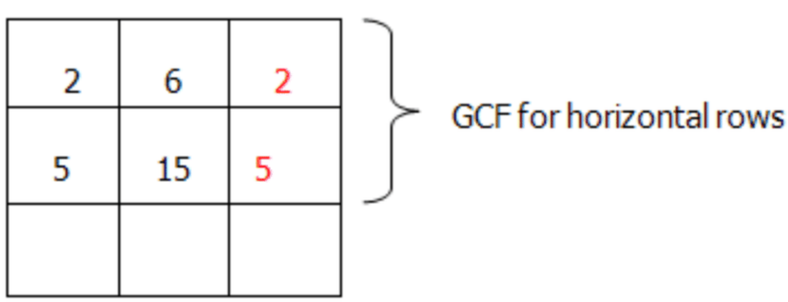
Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 2 and the 'c' value is 15. Start by making a box and placing these values in the box as shown.

2	
	15

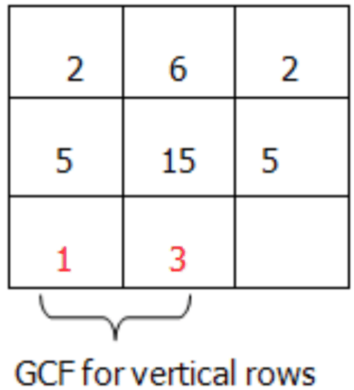
The product of 2 and 15 is 30. To continue filling in the box, you need to find two numbers that multiply to 30, but add up to +11 (the value of b in the original equation). The two numbers that work are 5 and 6: $5+6=11$ and $5\cdot 6=30$. Put 5 and 6 in the box.

2	6
5	15

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 2 and 6, has a GCF of 2. The second row, 5 and 15, has a GCF of 5.



The first column, 2 and 5, has a GCF of 1. The second column, 6 and 15, has a GCF of 3.



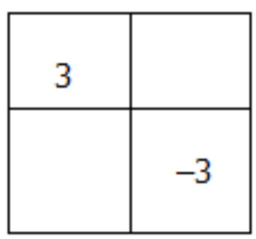
Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(1x+3)$ and $(2x+5)$. You can verify that those binomials multiply to create the original trinomial: $(x+3)(2x+5) = 2x^2+5x+6x+15 = 2x^2+11x+15$.

The factored form of $2x^2+11x+15$ is $(x+3)(2x+5)$.

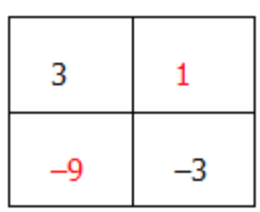
Example F

Factor: $3x^2-8x-3$

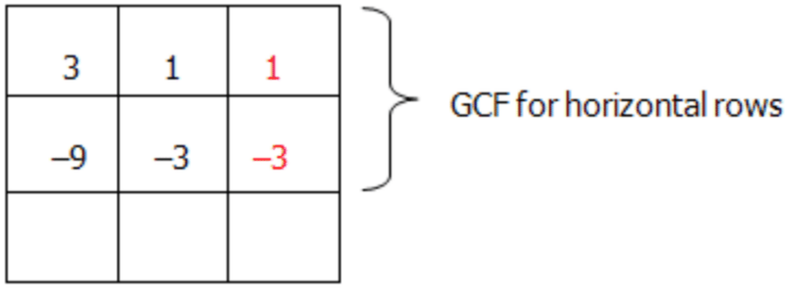
Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the '*a*' value is 3 and the '*c*' value is -3. Start by making a box and placing these values in the box as shown.



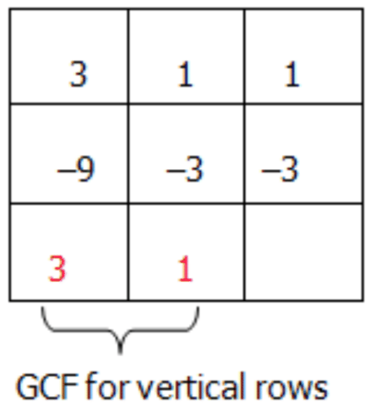
The product of 3 and -3 is -9. To continue filling in the box, you need to find two numbers that multiply to -9, but add up to -8 (the value of *b* in the original equation). The two numbers that work are -9 and 1. $-9+1=-8$ and $-9 \cdot 1 = -9$. Put -9 and 1 in the box.



Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 3 and 1, has a GCF of 1. The second row, -9 and -3, has a GCF of -3.



The first column, 3 and -9, has a GCF of 3. The second column, 1 and -3, has a GCF of 1.



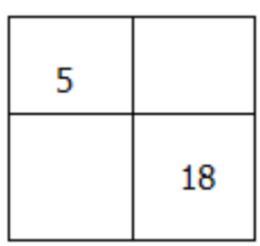
Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(3x+1)$ and $(1x-3)$. You can verify that those binomials multiply to create the original trinomial: $(3x+1)(x-3) = 3x^2 - 9x + 1x - 3 = 3x^2 - 8x - 3$.

The factored form of $3x^2 - 8x - 3$ is $(3x+1)(x-3)$.

Example G

Factor: $5w^2 - 21w + 18$

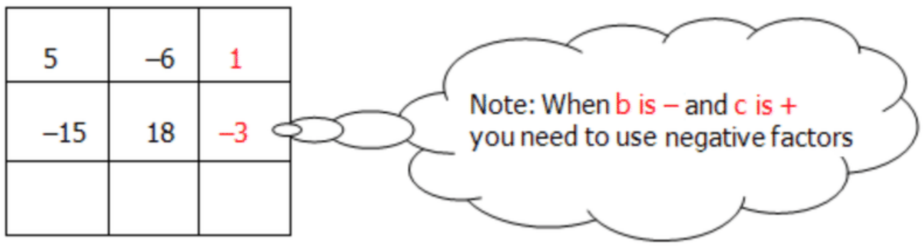
Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 5 and the 'c' value is 18. Start by making a box and placing these values in the box as shown.



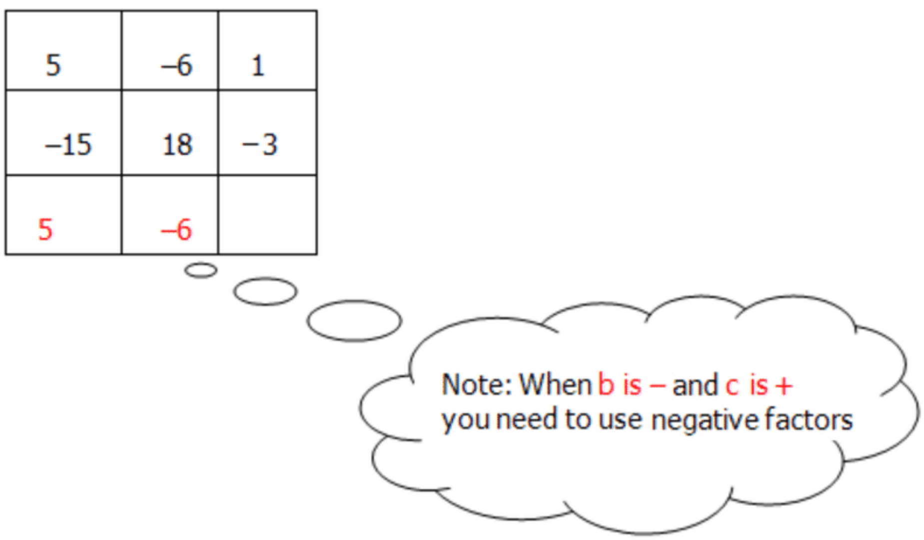
The product of 5 and 18 is 90. To continue filling in the box, you need to find two numbers that multiply to 90, but add up to -21 (the value of b is the original equation). The two numbers that work are -6 and -15. $-6 + (-15) = -21$ and $-6 \cdot -15 = 90$. Put -6 and -15 in the box.

5	–6
–15	18

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 5 and -6, has a GCF of 1. The second row, -15 and 18, has a GCF of 3.



The first column, 5 and -15, has a GCF of 5. The second column, -6 and 18, has a GCF of 6.



Notice that you need to make two of the GCFs negative in order to make the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(5w-6)$ and $(w-3)$. You can verify that those binomials multiply to create the original trinomial: $(5w-6)(w-3) = 5w^2 - 15w - 6w + 18 = 5w^2 - 21w + 18$.

The factored form of $5w^2 - 21w + 18$ is $(5w-6)(w-3)$.

Concept Problem Revisited

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?

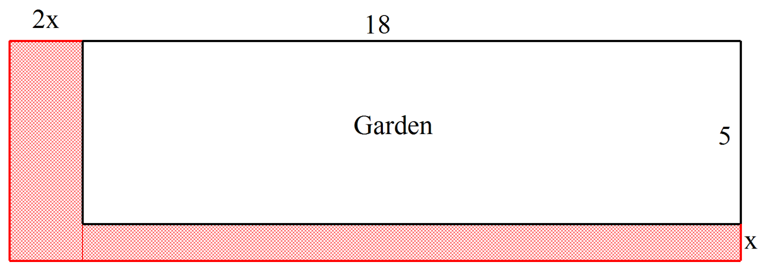


FIGURE 6.3

$$\begin{aligned}
\text{Area of Garden} &= 18 \times 5 = 90 \; yd^2 \\
\text{Area of border} &= 30 \; yd^2 \\
\text{Area of Garden} + \text{border} &= (18+2x)(5+x) \\
\text{Area of border} &= (\text{Area of garden} + \text{border}) - \text{Area of garden} \\
30 &= (18+2x)(5+x) - 90 \\
30 &= 90 + 18x + 10x + 2x^2 - 90 \\
30 &= 28x + 2x^2 \\
0 &= 2x^2 + 28x - 30
\end{aligned}$$

This trinomial has a common factor of 2. First, factor out this common factor:

$2x^2 + 28x - 30 = 2(x^2 + 14x - 15)$

Now, you can use the box method to factor the remaining trinomial. After using the box method, your result should be:

$2(x^2 + 14x - 15) = 2(x+15)(x-1)$

To find the dimensions of the border you need to solve a quadratic equation. This is explored in further detail in another concept:

$$\begin{aligned}
&2(x+15)(x-1) = 0 \\
&\swarrow \qquad \searrow \\
&x+15 = 0 \quad x-1 = 0 \\
&x = -15 \qquad x = 1
\end{aligned}$$

Our variable x represents a length, so it cannot be negative. x must equal 1.

Width of Border: $2x = 2(1) = 2 \; yd$

Length of Border: $x = 1 \; yd$

Vocabulary

Quadratic Expression

A **quadratic expression** is a polynomial of degree 2. The general form of a quadratic expression is ax^2+bx+c.

Guided Practice

1. Factor the following trinomial: $8c^2-2c-3$
2. Factor the following trinomial: $3m^2+3m-60$
3. Factor the following trinomial: $5e^3+30e^2+40e$

Answers:

1. Use the box method and you find that $8c^2-2c-3=(2c+1)(4c-3)$

2. First you can factor out the 3 from the polynomial. Then, use the box method. The final answer is $3m^2+3m-60=3(m-4)(m+5)$.

3. First you can factor out the $5e$ from the polynomial. Then, use the box method. The final answer is $5e^3+30e^2+40e=5e(e+2)(e+4)$.

Practice

Factor the following trinomials.

1. x^2+5x+4
2. $x^2+12x+20$
3. $a^2+13a+12$
4. $z^2+7z+10$
5. $w^2+8w+15$
6. $x^2-7x+10$
7. $x^2-10x+24$
8. m^2-4m+3
9. s^2-6s-7
10. $y^2-8y+12$
11. x^2-x-12
12. x^2+x-12
13. $x^2-5x-14$
14. $x^2-7x-44$
15. y^2+y-20
16. $3x^2+5x+2$
17. $5x^2+9x-2$
18. $4x^2+x-3$
19. $2x^2+7x+3$
20. $2y^2-15y-8$
21. $2x^2-5x-12$
22. $2x^2+11x+12$
23. $6w^2-7w-20$
24. $12w^2+13w-35$
25. $3w^2+16w+21$

26. $16a^2 - 18a - 9$
27. $36a^2 - 7a - 15$
28. $15a^2 + 26a + 8$
29. $20m^2 + 11m - 4$
30. $3p^2 + 17p - 20$

6.6 Special Cases of Quadratic Factorization

Learning Objectives

Here you'll learn to recognize two special kinds of quadratics and how to factor them quickly.

Concept Problem

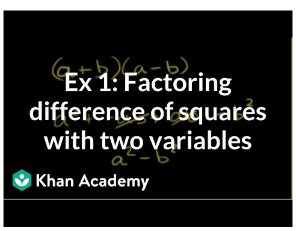
There are two special cases of factoring that appear frequently. Here is an example of each:

$$9x^2+24x+16$$
$$49k^2-4$$

You could factor these using the techniques laid out in the last section; however, if you can recognize them as special cases you can save yourself some effort.

Watch This

Khan Academy Factoring the Sum and Difference of Squares



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Guidance

When factoring quadratics, there are special cases that can be factored more quickly. There are two special quadratics that you should learn to recognize:

Perfect Square Trinomial

$x^2+2xy+y^2=(x+y)^2$
$x^2-2xy+y^2=(x-y)^2$

- Example: $x^2+10x+25=(x+5)^2$
- Example: $4x^2-32x+64=(2x-8)^2$

Difference of Squares

$x^2-y^2=(x+y)(x-y)$

- Example: $25x^2 - 100 = (5x + 10)(5x - 10)$
- Example: $4x^2 - 25 = (2x - 5)(2x + 5)$

Keep in mind that you can always use the box method to do the factoring if you don't notice the problem as a special case.

Example A

Factor $2x^2 + 28x + 98$.

Solution: First, notice that there is a common factor of 2. Factor out the common factor:

$2x^2 + 28x + 98 = 2(x^2 + 14x + 49)$

Next, notice that the first and last terms are both perfect squares ($x^2 = x \cdot x$ and $49 = 7 \cdot 7$, and the middle term is 2 times the product of the roots of the other terms ($14x = 2 \cdot x \cdot 7$). This means $x^2 + 14x + 49$ is a perfect square trinomial (Special Case 1). Using the pattern:

$x^2 + 14x + 49 = (x + 7)^2$

Therefore, the complete factorization is $2x^2 + 28x + 98 = 2(x + 7)^2$.

Example B

Factor $8a^2 - 24a + 18$.

Solution: First, notice that there is a common factor of 2. Factor out the common factor:

$8a^2 - 24a + 18 = 2(4a^2 - 12a + 9)$

Next, notice that the first and last terms are both perfect squares and the middle term is 2 times the product of the roots of the other terms ($12a = 2 \cdot 2a \cdot 3$). This means $4a^2 - 12a + 9$ is a perfect square trinomial (Special Case 1). Because the middle term is negative, there will be a negative in the binomial. Using the pattern:

$4a^2 - 12a + 9 = (2a - 3)^2$

Therefore, the complete factorization is $8a^2 - 24a + 18 = 2(2a - 3)^2$.

Example C

Factor $z^2 - 16$.

Solution: Notice that there are no common factors. The typical middle term of the quadratic is missing and each of the terms present are perfect squares and being subtracted. This means $z^2 - 16$ is a difference of squares (Special Case 2). Using the pattern:

$z^2 - 16 = (z - 4)(z + 4)$

Note that it would also be correct to say $z^2 - 16 = (z + 4)(z - 4)$. It does not matter whether you put the + version of the binomial first or the - version of the binomial first.

Example D

Factor $m^4 - n^4$

Solution: This expression involves subtraction and only has two terms, so your gut is probably telling you to use difference of squares... but there are no squares! Can it still work? Yes! In this case we can think of m^4 and n^4 as $(m^2)^2$ and $(n^2)^2$. Rewriting in this form shows us how we can apply the difference of squares rule:

$$(m^2)^2 - (n^2)^2 = (m^2 + n^2)(m^2 - n^2)$$

But, wait! There's more! Notice that the second factor is once again a difference of squares, so we can go one step further:

$$(m^2 + n^2)(m^2 - n^2) =$$
$$(m^2 + n^2)(m + n)(m - n)$$

Note that you cannot factor the $m^2 + n^2$ term any further. It is a sum of squares, not a difference of squares.

Concept Problem Revisited

$9x^2 + 24 + 16 = (3^2)x^2 + 2(3)(4)x + 4^2 = (3x + 4)^2$

$49k^2 - 4 = (7k)^2 - 2^2 = (7k - 2)(7k + 2)$

Guided Practice

1. Factor completely $s^2 - 18s + 81$

2. Factor completely $50 - 98x^2$

3. Factor completely $4x^2 + 48x + 144$

Answers:

1. This is Special Case 1. $s^2 - 18s + 81 = (s - 9)^2$

2. First factor out the common factor of 2. Then, it is Special Case 2. $50 - 98x^2 = 2(5 - 7x)(5 + 7x)$

3. First factor out the common factor of 4. Then, it is Special Case 1. $4x^2 + 48x + 144 = 4(x + 6)^2$

Practice

Factor each of the following:

1. $s^2 + 18s + 81$
2. $x^2 + 12x + 36$
3. $y^2 - 14y + 49$
4. $4a^2 + 20a + 25$
5. $9s^2 - 48s + 64$
6. $s^2 - 81$
7. $x^2 - 49$
8. $4t^2 - 25$
9. $25w^2 - 36$
10. $64 - 81a^2$
11. $y^2 - 22y + 121$
12. $16t^2 - 49$
13. $9a^2 + 30a + 25$
14. $100 - 25b^2$
15. $4s^2 - 28s + 49$

6.7 Complete Factorization of Polynomials

Learning Objectives

Here you will learn how to factor a polynomial completely by first looking for common factors and then factoring the resulting expression.

Concept Problem

Can you factor the following polynomial completely?

$8x^3+24x^2-32x$

Guidance

A cubic polynomial is a polynomial of degree equal to 3. Examples of cubics are:

- $9x^3+10x-5$
- $8x^3+2x^2-5x-7$

Recall that to factor a polynomial means to rewrite the polynomial as a product of other polynomials . You will not be able to factor all cubics at this point, but you will be able to factor some using your knowledge of common factors and factoring quadratics. In order to attempt to factor a cubic, you should:

1. Check to see if the cubic has any common factors. If it does, factor them out.
2. Check to see if the resulting expression can be factored, especially if the resulting expression is a quadratic. To factor the quadratic expression you could use the box method, or any method you prefer.

Anytime you are asked to **factor completely**, you should make sure that none of the pieces (factors) of your final answer can be factored any further. If you follow the steps above of first checking for common factors and then checking to see if the resulting expressions can be factored, you can be confident that you have factored completely.

Example A

Factor the following polynomial completely: $3x^3-15x$.

Solution: Look for the common factors in each of the terms. The common factor is $3x$. Therefore:

$3x^3-15x=3x(x^2-5)$

The resulting quadratic, x^2-5, cannot be factored any further (it is NOT a difference of perfect squares). Your answer is $3x(x^2-5)$.

Example B

Factor the following polynomial completely: $2a^3+16a^2+30a$.

Solution: Look for the common factors in each of the terms. The common factor is $2a$. Therefore:

$2a^3+16a^2+30a=2a(a^2+8a+15)$

The resulting quadratic, $a^2+8a+15$ can be factored further into $(a+5)(a+3)$. Your final answer is $2a(a+5)(a+3)$.

Example C

Factor the following polynomial completely: $6s^3+36s^2-18s-42$.

Solution: Look for the common factors in each of the terms. The common factor is 6. Therefore:

$6s^3+36s^2-18s-42=6(s^3+6s^2-3s-7)$

The resulting expression is a cubic, and you don't know techniques for factoring cubics without common factors at this point. Therefore, your final answer is $6(s^3+6s^2-3s-7)$.

Note: It turns out that the resulting cubic cannot be factored, even with more advanced techniques. Remember that not all expressions can be factored.

Concept Problem Revisited

Factor the following polynomial completely: $8x^3+24x^2-32x$.

Look for the common factors in each of the terms. The common factor is $8x$. Therefore:

$8x^3+24x^2-32x=8x(x^2+3x-4)$

The resulting quadratic can be factored further into $(x+4)(x-1)$. Your final answer is $8x(x+4)(x-1)$.

Guided Practice

Factor each of the following polynomials completely.

1. $9w^3+12w$.

2. y^3+4y^2+4y.

3. $2t^3-10t^2+8t$.

Answers:

1. The common factor is $3w$. Therefore, $9w^3+12w=3w(3w^2+4)$. The resulting quadratic cannot be factored any further, so your answer is $3w(3w^2+4)$.

2. The common factor is y. Therefore, $y^3+4y^2+4y=y(y^2+4y+4)$. The resulting quadratic can be factored into $(y+2)(y+2)$ or $(y+2)^2$. Your answer is $y(y+2)^2$.

3. The common factor is $2t$. Therefore, $2t^3-10t^2+8t=2t(t^2-5t+4)$. The resulting quadratic can be factored into $(t-4)(t-1)$. Your answer is $2t(t-4)(t-1)$.

Practice

Factor each of the following polynomials completely.

1. $6x^3-12$
2. $4x^3-12x^2$
3. $8y^3+32y$
4. $15a^3+30a^2$

5. $21q^3 + 63q$
6. $4x^3 - 12x^2 - 8$
7. $12e^3 + 6e^2 - 6e$
8. $15s^3 - 30s + 45$
9. $22r^3 + 66r^2 + 44r$
10. $32d^3 - 16d^2 + 12d$
11. $5x^3 + 15x^2 + 25x - 30$
12. $3y^3 - 18y^2 + 27y$
13. $12s^3 - 24s^2 + 36s - 48$
14. $8x^3 + 24x^2 - 80x$
15. $5x^3 - 25x^2 - 70x$

6.8 Zero Product Property for Quadratic Equations

Learning Objectives

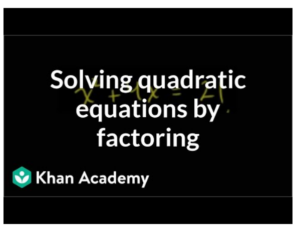
Here you'll learn how to solve a quadratic equation by factoring and using the zero product property.

Concept Problem

A rectangle's length is three meters more than twice its width. Its area is 119 m^2. What are its dimensions?

Watch This

Khan Academy Factoring Special Products



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Guidance

Recall that when solving an equation, you are trying to determine the values of the variable that make the equation true. For the equation $2x^2+10x+8=0$, $x=-1$ and $x=-4$ are both solutions. You can check this:

- $2(-1)^2+10(-1)+8=2(1)-10+8=0$
- $2(-4)^2+10(-4)+8=2(16)-40+8=32-40+8=0$

Here you will focus on solving quadratic equations. One of the methods for quadratic equations utilizes your factoring skills and a property called the **zero factor principle**.

If $a \cdot b=0$, what can you say about a or b? Either a or b must be equal to 0, because that is the only way that their product will be 0. If both a and b were non-zero, then their product would have to be non-zero. This is the idea of the zero factor principle. The zero factor principle states that if the product of two quantities is zero, then at least one of the quantities must be zero.

Zero is a special number in this regard. For example, if $a \cdot b=10$, it does not have to be the case that $a=10$ or $b=10$. It could be that $a=2$ and $b=5$. When you solve an equation using factoring, you **must** have zero on one side for the principle to hold.

The advantage of the zero factor principle is that it allows us to take one difficult problem and turn it into two simple problems. When you factor, you turn a quadratic expression into a product. If you have a quadratic expression equal to zero, you can factor it and then use the zero product property to solve. So, if you were given the equation $2x^2+5x-3=0$, first you would want to turn the quadratic expression into a product by factoring it:

$2x^2+5x-3=(x+3)(2x-1)$

You can rewrite the equation you are trying to solve as $(x+3)(2x-1)=0$.

Now, you have the product of two binomials equal to zero. This means at least one of those binomials must be equal to zero. So, you have two simple equations that you can solve to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means $x=-3$ OR
- $2x-1=0$, which means $x=\frac{1}{2}$

The two solutions to the equation $2x^2+5x-3=0$ are $x=-3$ and $x=\frac{1}{2}$.

Keep in mind that you can only use the zero product property if your equation is set equal to zero! If you have an equation not set equal to zero, first rewrite it so that it is set equal to zero. Then factor and use the zero product property.

Example A

Solve for x: $x^2+5x+6=0$.

Solution: First, change x^2+5x+6 into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+5x+6=(x+3)(x+2)$

Next, rewrite the equation you are trying to solve:

$x^2+5x+6=0$ becomes $(x+3)(x+2)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means that $x=-3$
- $x+2=0$, which means that $x=-2$

Before we say for sure, let's check the solutions in the original equation to see if they work:

- $(-3)^2+5(-3)+6=9-15+6=0 \quad \checkmark$
- $(-2)^2+5(-2)+6=4-10+6=0 \quad \checkmark$

The solutions are $x=-3$ or $x=-2$.

Example B

Solve for x: $6x^2+x-15=0$.

In order to solve for x you need to factor the polynomial.

Solution: First, change $6x^2+x-15$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$6x^2+x-15=(3x+5)(2x-3)$

Next, rewrite the equation you are trying to solve:

$6x^2+x-15=0$ becomes $(3x+5)(2x-3)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $3x+5=0$, which means that $x=-\frac{5}{3}$
- $2x-3=0$, which means that $x=\frac{3}{2}$

The solutions are $x=-\frac{5}{3}$ or $x=\frac{3}{2}$.

Example C

Solve for x: $x^2-35=-2x$.

Solution: First, get zero on one side by adding 2x to both sides:

$$x^2-35+2x=-2x+2x$$
$$x^2+2x-35=0$$

Next, change $x^2+2x-35$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+2x-35=(x+7)(x-5)$

Next, rewrite the equation you are trying to solve:

$x^2+2x-35=0$ becomes $(x+7)(x-5)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+7=0$, which means that $x=-7$
- $x-5=0$, which means that $x=5$

The solutions are $x=-7$ or $x=5$.

Concept Problem Revisited

A rectangle's length is three meters more than twice its width. Its area is 119 m^2. What are its dimensions?

Let w represent the width of our rectangle, so then its length is $2w+3$ (three more than twice the width). Now, draw a picture:

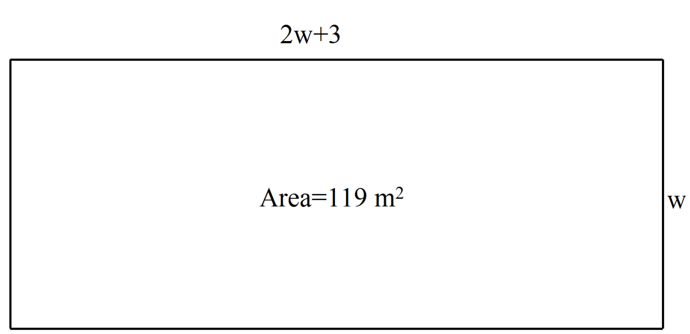


FIGURE 6.4

The formula for the area of a rectangle is $Area = Length \times Width$. In this case that gives us:

$$119 = (2w+3)w$$

To solve, we need to distribute the w on the left hand side and then move the 119 over so that we have 0 on one side:

$$119 = 2w^2 + 3w$$
$$0 = 2w^2 + 3w - 119$$

Now, factor the right hand side, then split the complicated equation into two simple ones:

$$0 = 2w^2 + 3w - 119$$
$$0 = (2w+17)(w-7)$$

$$\swarrow \qquad\qquad \searrow$$

$$2w + 17 = 0 \qquad\qquad amp; w - 7 = 0$$

$$amp; w = \frac{-17}{2} \qquad\qquad w = 7$$

So what's the answer? A negative value does not make sense for a physical length, so the width of the rectangle must be 7 meters, and its length is 17 meters.

Vocabulary

Quadratic Equation

A **quadratic equation** is an equation in which the highest power of a variable is 2. Standard form for a quadratic equation is $ax^2 + bx + c = 0$.

Zero factor principle

The **zero factor principle** states that if two factors are multiplied together and their product is zero, then one of the factors must equal zero

Guided Practice

1. Solve for the variable in the polynomial: $x^2 + 4x - 21 = 0$
2. Solve for the variable in the polynomial: $20m^2 + 11m = 4$
3. Solve for the variable in the polynomial: $2e^2 + 7e + 6 = 0$

Answers:

1. $x^2 + 4x - 21 = (x-3)(x+7)$

$$(x-3)(x+7) = 0$$
$$\swarrow \qquad \searrow$$
$$(x-3) = 0 \qquad (x+7) = 0$$
$$x = 3 \qquad x = -7$$

2. $20m^2 + 11m = 4 \longrightarrow 20m^2 + 11m - 4 = 0$
$20m^2 + 11m - 4 = (4m - 1)(5m + 4)$

$$\begin{array}{cc} \multicolumn{2}{c}{(4m-1)(5m+4) = 0} \\ \swarrow & \searrow \\ 4m - 1 = 0 & 5m + 4 = 0 \\ 4m = 1 & 5m = -4 \\ m = \frac{1}{4} & m = \frac{-4}{5} \end{array}$$

3. $2e^2 + 7e + 6 = (2e + 3)(e + 2)$

$$\begin{array}{cc} \multicolumn{2}{c}{(2e+3)(e+2) = 0} \\ \swarrow & \searrow \\ 2e + 3 = 0 & e + 2 = 0 \\ 2e = -3 & e = -2 \\ e = \frac{-3}{2} & \end{array}$$

Practice

Solve for the variable in each of the following equations.

1. $(x+1)(x-3) = 0$
2. $(a+3)(a+5) = 0$
3. $(x-5)(x+4) = 0$
4. $(2t-4)(t+3) = 0$
5. $(x-8)(3x-7) = 0$
6. $x^2 + x - 12 = 0$
7. $b^2 + 2b - 24 = 0$
8. $t^2 + 3t - 18 = 0$
9. $w^2 + 3w - 108 = 0$
10. $e^2 - 2e - 99 = 0$
11. $6x^2 - x - 2 = 0$
12. $2d^2 + 14d - 16 = 0$
13. $3s^2 + 20s + 12 = 0$
14. $18x^2 + 12x + 2 = 0$
15. $3j^2 - 17j + 10 = 0$

6.9 Applications of Solving Equations with Factoring

Concept Problem

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles, the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

Watch This

Khan Academy - An Application Using Solving by Factoring

Guidance

Many real world problems will result in equations that can be solved with factoring. Here are two hints for solving these kinds of word problems:

1. It is often helpful to start by drawing a picture in order to visualize what you are asked to solve.
2. Once you have solved the problem, it is important to make sure that your answers are realistic given the context of the problem. For example, if you are solving for the age of a person and one of your answers is a negative number, that answer does not make sense in the context of the problem and is not actually a solution.

Many geometric questions can be answered using the Pythagorean Theorem, which relates the sides lengths in a right triangle. The **Pythagorean Theorem** says that the sides of a right triangle are related according to the equation $a^2+b^2=c^2$ where a and b are legs of the triangle and c is the hypotenuse.

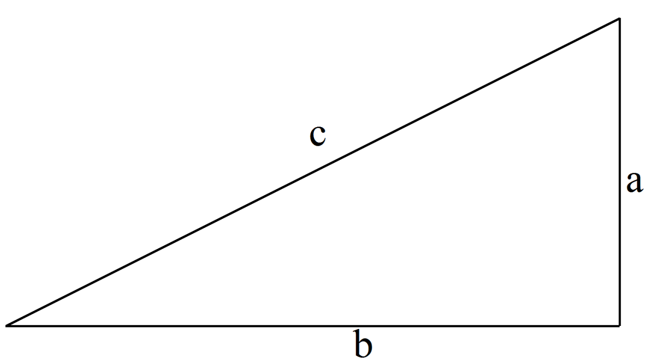


FIGURE 6.5

Example A

The number of softball games that must be scheduled in a league with n teams is given by $G(n)=\frac{n^2-n}{2}$. Each team can only play every other team exactly once. A league schedules 21 games. How many softball teams are in the league?

Solution: You are given the function $G(n) = \frac{n^2-n}{2}$ and you are asked to find n when $G(n) = 21$. This means, you have to solve the equation:

$21 = \frac{n^2-n}{2}$

First, multiply each side by 2 to clear the fractions:

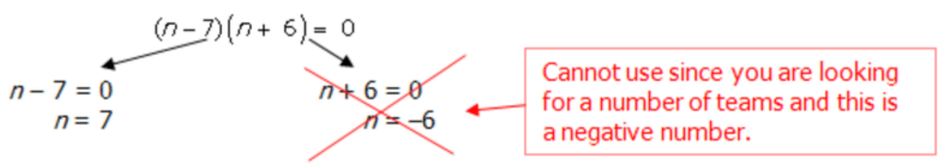
$$2 \cdot 21 = \left(\frac{n^2-n}{2}\right) \cdot 2$$
$$42 = n^2 - n$$

Next, set the equation equal to zero:

$$42 = n^2 - n$$
$$n^2 - n - 42 = 0$$

Now solve for n to find the number of teams (n) in the league. Start by factoring the left side of the equation and rewriting the equation:

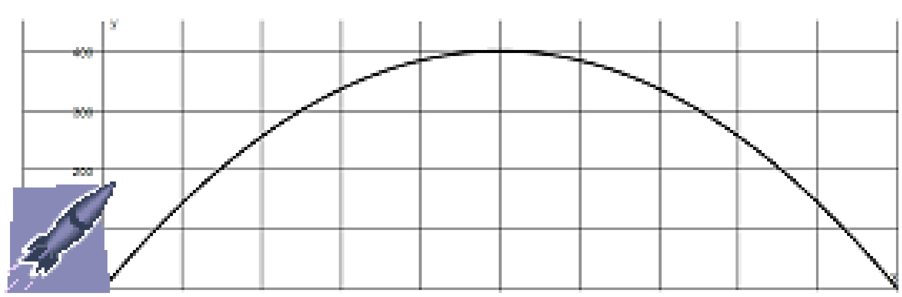
$n^2 - n - 42 = 0$ becomes $(n-7)(n+6) = 0$



There are 7 teams in the softball league.

Example B

When a home-made rocket is launched from the ground, it goes up and falls in the pattern of a parabola. The height, in feet, of a home-made rocket is given by the equation $h(t) = 160t - 16t^2$ where t is the time in seconds. How long will it take for the rocket to return to the ground?



Solution: The formula for the path of the rocket is $h(t) = 160t - 16t^2$. You are asked to find t when $h(t) = 0$, or when the rocket hits the ground and no longer has height. Start by factoring:

$160t - 16t^2 = 0$ becomes $16t(10-t) = 0$

This means $16t = 0$ (so $t = 0$) or $10 - t = 0$ (so $t = 10$). $t = 0$ represents the rocket being on the ground when it starts, so it is not the answer you are looking for. $t = 10$ represents the rocket landing back on the ground.

The rocket will hit the ground after 10 seconds.

Example C

Using the information in **Example B**, what is the height of the rocket after 2 seconds?

Solution: To solve this problem, you need to replace t with 2 in the quadratic function.

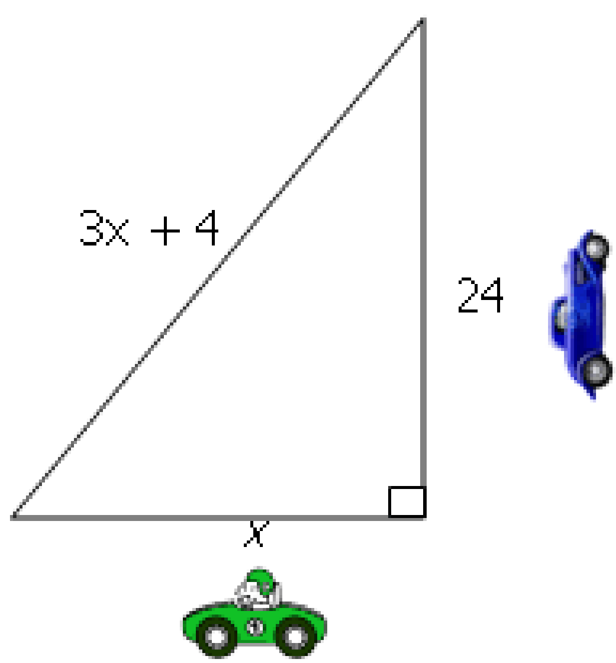
$$\begin{aligned} h(t) &= 160t - 16t^2 \\ h(2) &= 160(2) - 16(2)^2 \\ h(2) &= 320 - 64 \\ h(2) &= 256. \end{aligned}$$

Therefore, after 2 seconds, the height of the rocket is 256 feet.

Concept Problem Revisited

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

First draw a diagram. Since the cars are traveling north and west from the same starting position, the triangle made to connect the distance between them is a right triangle. Since you have a right triangle, you can use the Pythagorean Theorem to set up an equation relating the lengths of the sides of the triangle.

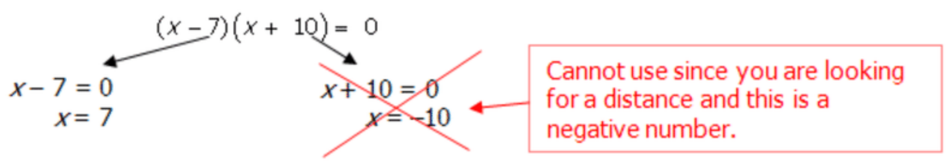


The Pythagorean Theorem is a geometry theorem that says that for all right triangles, $a^2 + b^2 = c^2$ where a and b are legs of the triangle and c is the longest side of the triangle, the hypotenuse. The equation for this problem is:

$$x^2 + 24^2 = (3x+4)^2$$
$$x^2 + 576 = (3x+4)(3x+4)$$
$$x^2 + 576 = 9x^2 + 24x + 16$$

Now set the equation equal to zero and factor the expression so that you can use the zero product property.

$$x^2 + 576 = 9x^2 + 24x + 16$$
$$0 = 8x^2 + 24x - 560$$
$$0 = 8(x^2 + 3x - 70)$$
$$0 = 8(x-7)(x+10)$$

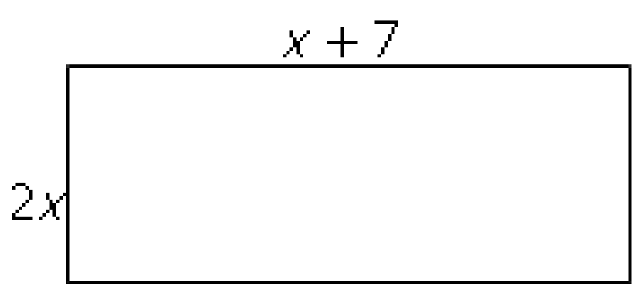


So you now know that $x = 7$. Since the distance between the cars is represented by the expression $3x+4$, the actual distance between the two cars after the car going north has traveled 24 miles is:

$$3x + 4 = 3(7) + 4$$
$$= 21 + 4$$
$$= 25 \ miles$$

Guided Practice

1. A rectangle is known to have an area of 520 square inches. The lengths of the sides are shown in the diagram below. Solve for both the length and the width.



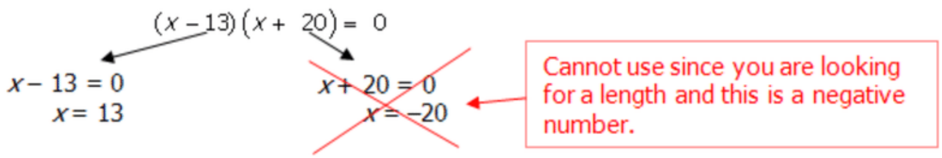
2. The height of a ball in feet can be found by the quadratic function $h(t) = -16t^2 + 80t + 5$ where t is the time in seconds that the ball is in the air. Determine the time(s) at which the ball is 69 feet high.

3. A manufacturer measures the number of cell phones sold using the binomial $0.015c + 2.81$. She also measures the wholesale price on these phones using the binomial $0.011c + 3.52$. Calculate her revenue if she sells 100,000 cell phones.

Answers:

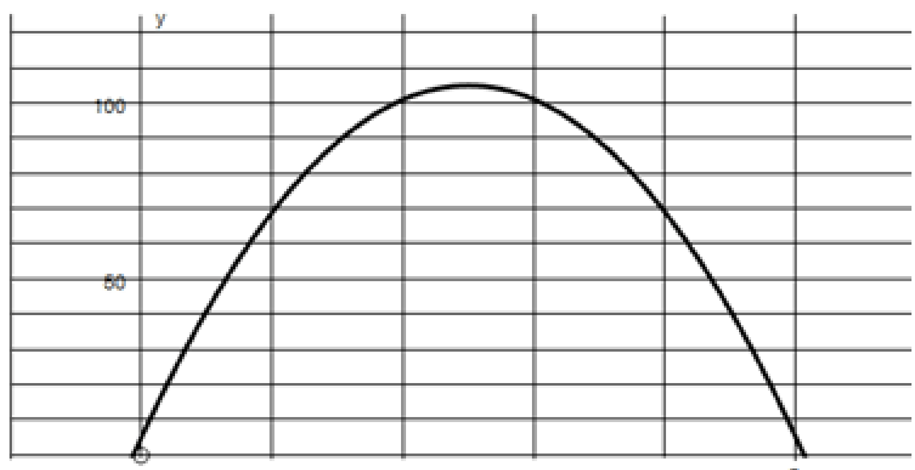
1. The rectangle has an area of 520 square inches and you know that the area of a rectangle has the formula: $A = l \times w$. Therefore:

$$\begin{aligned}520 &= (x+7)(2x)\\520 &= 2x^2+14x\\0 &= 2x^2+14x-520\\0 &= 2(x^2+7x-260)\\0 &= 2(x-13)(x+20)\end{aligned}$$



Therefore the value of x is 13. This means that the width is $2x$ or $2(13) = 26$ *inches*. The length is $x+7 = 13+7 = 20$ *inches*.

2. The equation for the ball being thrown is $h(t) = -16t^2+80t+5$. If you drew the path of the thrown ball, you would see something like that shown below.



You are asked to find the time(s) when the ball hits a height of 69 feet. In other words, solve for:

$$69 = -16t^2+80t+5$$

To solve for t, you have to factor the quadratic and then solve for the value(s) of t.

$$\begin{aligned}0 &= -16t^2+80t-64\\0 &= -16(t^2-5t+4)\\0 &= -16(t-1)(t-4)\end{aligned}$$

$$\swarrow \qquad \searrow$$

$$t-1=0 \qquad t-4=0$$
$$t=1 \qquad t=4$$

Since both values are positive, you can conclude that there are two times when the ball hits a height of 69 feet. These times are at 1 second and at 4 seconds.

3. The number of cell phones sold is the binomial $0.015c + 2.81$. The wholesale price on these phones is the binomial $0.011c + 3.52$. The revenue she takes in is the wholesale price times the number that she sells. Therefore:

$R(c) = (0.015c + 2.81)(0.011c + 3.52)$
First, let's expand the expression for R to get the quadratic expression. Therefore:

$$R(c) = (0.015c + 2.81)(0.011c + 3.52)$$
$$R(c) = 0.000165c^2 + 0.08371c + 9.8912$$

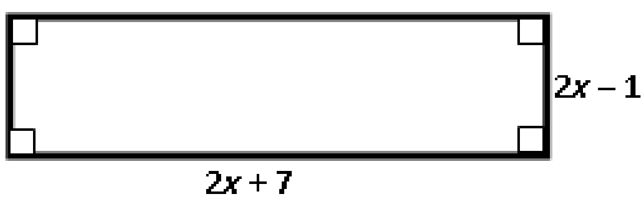
The question then asks if she sold 100,000 cell phones, what would her revenue be. Therefore what is $R(c)$ when $c = 100,000$.

$$R(c) = 0.000165c^2 + 0.08371c + 9.8912$$
$$R(c) = 0.000165(100,000)^2 + 0.08371(100,000) + 9.8912$$
$$R(c) = 1,658,380.89$$

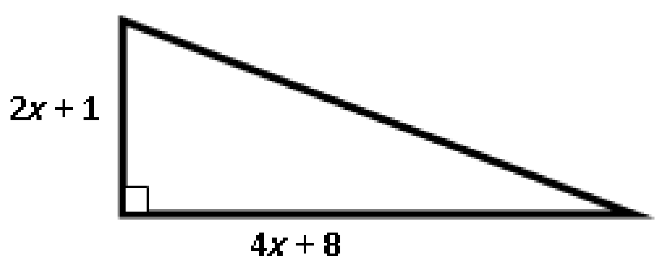
Therefore she would make $1,658,380.89 in revenue.

Practice

1. A rectangle is known to have an area of 234 square feet. The length of the rectangle is given by $x + 3$ and the width of the rectangle is given by $x + 8$. What is the value of x?
2. Solve for x in the rectangle below given that the area is 9 units.



3. Solve for x in the triangle below given that the area is 10 units.



A pool is treated with a chemical to reduce the amount of algae. The amount of algae in the pool t days after the treatment can be approximated by the function $A(t) = 40t^2 - 300t + 500$.

4. How many days after treatment will the pool have the no algae?

5. How much algae is in the pool before treatments are started?
6. How much algae is in the pool after 1 day?

A football is kicked into the air. The height of the football in meters can be found by the quadratic function $h(t) = -5t^2 + 25t$ where t is the time in seconds since the ball has been kicked.

7. How high is the ball after 3 seconds? At what other time is the ball the same height?
8. When will the ball be 20 meters above the ground?
9. After how many seconds will the ball hit the ground?

A ball is thrown into the air. The height of the ball in meters can be found by the quadratic function $h(t) = -5t^2 + 30t$ where t is the time in seconds since the ball has been thrown.

10. How high is the ball after 3 seconds?
11. When will the ball be 25 meters above the ground?
12. After how many seconds will the ball hit the ground?

Kim is drafting the windows for a new building. Their shape can be modeled by the function $h(w) = -w^2 + 4$, where h is the height and w is the width of points on the window frame, measured in meters.

13. Find the width of each window at its base.
14. Find the width of each window when the height is 3 meters.
15. What is the height of the window when the width is 1 meter?

6.10 Review - Division of a Polynomial by a Monomial

Learning Objectives

Here you'll learn how to divide a polynomial by a monomial.

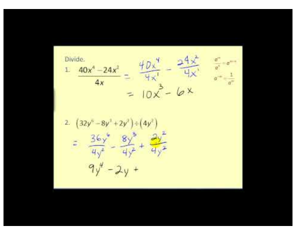
Concept Problem

Can you divide the polynomial by the monomial? How does this relate to factoring?

$$4e^4 + 6e^3 - 10e^2 \div 2e$$

Watch This

James Sousa: Dividing Polynomials by Monomials



MEDIA

Click image to the left or use the URL below.

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Guidance

Recall that a monomial is an algebraic expression that has only one term. So, for example, x, 8, -2, or $3ac$ are all monomials because they have only one term. The term can be a number, a variable, or a combination of a number and a variable. A polynomial is an algebraic expression that has more than one term.

When dividing polynomials by monomials, it is often easiest to separately divide each term in the polynomial by the monomial. When simplifying each mini-division problem, don't forget to use exponent rules for the variables. For example,

$$\frac{8x^5}{2x^3} = 4x^2$$

.

Remember that a fraction is just a division problem!

Example A

What is $(14s^2 - 21s + 42) \div (7)$?

Solution: This is the same as $\frac{14s^2-21s+42}{7}$. Divide each term of the polynomial numerator by the monomial denominator and simplify.

- $\frac{14s^4}{7} = 2s^4$
- $\frac{-21s}{7} = -3s$
- $\frac{42}{7} = 6$

Therefore, $(14s^2 - 21s + 42) \div (7) = 2s^4 - 3s + 6$.

Example B

What is $\frac{3w^3-18w^2-24w}{6w}$?

Solution: Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{3w^3}{6w} = \frac{w^2}{2}$
- $\frac{-18w^2}{6w} = -3w$
- $\frac{-24w}{6w} = -4$

Therefore, $\frac{3w^3-18w^2-24w}{6w} = \frac{w^2}{2} - 3w - 4$.

Example C

What is $(-27a^4b^5 + 81a^3b^4 - 18a^2b^3) \div (-9a^2b)$?

Solution: This is the same as $\frac{-27a^4b^5+81a^3b^4-18a^2b^3}{-9a^2b}$. Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{-27a^4b^5}{-9a^2b} = 3a^2b^4$
- $\frac{81a^3b^4}{-9a^2b} = -9ab^3$
- $\frac{-18a^2b^3}{-9a^2b} = 2b^2$

Therefore, $(-27a^4b^5 + 81a^3b^4 - 18a^2b^3) \div (-9a^2b) = 3a^2b^4 - 9ab^3 + 2b^2$.

Concept Problem Revisited

Can you divide the polynomial by the monomial? How does this relate to factoring?

$4e^4 + 6e^3 - 10e^2 \div 2e$

This process is the same as factoring out a $2e$ from the expression $4e^4 + 6e^3 - 10e^2$.

- $\frac{4e^4}{2e} = 2e^3$
- $\frac{6e^3}{2e} = 3e^2$
- $\frac{-10e^2}{2e} = -5e$

Therefore, $4e^4 + 6e^3 - 10e^2 \div 2e = 2e^3 + 3e^2 - 5e$.

Guided Practice

Complete the following division problems.

1. $(3a^5 - 5a^4 + 17a^3 - 9a^2) \div (a)$
2. $(-40n^3 - 32n^7 + 88n^{11} + 8n^2) \div (8n^2)$
3. $\frac{16m^6 - 12m^4 + 4m^2}{4m^2}$

Answers:

1. $(3a^5 - 5a^4 + 17a^3 - 9a^2) \div (a) = 3a^4 - 5a^3 + 17a^2 - 9a$
2. $(-40n^3 - 32n^7 + 88n^{11} + 8n^2) \div (8n^2) = -5n - 4n^5 + 11n^9 + 1$
3. $\frac{(16m^6 - 12m^4 + 4m^2)}{(4m^2)} = 4m^4 - 3m^2 + 1$

Practice

Complete the following division problems.

1. $(6a^3 + 30a^2 + 24a) \div 6$
2. $(15b^3 + 20b^2 + 5b) \div 5$
3. $(12c^4 + 18c^2 + 6c) \div 6c$
4. $(60d^{12} + 90d^{11} + 30d^8) \div 30d$
5. $(33e^7 + 99e^3 + 22e^2) \div 11e$
6. $(-8a^4 + 8a^2) \div (-4a)$
7. $(-3b^4 + 6b^3 - 30b^2 + 15b) \div (-3b)$
8. $(-40c^{12} - 20c^{11} - 25c^9 - 30c^3) \div 5c^2$
9. $(32d^{11} + 16d^7 + 24d^4 - 64d^2) \div 8d^2$
10. $(14e^{12} - 18e^{11} - 12e^{10} - 18e^7) \div -2e^5$
11. $(18a^{10} - 9a^8 + 72a^7 + 9a^5 + 3a^2) \div 3a^2$
12. $(-24b^9 + 42b^7 + 42b^6) \div -6b^3$
13. $(24c^{12} - 42c^7 - 18c^6) \div -2c^5$
14. $(14d^{12} + 21d^9 + 42d^7) \div -7d^4$
15. $(-40e^{12} + 30e^{10} - 10e^4 + 30e^3 + 80e) \div -10e^2$

Summary

You learned that adding, subtracting, and multiplying polynomials all rely on the distributive property and the rules of combining like terms. You also learned that factoring is the reverse of multiplying because when you factor a polynomial you are trying to rewrite the polynomial as a product of other polynomials. You learned how to factor completely by first looking for common factors and then using other methods to factor the remaining expression. You learned special cases of factoring to watch out for including the difference of perfect squares, perfect square trinomials, and the sum and difference of cubes. You learned how factoring can allow you to solve equations with the help of the zero product property.

CHAPTER 7 Unit 7 - Rational Expressions and Rational Functions

Chapter Outline

Introduction

Here you'll learn all about rational expressions. You'll start by learning how to apply your factoring skills to simplifying rational expressions. You'll then learn how operations with fractions generalize to operations with rational expressions. Finally, you will learn about rational functions. You will learn what the graphs of rational functions look like and how to find the asymptotes for rational functions.

7.1 Rational Expression Simplification

Learning Objectives

Here you'll learn how to simplify rational expressions.

Concept Problem

How could you use factoring to help simplify the following expression?

$$\frac{3x^2-27}{x^2+7x+12}$$

Watch This

Simplifying Rationals Expressions

Guidance

Recall that a rational number is a number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. So the numbers $\frac{2}{3}, -\frac{7}{11}$, and 4 (which can be written as $\frac{4}{1}$)are rational numbers.

Similarly, a **rational expression** is an algebraic expression of the form $\frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomials and $b(x) \neq 0$. The expressions below are examples of rational expressions:

$$\frac{x+3}{x-4} \qquad \frac{3}{2x+5} \qquad \frac{4x^2+20x+24}{2x^2+8x+8}$$

When working with rational numbers, we usually prefer to make them as simple as possible. For instance, given the number $\frac{36}{27}$, we would simplify it to $\frac{4}{3}$ by dividing the top and bottom by 9. Let's take a closer look at this process to see what's going on in the background.

Cancellation

Because a number divided by itself is 1 ($\frac{5}{5}=1$ for example), we can say the following:

$$\frac{ab}{ac}=\frac{a}{a}\cdot\frac{b}{c}=1\cdot\frac{b}{c}=\frac{b}{c}$$

Thus, fractions have the property of *cancellation:*

$$\frac{ab}{ac} = \frac{b}{c}$$

Notice that on the left side of the equation the numerator and denominator are written in terms of multiplication. In other words, they are factored. In fact, when we simplify a numeric fraction, we are really factoring behind the scenes.

For example, with the number $\frac{36}{27}$, we notice that the numerator and denominator are both divisible by 9. What we are really doing is factoring:

$$\frac{36}{27} = \frac{9 \cdot 4}{9 \cdot 3} = \frac{4}{3}$$

Simplifying Expressions

The same cancellation property holds for rational expressions. We can cancel any factors that the numerator and denominator have in common. Consider the following rational expression:

$$\frac{4x^2 + 20x + 24}{2x^2 + 8x + 8}$$

Factor both the numerator and denominator completely:

$$\frac{4(x+2)(x+3)}{2(x+2)(x+2)}$$

Notice that there is one factor of $x+2$ in both the numerator and denominator. These factors "cancel out" (the second factor of $(x+2)$ in the denominator will remain there).

$$\frac{4\cancel{(x+2)}(x+3)}{2\cancel{(x+2)}(x+2)}$$

Also, the $\frac{4}{2}$ reduces to just 2. The simplified expression is:

$$\frac{2(x+3)}{x+2}$$

Keep in mind that you cannot "cancel out" common factors until both the numerator and denominator have been factored.

A rational expression is like any other fraction in that it is said to be undefined if the denominator is equal to zero. Values of the variable that cause the denominator of a rational expression to be zero are referred to as ***restrictions*** and must be excluded from the set of possible values for the variable. For the original expression above, the restriction is $x \neq -2$ because if $x = -2$ then the denominator would be equal to zero. Note that to determine the restrictions you must look at the **original** expression before any common factors have been cancelled.

Example A

Simplify the following and state any restrictions on the denominator.

$$\frac{x-2}{x^2-10x+16}$$

Solution: To begin, factor both the numerator and the denominator:

$$\frac{x-2}{(x-8)(x-2)}$$

Cancel out the common factor of $x-2$ to create the simplified expression:

$$\frac{\cancel{(x-2)}}{(x-8)\cancel{(x-2)}} = \frac{1}{x-8}$$

The restrictions are $x \neq 2$ and $x \neq 8$ because both of those values for x would have made the denominator of the original expression equal to zero.

Example B

Simplify the following and state any restrictions on the denominator.

$$\frac{x^2+7x+12}{x^2-16}$$

Solution: To begin, factor both the numerator and the denominator:

$$\frac{(x+4)(x+3)}{(x-4)(x+4)}$$

Cancel out the common factor of $x+4$ to create the simplified expression:

$$\frac{\cancel{(x+4)}(x+3)}{(x-4)\cancel{(x+4)}} = \frac{x+3}{x-4}$$

The restrictions are $x \neq 4$ and $x \neq -4$ because both of those values for x would have made the denominator of the original expression equal to zero.

Example C

Simplify the following and state any restrictions on the denominator.

$$\frac{3x^2-7x-6}{4x^2-13x+3}$$

Solution: To begin, factor both the numerator and the denominator:

$$\frac{(x-3)(3x+2)}{(x-3)(4x-1)}$$

Cancel out the common factor of $x-3$ to create the simplified expression:

$$\frac{\cancel{(x-3)}(3x+2)}{\cancel{(x-3)}(4x-1)} = \frac{3x+2}{4x-1}$$

The restrictions are $x \neq 3$ and $x \neq \frac{1}{4}$ because both of those values for x would have made the denominator of the original expression equal to zero.

Concept Problem Revisited

$$\begin{aligned}
&\frac{3x^2-27}{x^2+7x+12}\\
=&\frac{3(x^2-9)}{(x+3)(x+4)}\\
=&\frac{3(x+3)(x-3)}{(x+3)(x+4)}\\
=&\frac{3\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+4)}\\
=&\frac{3(x-3)}{x+4}
\end{aligned}$$

where $x \neq -3$ and $x \neq -4$

Vocabulary

Rational Expression

A **rational expression** is an algebraic expression that can be written in the form $\frac{a(x)}{b(x)}$ where $a(x)$ and $b(x)$ are polynomials and $b(x) \neq 0$.

Guided Practice

Simplify each of the following and state the restrictions.

1. $\frac{m^2-9m+18}{4m^2-24m}$
2. $\frac{2x^2-8}{4x+8}$
3. $\frac{c^2+4c-5}{c^2-2c-35}$

Answers:

1. $\frac{m^2-9m+18}{4m^2-24m} = \frac{(m-6)(m-3)}{(4m)(m-6)} = \frac{(m-3)}{4m}, m \neq 0; m \neq 6$
2. $\frac{2x^2-8}{4x+8} = \frac{(2)(x-2)(x+2)}{(4)(x+2)} = \frac{(x-2)}{2}, x \neq -2$
3. $\frac{c^2+4c-5}{c^2-2c-35} = \frac{(c+5)(c-1)}{(c-7)(c+5)} = \frac{(c-1)}{(c-7)}, c \neq -5; c \neq 7$

Practice

For each of the following rational expressions, state the restrictions.

1. $\frac{7}{x+4}$
2. $\frac{-3}{x-5}$
3. $\frac{5x+1}{5x-1}$
4. $\frac{6}{4x-3}$
5. $\frac{(x+1)}{x^2-4}$
6. $\frac{x-8}{x^2+3x+2}$
7. $\frac{x+6}{x^2-5x-24}$
8. $\frac{5x+2}{2x^2+5x+2}$

Simplify each of the following rational expressions and state the restrictions.

9. $\frac{4}{4x+12}$
10. $\frac{4c^2}{8c^2-4c}$
11. $\frac{10x+5}{2x+1}$
12. $\frac{x-4}{x^2-16}$
13. $\frac{y+1}{y^2+5y+4}$
14. $\frac{c+2}{c^2-5c-14}$
15. $\frac{(b-3)^2}{b^2-6b+9}$
16. $\frac{3n^2-27}{6n+18}$
17. $\frac{6k^2+7k-20}{12k^2-19k+4}$
18. $\frac{4x^2-4x-3}{2x^2+3x-9}$

7.2 Rational Expression Multiplication and Division

Learning Objectives

Here you'll learn how to multiply and divide rational expressions.

Concept Problem

How can you use your knowledge of multiplying fractions to multiply the following rational expressions?

$$\frac{10y+20}{5y-15}\cdot\frac{y-3}{y^2+10y+16}$$

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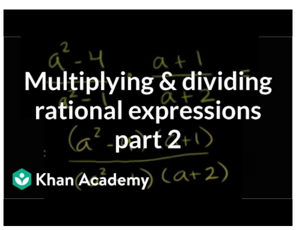
Khan Academy Multiplying and Dividing Rational Expressions



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Guidance

To see how we multiply and divide rational expressions, we will again start by looking at numeric fractions. Suppose we want to multiply $\frac{130}{33}\cdot\frac{63}{1274}$. There are two options for doing this.

Option A

Multiply first and then simplify:

$$\frac{130}{33}\cdot\frac{63}{1274}=\frac{8190}{42042}=\frac{546\cdot 15}{546\cdot 77}=\frac{15}{77}$$

This option is straightforward; however, it can be difficult to factor the large numbers.

Option B

Factor all the numbers first and then cancel before multiplying:

$$\frac{130}{33}\cdot\frac{63}{1274}=\frac{2\cdot5\cdot13}{3\cdot11}\cdot\frac{3\cdot3\cdot7}{2\cdot7\cdot7\cdot13}=\frac{\cancel{2}\cdot5\cdot\cancel{13}}{\cancel{3}\cdot11}\cdot\frac{\cancel{3}\cdot3\cdot\cancel{7}}{\cancel{2}\cdot\cancel{7}\cdot7\cdot\cancel{13}}=\frac{5\cdot3}{11\cdot7}=\frac{15}{77}$$

The computation is easier because we do the factoring before the multiplication. This is the method that we will extend to use with rational expressions

Multiplying Rational Expressions

To multiply rational expressions we will take the same strategy as we did with Option B above: factor everything and then see what cancels. As with numbers, after multiplying or dividing we will want to simplify your result.

To multiply:

$$\frac{x^2+2x}{x+3}\cdot\frac{x^2+4x+3}{x}$$

First, factor all expressions that can be factored:

$$\frac{x(x+2)}{x+3}\cdot\frac{(x+3)(x+1)}{x}$$

Next, multiply the numerators and multiply the denominators to create one big rational expression. Leave in factored form:

$$\frac{x(x+2)(x+3)(x+1)}{x(x+3)}$$

Simplify:

$$\frac{\cancel{x}(x+2)\cancel{(x+3)}(x+1)}{\cancel{x}\cancel{(x+3)}}$$

$$=(x+2)(x+1)$$

$$=x^2+3x+2$$

To divide rational expressions, recall that dividing one fraction by another is the same as multiplying the first fraction by the reciprocal of the second fraction. For example, $\frac{x^2+2x}{x+3}\div\frac{x}{x^2+4x+3}$ is equivalent to, and can be rewritten as, $\frac{x^2+2x}{x+3}\cdot\frac{x^2+4x+3}{x}$, which can then be solved using the same steps as above.

Example A

Multiply the following rational expressions and state the restrictions.

$$\frac{4x-8}{x^2-7x+10} \cdot \frac{x^2-3x-10}{x^2-4}$$

Solution: Begin by factoring the numerator and denominator of each expression:

$$\frac{4(x-2)}{(x-5)(x-2)} \cdot \frac{(x-5)(x+2)}{(x-2)(x+2)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$$\frac{4(x-2)(x-5)(x+2)}{(x-5)(x-2)(x-2)(x+2)}$$

Simplify by removing common factors from the numerator and denominator:

$$\frac{4\cancel{(x-2)}\cancel{(x-5)}\cancel{(x+2)}}{\cancel{(x-5)}\cancel{(x-2)}\cancel{(x+2)}(x-2)}$$

The final answer is: $\frac{4}{(x-2)}$.

Example B

Divide the following rational expressions and state the restrictions.

$$\frac{m^2-4}{m^2+9m+14} \div \frac{3m^2-6m}{m^2-49}$$

Solution: To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$$\frac{m^2-4}{m^2+9m+14} \cdot \frac{m^2-49}{3m^2-6m}$$

Begin by factoring the numerator and denominator of each expression.

$$\frac{(m-2)(m+2)}{(m+7)(m+2)} \cdot \frac{(m-7)(m+7)}{3m(m-2)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$$\frac{(m-2)(m+2)(m-7)(m+7)}{3m(m+7)(m+2)(m-2)}$$

Simplify by removing common factors:

$$\frac{\cancel{(m+2)}\cancel{(m-2)}\cancel{(m+7)}(m-7)}{3m\cancel{(m-2)}\cancel{(m+7)}\cancel{(m+2)}}$$

The final answer is: $\frac{(m-7)}{3m}$.

Example C

Simplify the following rational expressions and state the restrictions.

$$\frac{12x^2+13x-35}{5x^2-21x+18} \div \frac{3x^2+16x+21}{5x^2+9x-18}$$

Solution: To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$$\frac{12x^2+13x-35}{5x^2-21x+18} \times \frac{5x^2+9x-18}{3x^2+16x+21}$$

Begin by factoring the numerator and denominator of each expression.

$$\frac{(4x-5)(3x+7)}{(5x-6)(x-3)} \times \frac{(5x-6)(x+3)}{(3x+7)(x+3)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$$\frac{(4x-5)(3x+7)(5x-6)(x+3)}{(5x-6)(x-3)(3x+7)(x+3)}$$

Simplify by removing common factors:

$$\frac{(4x-5)\cancel{(3x+7)}\cancel{(5x-6)}\cancel{(x+3)}}{\cancel{(5x-6)}(x-3)\cancel{(3x+7)}\cancel{(x+3)}}$$

The final answer is: $\frac{(4x-5)}{(x-3)}$.

Concept Problem Revisited

$$\frac{10y+20}{5y-15} \cdot \frac{y-3}{y^2+10y+16}$$

Begin by factoring the numerator and denominator of each expression.

$$\frac{10(y+2)}{5(y-3)} \cdot \frac{y-3}{(y+8)(y+2)}$$

Express the factored rational expressions as a single rational expression.

$$\frac{10(y+2)(y-3)}{5(y-3)(y+8)(y+2)}$$

Cancel the common factors that exist in the numerator and the denominator.

$$\frac{\overset{2}{\cancel{10}}\cancel{(y+2)}\cancel{(y-3)}}{\cancel{5}\cancel{(y-3)}(y+8)\cancel{(y+2)}}$$

The result of cancelling the common factors is the answer.

$$\boxed{\frac{2}{y+8}}$$

Guided Practice

Multiply or divide each of the following and state the restrictions.

1. $\frac{x+7}{x^2-5x-36} \div \frac{x^2-2x-63}{x+4} \cdot \frac{x^2-15x+54}{x^2-36}$
2. $\frac{y^2-25}{y^2-6y} \cdot \frac{y^2-12y+36}{y^2+2y-15} \div \frac{y^2-11y+30}{y^2+4y-21}$
3. $\frac{2x^2+7x-4}{6x^2+x-2} \cdot \frac{15x^2+7x-2}{5x^2+19x-4}$

Answers:

1. Write the term after the division sign as a reciprocal and multiply.

$\frac{x+7}{x^2-5x-36} \cdot \frac{x+4}{x^2-2x-63} \cdot \frac{x^2-15x+54}{x^2-36}$

Factor the numerator and denominator of each expression.

$\frac{x+7}{(x-9)(x+4)} \cdot \frac{x+4}{(x-9)(x+7)} \cdot \frac{(x-9)(x-6)}{(x+6)(x-6)}$

Express the factored rational expressions as a single rational expression.

$\frac{(x+7)(x+4)(x-9)(x-6)}{(x-9)(x+4)(x-9)(x+7)(x+6)(x-6)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\overbrace{\cancel{(x+7)}\cancel{(x+4)}\cancel{(x-9)}\cancel{(x-6)}}^{1}}{\cancel{(x-9)}\cancel{(x+4)}(x-9)\cancel{(x+7)}(x+6)\cancel{(x-6)}}$

The result of cancelling the common factors is the answer.

$$\boxed{=\frac{1}{(x-9)(x+6)}}$$

2. Write the term after the division sign as a reciprocal and multiply.

$\frac{y^2-25}{y^2-6y}\cdot\frac{y^2-12y+36}{y^2+2y-15}\cdot\frac{y^2+4y-21}{y^2-11y+30}$

Factor the numerator and denominator of each expression.

$\frac{(y+5)(y-5)}{y(y-6)}\cdot\frac{(y-6)(y-6)}{(y+5)(y-3)}\cdot\frac{(y+7)(y-3)}{(y-6)(y-5)}$

Express the factored rational expressions as a single rational expression.

$\frac{(y+5)(y-5)(y-6)(y-6)(y+7)(y-3)}{y(y-6)(y+5)(y-3)(y-6)(y-5)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\cancel{(y+5)}\cancel{(y-5)}\cancel{(y-6)}\cancel{(y-6)}(y+7)\cancel{(y-3)}}{y\cancel{(y-6)}\cancel{(y+5)}\cancel{(y-3)}\cancel{(y-6)}\cancel{(y-5)}}$

The result of cancelling the common factors is the answer.

$$\boxed{=\frac{y+7}{y}}$$

3. Factor the numerator and denominator of each expression.

$\frac{(2x-1)(x+4)}{(2x-1)(3x+2)}\cdot\frac{(5x-1)(3x+2)}{(5x-1)(x+4)}$

Express the factored rational expressions as a single rational expression.

$\frac{(2x-1)(x+4)(5x-1)(3x+2)}{(2x-1)(3x+2)(5x-1)(x+4)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\overbrace{\cancel{(2x-1)}\cancel{(x+4)}\cancel{(5x-1)}\cancel{(3x+2)}}^{1}}{\cancel{(2x-1)}\cancel{(3x+2)}\cancel{(5x-1)}\cancel{(x+4)}}$

The result of cancelling the common factors is the answer. In this case every factor cancels so the result is

$$\boxed{\frac{1}{1}=1}$$

Practice

Multiply or divide each of the following.

1. $\frac{3x+9}{6x}\cdot\frac{x^2}{x^2-9}$
2. $\frac{c^2+5c+6}{c-1}\cdot\frac{c^2-1}{c+3}$
3. $\frac{a^2+3a}{3a-9}\cdot\frac{a^2-a-6}{2a^2+6a}$

4. $\frac{y-3}{y+3} \cdot \frac{y^2-9}{y+3} \cdot \frac{y^2+6y+9}{y^2-6y+9}$
5. $\frac{m^2-4m-5}{m^2-5m} \cdot \frac{m^2-6m+5}{m^2-1} \cdot \frac{m}{m-5}$
6. $\frac{x^2-x-20}{x^2-25} \div \frac{3x+12}{x+5}$
7. $\frac{d^2-9}{3-3d} \div \frac{d^2+5d+6}{d^2+3d-4}$
8. $\frac{4x^2-20x}{3x+6} \div \frac{x-5}{x^2-x-6}$
9. $\frac{4n^2-9}{2n^3+2n^2-4n} \div \frac{2n^2-n-3}{3n^2-6n+3}$
10. $\frac{e^2+10e+21}{2e^2+7e-15} \div \frac{e^2+8e+15}{e^2+10e+25}$
11. $\frac{x^2+2x-15}{x^2-6x+8} \cdot \frac{x^2+2x-8}{x^2-6x+9} \cdot \frac{x^2-7x+12}{x^2-x-30}$
12. $\frac{2x^2+5x-3}{4x^2-12x+5} \div \frac{3x^2+13x+12}{6x^2-7x-20}$
13. $\frac{5m^2-20}{m^2+14m+33} \cdot \frac{m^2+10m-11}{m^2-8m+12} \cdot \frac{m^2-3m-18}{m^2+m-2}$
14. $\frac{2y^2+5y-12}{y^2+9y+14} \div \frac{6y^2-7y-3}{3y^2+25y+8} \cdot \frac{y^2+3y-28}{y^2-16}$
15. $\frac{x^2-49}{x^2+3x-88} \cdot \frac{x^2+6x-55}{x^2-11x+28} \div \frac{x^2+2x-35}{x^2-12x+32}$

7.3 Rational Expression Addition and Subtraction

Learning Objectives

Here you'll learn how to add and subtract rational expressions.

Concept Problem

Can you use your knowledge of rational expressions and adding fractions to add the following rational expressions?

$$\frac{3x}{x^2+6x-16}+\frac{2x}{x-2}$$

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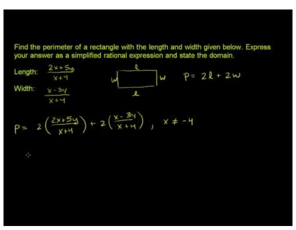
Khan Academy Adding and Subtracting Rational Expressions



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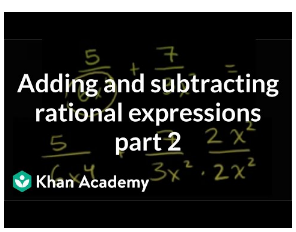

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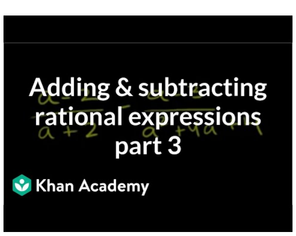

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Guidance

Rational expressions are examples of fractions, so you add and subtract rational expressions in the same way that you add and subtract numeric fractions. As with fractions, you will need a common denominator, ideally the lowest common denominator (LCD), in order to add or subtract.

For example, to add:

$$\frac{x^2+2x}{x+3}+\frac{x}{x^2+4x+3}$$

First, factor the denominators to get:

$$\frac{x^2+2x}{x+3}+\frac{x}{(x+3)(x+1)}$$

.

Next, find the lowest common denominator (LCD). This will be the product of each unique factor in the denominator raised to its highest power. In this case, the LCD is $(x+3)(x+1)$.

Note that the LCD is **NOT** simply the product of the denominators. In this case that would be $(x+3)^2(x+1)$.

Multiply the numerator and denominator of each fraction by the factors necessary to create the common denominator. In this case, you only need to multiply the fraction on the left by $\left(\frac{x+1}{x+1}\right)$. The expression becomes:

$$\frac{x^2+2x}{x+3}\left(\frac{x+1}{x+1}\right)+\frac{x}{(x+3)(x+1)}$$

$$=\frac{(x^2+2x)(x+1)}{(x+3)(x+1)}+\frac{x}{(x+3)(x+1)}$$

Now add the numerators and write as one rational expression:

$$=\frac{(x^2+2x)(x+1)+x}{(x+3)(x+1)}$$

Simplify the numerator by multiplying, combining like terms, and factoring if possible (the denominator is left in factored form):

$$\frac{x^3+3x^2+3x}{(x+3)(x+1)}$$

$$=\frac{x(x^2+3x+3)}{(x+3)(x+1)}$$

The rational expression cannot be simplified any further so this is your answer. The restrictions are $x \neq -3$ and $x \neq -1$ because those values would cause one or both of the original denominators to be equal to zero.

Example A

Identify the lowest common denominator (LCD) in factored form.

i) $\frac{2x-3}{x^2-7x+10} - \frac{x-5}{x^2-2x-15}$

ii) $\frac{2x+1}{x^2+6x+9} + \frac{3x-2}{x^2+x-6}$

Solution: To determine the LCD, begin by factoring the denominators.

i) $\frac{2x-3}{x^2-7x+10} - \frac{x-5}{x^2-2x-15} = \frac{2x-3}{(x-5)(x-2)} - \frac{x-5}{(x-5)(x+3)}$

The LCD is

$$\boxed{(x-5)(x-2)(x+3)}$$

ii) $\frac{2x+1}{x^2+6x+9} + \frac{3x-2}{x^2+x-6} = \frac{2x+1}{(x+3)(x+3)} + \frac{3x-2}{(x+3)(x-2)}$

The LCD is

$$\boxed{(x+3)(x+3)(x-2)}$$

Example B

Add the following rational expressions and state the restrictions.

$$\frac{3x+1}{x^2+8x+16} + \frac{2x-3}{x^2+x-12}$$

Solution: Begin by determining the LCD. Factor the denominators of each expression.

$$\frac{3x+1}{x^2+8x+16} + \frac{2x-3}{x^2+x-12} = \frac{3x+1}{(x+4)(x+4)} + \frac{2x-3}{(x+4)(x-3)}$$

The LCD is

$$\boxed{(x+4)(x+4)(x-3)}$$

Multiply the numerators and denominators of each expression by the necessary factors to create the LCD.

$$\frac{3x+1}{(x+4)(x+4)}\left(\frac{x-3}{x-3}\right) + \frac{2x-3}{(x+4)(x-3)}\left(\frac{x+4}{x+4}\right)$$

Multiply the numerators. Keep the denominators in factored form.

$$\frac{3x^2-8x-3}{(x+4)(x+4)(x-3)} + \frac{2x^2+5x-12}{(x+4)(x+4)(x-3)}$$

Write the two expressions as one rational expression.

$$\frac{3x^2-8x-3+2x^2+5x-12}{(x+4)(x+4)(x-3)}$$

Simplify the numerator by combining like terms.

$$\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)}$$

The numerator cannot be factored so the expression cannot be further simplified. The answer in lowest terms is:

$$\boxed{\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)}}$$

Example C

Subtract the following rational expressions.

$$\frac{x}{x^2-9x+18}-\frac{x-2}{x^2-10x+24}$$

Solution: Begin by determining the LCD. Factor the denominators of each expression.

$$\frac{x}{(x-6)(x-3)}-\frac{x-2}{(x-6)(x-4)}$$

The LCD is

$$\boxed{(x-6)(x-3)(x-4)}$$

Multiply the numerators and denominators of each expression to get the LCD.

$$\frac{x}{(x-6)(x-3)}\left(\frac{x-4}{x-4}\right)-\frac{x-2}{(x-6)(x-4)}\left(\frac{x-3}{x-3}\right)$$

Multiply the numerators.

$$\frac{x^2-4x}{(x-6)(x-3)(x-4)}-\frac{x^2-5x+6}{(x-6)(x-3)(x-4)}$$

Write the expressions as one rational expression.

$$\frac{x^2-4x-(x^2-5x+6)}{(x-6)(x-3)(x-4)}$$

$$= \frac{x^2 - 4x - x^2 + 5x - 6}{(x-6)(x-3)(x-4)}$$

Simplify the numerator by combining like terms.

$$\frac{x-6}{(x-6)(x-3)(x-4)}$$

The term $(x-6)$ is common to both the numerator and the denominator. This term can be "cancelled." The solution is:

$$\boxed{\frac{1}{(x-3)(x-4)}}$$

Concept Problem Revisited

$$\frac{3x}{x^2 + 6x - 16} + \frac{2x}{x-2}$$

Factor the denominator of the first fraction and rewrite the problem:

$$\frac{3x}{(x+8)(x-2)} + \frac{2x}{x-2}$$

The LCD is $(x+8)(x-2)$.

$$\frac{3x}{(x+8)(x-2)} + \frac{2x}{x-2}\left(\frac{x+8}{x+8}\right)$$

Multiply the numerators.

$$\frac{3x}{(x+8)(x-2)} + \frac{2x^2 + 16x}{(x-2)(x+8)}$$

Write the two expressions a one rational expression.

$$\frac{3x + 2x^2 + 16x}{(x+8)(x-2)}$$

Simplify the numerator by combining like terms. Your final answer is:

$$\boxed{\frac{2x^2 + 19x}{(x+8)(x-2)}}$$

Guided Practice

Add or subtract the following and state the restrictions.

1. $\frac{2x}{x^2-4} - \frac{1}{x-2}$
2. $\frac{-2}{3y^2+5y+2} + \frac{3}{y^2-7y-8}$
3. $\frac{3m-1}{9m^3-36m^2} + \frac{2m+1}{2m^2-5m-12}$

Answers:

1.

$$\begin{aligned}\frac{2x}{x^2-4} - \frac{1}{x-2} &= \frac{2x}{(x-2)(x+2)} - \frac{x+2}{(x-2)(x+2)}\\ &= \frac{2x-(x+2)}{(x-2)(x+2)}\\ &= \frac{(x-2)}{(x-2)(x+2)}\\ &= \frac{1}{(x+2)}\end{aligned}$$

2.

$$\begin{aligned}\frac{-2}{3y^2+5y+2} + \frac{3}{y^2-7y-8} &= \frac{-2}{(3y+2)(y+1)} + \frac{3}{(y-8)(y+1)}\\ &= \frac{-2(y-8)}{(3y+2)(y+1)(y-8)} + \frac{3(3y+2)}{(3y+2)(y-8)(y+1)}\\ &= \frac{-2y+16+9y+6}{(3y+2)(y-8)(y+1)}\\ &= \frac{7y+22}{(3y+2)(y-8)(y+1)}\end{aligned}$$

3.

$$\begin{aligned}\frac{3m-1}{9m^3-36m^2} + \frac{2m+1}{2m^2-5m-12} &= \frac{3m-1}{9m^2(m-4)} + \frac{2m+1}{(2m+3)(m-4)}\\ &= \frac{(3m-1)(2m+3)}{9m^2(m-4)(2m+3)} + \frac{9m^2(2m+1)}{9m^2(2m+3)(m-4)}\\ &= \frac{6m^2-2m+9m-3+18m^3+9m^2}{9m^2(m-4)(2m+3)}\\ &= \frac{18m^3+15m^2+7m-3}{9m^2(m-4)(2m+3)}\end{aligned}$$

Practice

For each of the following rational expressions, determine the LCD.

1. $\frac{2a-3}{4} + \frac{3a-1}{5} - \frac{a-5}{2}$

2. $\frac{5}{3x^2} - \frac{1}{2x} + \frac{3}{5x^3}$
3. $\frac{x}{a^2b} - \frac{y}{ab^2} + \frac{z}{3a^3b^2}$
4. $\frac{2w}{w^2-6w+5} - \frac{3w}{w^2-11w+30}$
5. $\frac{1}{y^2+5y} - \frac{2}{y^2+12y+35} - \frac{3}{y^3+7y^2}$

Perform the indicated operation for each of the following rational expressions.

6. $\frac{3}{x^2-5x+4} + \frac{4}{x^2-16}$
7. $\frac{5}{a^2+a} - \frac{2}{a^2+3a+2}$
8. $\frac{6}{m^2-5m} + \frac{7}{m^2-4m-5}$
9. $\frac{3n}{n^2+2n-3} - \frac{4n}{n^2+n-6}$
10. $\frac{6}{y^2-4} + \frac{4}{y^2+4y+4}$

11. $\frac{2a-3}{4} + \frac{3a-1}{5} - \frac{a-5}{2}$
12. $\frac{5}{3x^2} - \frac{1}{2x} + \frac{3}{5x^3}$
13. $\frac{x}{a^2b} - \frac{y}{ab^2} + \frac{z}{3a^3b^2}$
14. $\frac{2w}{w^2-6w+5} - \frac{3w}{w^2-11w+30}$
15. $\frac{1}{y^2+5y} - \frac{2}{y^2+12y+35} - \frac{3}{y^3+7y^2}$

7.4 Solving Rational Equations

Concept Problem

A woman can bike 96 miles one hour faster than it takes her to walk 18 miles. If she bikes 10 miles per hour faster than she walks, how fast can she walk?

We have seen problems involving distance, rate, and time before, but, as we will see in this section, the information that is given in this case results in an equation that involves rational expressions. How can we solve such an equation?

Solving Rational Equations

In the section on solving linear equations, we learned how to clear the denominators from numerical fractions from an equation by multiplying both sides by the LCD. The same technique will work with more complicated denominators. We can use the LCD of all the rational expressions within the equation and eliminate the fractions. To demonstrate, we will walk through a few problems.

Solve the following rational equation:

$$\frac{5}{2}+\frac{1}{x}=3$$

The LCD for 2 and x is $2x$. Multiply each term by $2x$, so that the denominators are eliminated. We can write the $2x$ as $\frac{2x}{1}$ when multiplying it by the fractions, so that it is easier to line up and cross-cancel.

$$\begin{aligned}\frac{5}{2}+\frac{1}{x}&=3\\ \frac{\cancel{2}x}{1}\cdot\frac{5}{\cancel{2}}+\frac{2\cancel{x}}{1}\cdot\frac{1}{\cancel{x}}&=2x\cdot 3\\ 5x+2&=6x\\ 2&=x\end{aligned}$$

Checking the answer, we have $\frac{5}{2}+\frac{1}{2}=3\rightarrow\frac{6}{2}=3 \quad \checkmark$

Example A

Solve the following rational equation:

$$\frac{5x}{x-2}=7+\frac{10}{x-2}$$

Because the denominators are the same, we need to multiply all three terms by $x-2$.

$$\begin{aligned}
\frac{5x}{x-2} &= 7+\frac{10}{x-2} \\
\cancel{(x-2)}\cdot\frac{5x}{\cancel{x-2}} &= (x-2)\cdot 7+\cancel{(x-2)}\cdot\frac{10}{\cancel{x-2}} \\
5x &= 7x-14+10 \\
-2x &= -4 \\
x &= 2
\end{aligned}$$

Checking our answer, we have: $\frac{5\cdot 2}{2-2} = 7+\frac{10}{2-2} \rightarrow \frac{10}{0} = 7+\frac{10}{0}$. Because division by zero is undefined, $x=2$ is an extraneous solution. Therefore, there is no solution to this problem.

Example B

Solve the following rational equation:

$$\frac{3}{x}+1=\frac{6}{x-2}$$

Determine the LCD for x, and $x-2$. It would be the two terms multiplied together: $x(x-2)$. Multiply each term by the LCD.

$$\begin{aligned}
\frac{3}{x}+1 &= \frac{6}{x-2} \\
\frac{\cancel{x}(x-2)}{1}\cdot\frac{3}{\cancel{x}}+\frac{x(x-2)}{1}\cdot 1 &= \frac{x\cancel{(x-2)}}{1}\cdot\frac{6}{\cancel{x-2}} \\
3(x-2)+x(x-2) &= 6x
\end{aligned}$$

Multiplying each term by the entire LCD cancels out each denominator, so that we have an equation that we have learned how to solve in previous concepts. Distribute the 15 and $4x$, combine like terms and solve.

$$\begin{aligned}
3x-6+x^2-2x &= 6x \\
x^2-5x-6 &= 0
\end{aligned}$$

This polynomial is factorable!

$$\begin{aligned}
(x-6)(x+1) &= 0 \\
x = 6 \ \ or \ \ x &= -1
\end{aligned}$$

Now, we check these potential solutions by plugging them into the original equation:

Check x = 6:

$$\frac{3}{6}+1=\frac{6}{6-2}$$
$$\frac{3}{6}+1=\frac{6}{4}$$
$$\frac{1}{2}+1=\frac{3}{2}$$
$$\frac{3}{2}=\frac{3}{2}$$

So x = 6 is a valid solution!

Next, check x = -1:

$$\frac{3}{-1}+1=\frac{6}{-1-2}$$
$$\frac{3}{-1}+1=\frac{6}{-3}$$
$$\frac{-3}{1}+1=\frac{-2}{1}$$
$$-3+1=-2$$
$$-2=-2$$

So both x = 6 and x = -1 are solutions to this problem.

More Examples

Concept Problem Revisited

We can use the distance formula $d=rt$ to answer this question. In this case, though, it makes more sense to rearrange the formula so that the items we have information about (distance and rate) are on the same side: $\frac{d}{r}=t$. The problem tells us that

$$biking\ \ time=walking\ \ time-1$$

so we just need to come up with expressions for her biking and walking time. Since the question asks us to find the walking rate, we set $x=walking\ \ rate$. This means that $x+10=biking\ \ rate$. From these we create the equation:

$$\frac{96}{x+10}=\frac{18}{x}-1$$

Now, we multiply both sides by the LCD$x(x+10)$ and solve the resulting equation:

$$x(x+10)\left(\frac{96}{x+10}\right) = \left(\frac{18}{x}-1\right)x(x+10)$$
$$96x = 18(x+10) - x(x+10)$$
$$96x = 18x + 180 - x^2 - 10x$$
$$x^2 + 88x - 180 = 0$$
$$(x+90)(x-2) = 0$$
$$x = -90 \ \ or \ \ x = 2$$

In this case, a walking speed of -90 miles per hour doesn't make sense, so the woman must have a walking speed of **2 miles per hour** (and a biking speed of 12 miles per hour).

Example C

$$\frac{2x}{x-3} = 2 + \frac{3x}{x^2-9}$$

The LCD is x^2-9. Multiply each term by its factored form to cross-cancel.

$$\frac{2x}{x-3} = 2 + \frac{3x}{x^2-9}$$
$$\frac{\cancel{(x-3)}(x+3)}{1}\cdot\frac{2x}{\cancel{x-3}} = (x-3)(x+3)\cdot 2 + \frac{\cancel{(x-3)(x+3)}}{1}\cdot\frac{3x}{\cancel{x^2-9}}$$
$$2x(x+3) = 2(x^2-9) + 3x$$
$$2x^2 + 6x = 2x^2 - 18 + 3x$$
$$3x = -18$$
$$x = -6$$

Checking our answer, we have: $\frac{2(-6)}{-6-3} = 2 + \frac{3(-6)}{(-6)^2-9} \rightarrow \frac{-12}{-9} = 2 + \frac{-18}{27} \rightarrow \frac{4}{3} = 2 - \frac{2}{3}$

Example D

$$\frac{4}{x-3} + 5 = \frac{9}{x+2}$$

The LCD is $(x-3)(x+2)$. Multiply each term by the LCD.

$$\frac{4}{x-3} + 5 = \frac{9}{x+2}$$
$$\cancel{(x-3)}(x+2)\cdot\frac{4}{\cancel{x-3}} + (x-3)(x+2)\cdot 5 = (x-3)\cancel{(x+2)}\cdot\frac{9}{\cancel{x+2}}$$
$$4(x+2) + 5(x-3)(x+2) = 9(x-3)$$
$$4x + 8 + 5x^2 - 5x - 30 = 9x - 27$$
$$5x^2 - 10x + 5 = 0$$
$$5(x^2 - 2x + 1) = 0$$

This polynomial factors to be $5(x-1)(x-1) = 0$, so $x = 1$ is a repeated solution. Checking our answer, we have $\frac{4}{1-3} + 5 = \frac{9}{1+2} \rightarrow -2 + 5 = 3$

Example E

$$\frac{4}{(x+1)^2} + \frac{1}{x-1} = \frac{3}{x+1}$$

The LCD is $(x+1)^2(x-1)$. We multiply both sides by the LCD and then solve:

$$\begin{aligned}
(x+1)^2(x-1)\left(\frac{4}{(x+1)^2} + \frac{1}{x-1}\right) &= \left(\frac{3}{x+1}\right)(x+1)^2(x-1) \\
4 \cdot (x-1) + 1 \cdot (x+1)^2 &= 3 \cdot (x+1)(x-1) \\
4x - 4 + x^2 + 2x + 1 &= 3x^2 - 3 \\
x^2 + 6x - 3 &= 3x^2 - 3 \\
-2x^2 + 6x &= 0 \\
-2x(x-3) &= 0 \\
x = 0 \ \ or \ \ x &= 3
\end{aligned}$$

In this case both x=0 and x=3 satisfy the original equation (you should check that yourself, but it's true), so both are solutions to the problem.

Practice

Determine if the following values for x are solutions for the given equations.

1. $\frac{4}{x-3} + 2 = \frac{3}{x+4}$, $x = -1$
2. $\frac{2x-1}{x-5} - 3 = \frac{x+6}{2x}$, $x = 6$

What is the LCD for each set of numbers?

3. $4 - x,\ x^2 - 16$
4. $2x,\ 6x - 12,\ x^2 - 9$
5. $x - 3,\ x^2 - x - 6,\ x^2 - 4$

Solve the following equations.

6. $\frac{6}{x+2} + 1 = \frac{5}{x}$
7. $\frac{5}{3x} - \frac{2}{x+1} = \frac{4}{x}$
8. $\frac{12}{x^2-9} = \frac{8x}{x-3} - \frac{2}{x+3}$
9. $\frac{6x}{x^2-1} + \frac{2}{x+1} = \frac{3x}{x-1}$
10. $\frac{5x-3}{4x} - \frac{x+1}{x+2} = \frac{1}{x^2+2x}$
11. $\frac{4x}{x^2+6x+9} - \frac{2}{x+3} = \frac{3}{x^2-9}$
12. $\frac{x^2}{x^2-8x+16} = \frac{x}{x-4} + \frac{3x}{x^2-16}$
13. $\frac{5x}{2x-3} + \frac{x+1}{x} = \frac{6x^2+x+12}{2x^2-3x}$
14. $\frac{3x}{x^2+2x-8} = \frac{x+1}{x^2+4x} + \frac{2x+1}{x^2-2x}$
15. $\frac{x+1}{x^2+7x} + \frac{x+2}{x^2-3x} = \frac{x}{x^2+4x-21}$

Summary

You learned that operations with rational expressions rely on factoring and operations with fractions. To multiply rational expressions, multiply across and simplify. To divide rational expressions, change the problem to a multiplication problem by multiplying the first fraction by the reciprocal of the second fraction. To add or subtract, find the lowest common denominator in order to combine the expressions.

Chapter 8 Unit 8 - Linear Inequalities

Chapter Outline

Introduction

Here you'll learn how to solve inequalities in one and two variables. You will learn how to represent your solutions graphically on a number line and the Cartesian plane.

8.1 One Variable Inequalities

Learning Objectives

Here you are going to learn about one variable linear inequalities.

Concept Problem

Life is full of uncertainties. For instance, when making a big purchase like a car, you generally would not say "I'm going to spend exactly \$18,721.09 on a new car". That's too specific. A more likely scenario is that you would say "I want to spend less than \$19,000 on my car purchase." This situation is more ambiguous. Instead of a single possible value, there are now many possibilities for what you might spend. An equation does not make sense as a model here. Instead, we can use an inequality.

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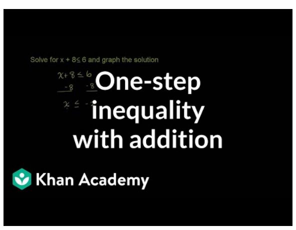

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One variable linear inequalities look similar to one variable linear equations. Linear equations have the general form of $ax+b=c$, where $a \neq 0$. Linear inequalities have the same form, except they replace the equals symbol with one of the four inequality symbols:

$$ax+b>c$$
$$ax+b<c$$
$$ax+b\geq c$$
$$ax+b\leq c$$

What's the difference between the first two symbols and the last two? The first two are called *strict* inequalities, because they do not include the case where the left side is exactly equal to the right side. The last two are not strict inequalities; they include the case where the left side is exactly equal to the right. For example, $5 \leq 5$ would be considered a true statement, while $5 < 5$ would be considered false.

When you solve for a linear inequality, you follow the same rules as you would for a linear equation; however, there is one additional rule to remember: ***If you divide or multiply by a negative number while solving, you must reverse the sign of the inequality.***

To see why we need this rule, consider the inequality $1 < 3$. This is a true statement; however, if we multiply both sides by -1 without changing the sign it becomes $-1 < -3$, which is not true! To maintain truthiness, we must switch the direction of the sign.

Example A

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 8 in for p?

TABLE 8.1:

Equation	Inequality	Is the inequality still true?
$2p+4=20$	$2p+4<20$	?
$2p+4-4=20-4$		
$2p=16$		
$\frac{2p}{2}=\frac{16}{2}$		
$p=8$		

Solution:

TABLE 8.2:

Equation	Inequality	Is the inequality still true?
$2p+4=20$	$2p+4<20$	no
$2p+4-4=20-4$	$2p+4-4<20-4$	
$2p=16$	$2p<16$	
$\frac{2p}{2}=\frac{16}{2}$	$\frac{2p}{2}<\frac{16}{2}$	
$p=8$	$p<8$	

Yes, the steps are the same to find the two solutions.

Example B

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 6 in for x?

TABLE 8.3:

Equation	Inequality	Is the inequality still true?
$3x+5=23$	$3x+5\geq 23$	?
$3x+5-5=23-5$		
$3x=18$		
$\frac{3x}{3}=\frac{18}{3}$		
$x=6$		

Solution:

TABLE 8.4:

Equation	Inequality	Is the inequality still true?
$3x+5=23$	$3x+5\geq 23$	yes

TABLE 8.4: (continued)

Equation	Inequality	Is the inequality still true?
$3x+5-5=23-5$	$3x+5-5 \geq 23-5$	
$3x=18$	$3x \geq 18$	
$\frac{3x}{3}=\frac{18}{3}$	$\frac{3x}{3} \geq \frac{18}{3}$	
$x=6$	$x \geq 6$	

Yes, the steps are the same to find the two solutions.

Example C

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 3 in for c?

TABLE 8.5:

Equation	Inequality	Is the inequality still true?
$5-3c=-4$	$5-3c \leq -4$	?
$5-5-3c=-4-5$		
$-3c=-9$		
$\frac{-3c}{-3}=\frac{-9}{-3}$		
$c=3$		

Solution:

TABLE 8.6:

Equation	Inequality	Is the inequality still true?
$5-3c=-4$	$5-3c \leq -4$	yes
$5-5-3c=-4-5$	$5-5-3c \leq -4-5$	
$-3c=-9$	$-3c \leq -9$	
$\frac{-3c}{-3}=\frac{-9}{-3}$	$\frac{-3c}{-3} \geq \frac{-9}{-3}$	
$c=3$	$c \geq 3$	

No, the steps are different between the two solutions. When dividing by -3, the sign of the inequality was reversed.

Concept Problem Revisited

Let's go back to the car shopping discussed at the beginning of this section and add some additional details. Suppose that you want to spend less that \$19,000 on your purchase, but the stated price of a car does not include the 5% sales tax and \$200 registration fee. What prices are actually available to you? Let x represent the stated price of a car. You can make the following inequality:

$$\underbrace{x}_{\text{car price}} + \underbrace{.05x}_{tax} + \underbrace{200}_{\text{reg. fee}} < \underbrace{19,000}_{budget}$$

The first two terms combine to become $1.05x$, and you get an inequality to solve:

$$1.05x + 200 < 19,000$$
$$1.05x < 18,800$$
$$x < 17,904.76$$

So you need to choose a car that has a price under $17,904.76 to stay within your budget.

Compound Inequalities

In the car purchase example, the cost of your new car was only bounded above, on the upper end, by the budget you set. In many cases, however, a quantity may be bounded on both the upper and lower ends. For instance, a meteorologist may say "The low temperature today will be 70 degrees, and the high will be 83." Here, the temperature has both an upper and a lower bound. If x is the temperature during the day, then the meteorologist's statement translates into two inequalities:

$$x \geq 70$$
$$x \leq 83$$

We can work with these separately, but it is also possible to combine them into a single mathematical statement. (Remember that $x \geq 70$ is the same as $70 \leq x$)

$$70 \leq x$$
$$x \leq 83$$
$$\downarrow$$
$$70 \leq x \leq 83$$

This consolidates the two pieces of information into a single item. An inequality in this form is called a **compound inequality**.

Example D

Solve the inequality: $8 \leq 2x + 12 \leq 22$

Solution:

When solving a compound inequality, the goal is to get the variable by itself in the middle. The rule to remember is that any operation you do on one part of the inequality needs to also be done to the other two parts.

In this problem, we would like to get the x by itself in the middle part of the inequality. A good first step would be to remove the $+12$ by subtracting 12. We can do that as long as we also subtract 12 from the other two parts:

$$8 \leq 2x + 12 \leq 22$$

$$8 - 12 \leq 2x + 12 - 12 \leq 22 - 12$$

$$-4 \leq 2x \leq 10$$

The next step is to divide by 2. Again, we must do it to each part of the inequality:

$$-4 \leq 2x \leq 10$$

$$\frac{-4}{2} \leq \frac{2x}{2} \leq \frac{10}{2}$$

$$-2 \leq x \leq 5$$

This tells us that in order to satisfy the original inequality x can be no less than -2 and no greater than 5.

Example E

Solve the inequality: $10 \leq \frac{1}{3}n - 5 < 18$

Solution

Again, the goal is to get the n by itself in the middle of the inequality. First, we should add 5 to each part:

$$10 \leq \frac{1}{3}n - 5 < 18$$

$$10+5 \leq \frac{1}{3}n - 5+5 < 18+5$$

$$15 \leq \frac{1}{3}n < 23$$

Now, multiply by 3:

$$15 \leq \frac{1}{3}n < 18$$

$$3 \cdot 15 \leq 3 \cdot \frac{1}{3}n < 3 \cdot 18$$

$$45 \leq n < 54$$

Interval Notation

You have probably noticed that solutions to inequalities come in a few common forms like

$$x \geq N, x < M, a \leq x \leq b, \text{etc...}$$

Mathematicians have developed a notation that strips away all the excess symbols from statements like these and condenses them to basic elements.

Take the inequality $5 \leq x < 11$. There is a lot of stuff in the middle; let's see if we can boil it down. The important data are the lowest possible value (5), the largest possible value (11), and whether or not those values are included as part of the set. Instead of putting inequalities and variables between the lowest and highest values, we can simply separate them with a comma:

$$5, 11$$

Next, we need to indicate whether each value is included in the set or not. We can do this by putting a different symbol next to each one. To include the 5 we use the [bracket. To exclude the 11 we use the) parenthesis.

$$[5, 11)$$

Writing it this way conveys the same information as writing $5 \leq x < 11$ but is more succinct (and cuter).

This notation is called **interval notation**. In general interval notation has the form $[a, b]$ where a is the smallest value, b is the largest value, and we use [or (to indicate if a value is included or excluded, respectively. In cases where there is no largest or smallest value we use the ∞ or $-\infty$ symbol and the (character.

Example F

Write the inequalities using interval notation:

a. -7

b. $x < 28$

c. $5 < 21$

d. $x \geq \frac{3}{4}$

Solution

a. In interval notation this inequality becomes $(-7, 4]$.

b. In this case, 28 is the largest value (since x is smaller than it) and there is no lowest value, so the interval is $(-\infty, 28)$.

c. In interval notation this inequality becomes $(5, 21)$. Notice that this looks the same as the notation for a point. In a situation like this you need to determine from the context whether it is referring to a point or an interval.

d. In this case $\frac{3}{4}$ is the smallest possible value, and there is no largest value, so the interval is $\left[\frac{3}{4}, \infty\right)$.

Example G

Solve the inequality and put your answer in interval notation: $-20 \leq -2x + 8 \leq -10$.

Solution

First, remove the $+8$ by subtracting 8 from both sides:

$$-20 \leq -2x + 8 \leq -10$$

$$-20 - 8 \leq -2x + 8 - 8 \leq -10 - 8$$

$$-28 \leq -2x \leq -18$$

Next, divide by -2. Note that the rule for inequalities still applies: when we divide by the negative value the inequalities will change their directions:

$$-28 \leq -2x \leq -18$$

$$\frac{-28}{-2} \leq \frac{-2x}{-2} \leq \frac{-18}{-2}$$

$$14 \geq x \geq 9$$

So is the answer $[14, 9]$? No, this does not fit the interval notation format, since the smallest value needs to come first. In cases like this it can be helpful to flip the inequality around:

$$14 \geq x \geq 9$$
$$\downarrow$$
$$9 \leq x \leq 14$$

Now we see that the interval notation should be $[9, 14]$.

Guided Practice

1. In the following table, a linear equation has been solved. Solve for the inequality using similar steps, but remember if you multiply or divide by a negative number you should reverse the inequality sign.

TABLE 8.7:

Equation	Inequality	Is the inequality still true?
$4.6a+8.2=2.4a-13.8$	$4.6a+8.2>2.4a-13.8$	?
$4.6a+8.2+13.8=2.4a-13.8+13.8$		
$4.6a+22=2.4a$		
$4.6a-4.6a+22=2.4a-4.6a$		
$22=-2.2a$		
$\frac{22}{-2.2}=\frac{-2.2a}{-2.2}$		
$a=-10$		

2. In the following table, a linear equation has been solved. Solve for the inequality using similar steps.

TABLE 8.8:

Equation	Inequality	Is the inequality still true?
$3(w+4)=2(3+2w)$	$3(w+4)<2(3+2w)$	?
$3w+12=6+4w$		
$3w+12-12=6-12+4w$		
$3w=-6+4w$		
$3w-4w=-6+4w-4w$		
$-w=-6$		
$\frac{-w}{-1}=\frac{-6}{-1}$		

TABLE 8.8: (continued)

Equation	Inequality	Is the inequality still true?
$w = 6$		

3. In the following table, a linear equation has been solved. Solve for the inequality using similar steps.

TABLE 8.9:

Equation	Inequality	Is the inequality still true?
$\frac{1}{3}(2-h) = 4$ $\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$ $\frac{1}{3}(2-h) = \frac{12}{3}$ $2-h = 12$ $2-2-h = 12-2$ $-h = 10$ $\frac{-h}{-1} = \frac{10}{-1}$ $h = -10$	$\frac{1}{3}(2-h) \geq 4$	?

4. Seven less than two times a number is less than 33. What numbers satisfy this statement?

5. $4t+3 > 11$

6. $2z+7 \leq 5z+28$

7. $9(j-2) \geq 6(j+3)-9$

Answers:

1.

TABLE 8.10:

Equation	Inequality	Is the inequality still true?
$4.6a+8.2 = 2.4a-13.8$ $4.6a+8.2+13.8 = 2.4a-13.8+13.8$ $4.6a+22 = 2.4a$ $4.6a-4.6a+22 = 2.4a-4.6a$ $22 = -2.2a$ $\frac{22}{-2.2} = \frac{-2.2a}{-2.2}$ $a = -10$	$4.6a+8.2 > 2.4a-13.8$ $4.6a+8.2+13.8 > 2.4a-13.8+13.8$ $4.6a+22 > 2.4a$ $4.6a-4.6a+22 > 2.4a-4.6a$ $22 > -2.2a$ $\frac{22}{-2.2} < \frac{-2.2a}{-2.2}$ $a > -10$	no

2.

TABLE 8.11:

Equation	Inequality	Is the inequality still true?
$3(w+4) = 2(3+2w)$ $3w+12 = 6+4w$ $3w+12-12 = 6-12+4w$ $3w = -6+4w$ $3w-4w = -6+4w-4w$ $-w = -6$ $\frac{-w}{-1} = \frac{-6}{-1}$	$3(w+4) < 2(3+2w)$ $3w+12 = 6 < 4w$ $3w+12-12 < 6-12+4w$ $3w < -6+4w$ $3w-4w < -6+4w-4w$ $-w < -6$ $\frac{-w}{-1} > \frac{-6}{-1}$	no

TABLE 8.11: (continued)

Equation	Inequality	Is the inequality still true?
$w = 6$	$w > 6$	

3.

TABLE 8.12:

Equation	Inequality	Is the inequality still true?
$\frac{1}{3}(2-h) = 4$	$\frac{1}{3}(2-h) \geq 4$	yes
$\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$	$\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$	
$\frac{1}{3}(2-h) = \frac{12}{3}$	$\frac{1}{3}(2-h) \geq \frac{12}{3}$	
$2-h = 12$	$2-h \geq 12$	
$2-2-h = 12-2$	$2-2-h \geq 12-2$	
$-h = 10$	$-h \geq 10$	
$\frac{-h}{-1} = \frac{10}{-1}$	$\frac{-h}{-1} \leq \frac{10}{-1}$	
$h = -10$	$h \leq -10$	

4. Use n to represent 'a number'. The statement translates into the inequality $2n-7 < 33$, which simplifies to $n < 20$. So any number less than 20 satisfies the statement.

5. $t > 2$. Here are the steps:

$$
\begin{aligned}
4t+3 &> 11 \\
4t+3-3 &> 11-3 && \text{Subtract 3 from both sides to isolate the variable} \\
4t &> 8 && \text{Simplify} \\
\frac{4t}{4} &> \frac{8}{4} && \text{Divide by 4 to solve for the variable} \\
t &> 2
\end{aligned}
$$

6. $z \geq -7$. Here are the steps:

$$
\begin{aligned}
2z+7 &\leq 5z+28 \\
2z-2z+7 &\leq 5z-2z+28 && \text{Subtract } 2z \text{ from both sides to get variables on same side of the inequality sign.} \\
7 &\leq 3z+28 && \text{Simplify} \\
7-28 &\leq 3z+28-28 && \text{Subtract 28 from both sides to isolate the variable} \\
-21 &\leq 3z && \text{Simplify} \\
\frac{3z}{3} &\geq \frac{-21}{3} && \text{Divide by 3 to solve for the variable} \\
z &\geq -7
\end{aligned}
$$

7. $j \geq 9$. Here are the steps:

$9(j-2) \geq 6(j+3)-9$	
$9j-18 \geq 6j+18-9$	Remove parentheses
$9j-18 \geq 6j+9$	Combine like terms on each side of inequality sign
$9j-6j-18 \geq 6j-6j+9$	Subtract $6j$ from both sides to get variables on same side of the inequality sign.
$3j-18 \geq 9$	Simplify
$3j-18+18 \geq 9+18$	Add 18 to both sides to isolate the variable
$3j \geq 27$	Simplify
$\frac{3j}{3} \geq \frac{27}{3}$	Divide by 3 to solve for the variable
$j \geq 9$	

Practice

In the following table, a linear equation has been solved.

TABLE 8.13:

Equation	Inequality	Is the inequality still true?
$5.2+x+3.6=4.3$ $8.8+x=4.3$ $8.8-8.8+x=4.3-8.8$ $x=-4.5$	$5.2+x+3.6 \geq 4.3$	?

Solve for the inequality using similar steps.

In the following table, a linear equation has been solved.

TABLE 8.14:

Equation	Inequality	Is the inequality still true?
$\frac{n}{4}-5=-3$ $\frac{n}{4}-5\left(\frac{4}{4}\right)=-3\left(\frac{4}{4}\right)$ $\frac{n}{4}-\frac{20}{4}=\frac{-12}{4}$ $n-20=-12$ $n-20+20=-12+20$ $n=8$	$\frac{n}{4}-5<-3$	?

Solve for the inequality using similar steps.

In the following table, a linear equation has been solved.

TABLE 8.15:

Equation	Inequality	Is the inequality still true?
$1-z=5(3+2z)+8$ $1-z=15+10z+8$ $1-z=23+10z$	$1-z<5(3+2z)+8$	?

TABLE 8.15: (continued)

Equation	Inequality	Is the inequality still true?
$1-z+z=23+10z+z$ $1=23+11z$ $1-23=23-23+11z$ $-22=11z$ $\frac{-22}{11}=\frac{11z}{11}$ $z=-2$		

Solve for the inequality using similar steps.

1. The sum of two numbers is greater than 764. If one of the numbers is 416, what could the other number be?
2. 205 less a number is greater than or equal to 112. What could that number be?
3. Five more than twice a number is less than 20. If the number is a whole number, what could the number be?
4. The product of 7 and a number is greater than 42. If the number is a whole number less than 10, what could the number be?
5. Three less than 5 times a number is less than or equal to 12. If the number is a whole number, what could the number be?
6. Double a number and add 12 and the result will be greater than 20. The number is less than 6. What is the number?

8.2 Graphical Solutions to One Variable Inequalities

Learning Objectives

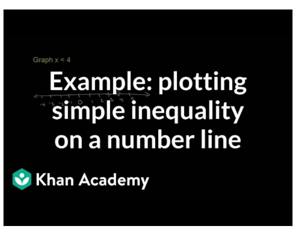
Here you'll learn how to represent your solution to an inequality on a number line.

Concept Problem

Jack is 3 years older than his brother. How old are they if the sum of their ages is greater than 17? Write an inequality and solve. Represent the solution set on a number line.

Watch This

Khan Academy Inequalities on a Number Line



MEDIA

Click image to the left or use the URL below.

URL: http://www.ck12.org/flx/render/embeddedobject/5533

Guidance

When you solve a linear inequality you can represent your solution graphically with a number line. Your job is to show which of the numbers on the number line are solutions to the inequality.

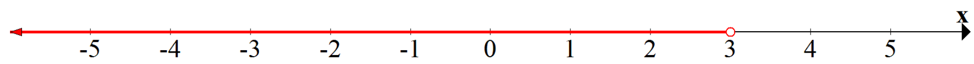
When graphing solutions to inequalities, use an open circle to show that the endpoint is not included as part of the solution and a closed circle to show that the endpoint is included as part of the solution. The line above the number line shows all of the numbers that are possible solutions to the inequality.

Example A

Represent the solution set to the following inequality on a number line: $x+2<5$.

Solution:

$$\begin{aligned} x+2&<5 \\ x+2-2&<5-2 && \text{(Subtract 2 from both sides to isolate the variable)} \\ x&<3 && \text{(Simplify)} \end{aligned}$$

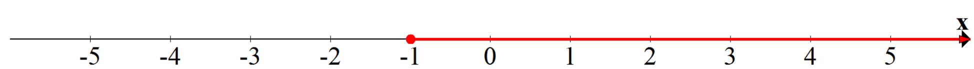


Example B

Represent the solution set to the following inequality on a number line: $2x+6 \geq 4$.

Solution:

$$\begin{aligned} 2x+6 &\geq 4 \\ 2x+6-6 &\geq 4-6 && \text{(Subtract 6 from both sides to isolate the variable)} \\ 2x &\geq -2 && \text{(Simplify)} \\ \frac{2x}{2} &\geq \frac{-2}{2} && \text{(Divide by 2 to solve for the variable)} \\ x &\geq -1 \end{aligned}$$

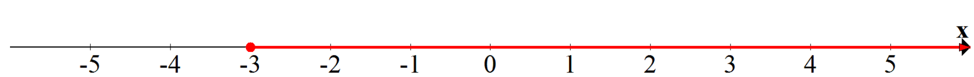


Example C

Represent the solution set to the following inequality on a number line: $-3x+8 \leq 17$.

Solution:

$$\begin{aligned} -3x+8 &\leq 17 \\ -3x+8-8 &\leq 17-8 && \text{(Subtract 8 from both sides to isolate the variable)} \\ -3x &\leq 9 && \text{(Simplify)} \\ \frac{-3x}{-3} &\geq \frac{9}{-3} && \text{(Divide both sides by -3 to solve for the variable, reverse sign of inequality)} \\ x &\geq -3 \end{aligned}$$



Concept Problem Revisited

Jack is 3 years older than his brother. How old are they if the sum of their ages is greater than 17? Write an inequality and solve. Represent the solution set on a number line.

If first let's write down what you know:

Let $x =$ Jack's brother's age

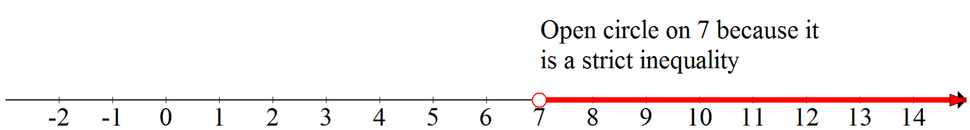
Let $x+3 =$ Jack's age

The equation would therefore be:

$$\begin{aligned}
x+x+3 &> 17 \\
2x+3 &> 17 && \text{(Combine like terms)} \\
2x+3-3 &> 17-3 && \text{(Subtract 3 from both sides to solve for the variable)} \\
2x &> 14 && \text{(Simplify)} \\
\frac{2x}{2} &> \frac{14}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &> 7
\end{aligned}$$

Therefore if Jack's brother is 8 (since $8 > 7$), Jack would be 11. If Jack's brother is 10, Jack would be 13.

Representing Jack's brother's age on a number line:



Guided Practice

1. Represent the solution set to the inequality $3(a-1) < 9$ on a number line.

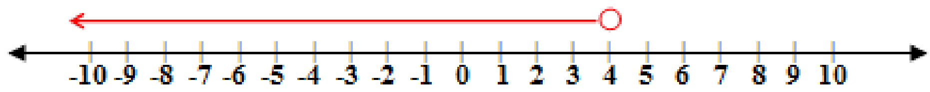
2. Represent the solution set to the inequality $2b+4 \geq 5b+19$ on a number line.

3. Represent the solution set to the inequality $0.6c+2 \geq 5.6$ on a number line.

Answers:

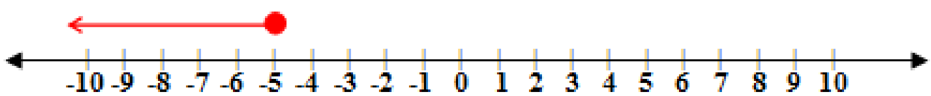
1.

$$\begin{aligned}
3(a-1) &< 9 \\
3a-3 &< 9 && \text{(Distribute 3 to remove parentheses)} \\
3a-3+3 &< 9+3 && \text{(Add 3 to both sides to isolate the variable)} \\
3a &< 12 && \text{(Simplify)} \\
\frac{3a}{3} &< \frac{12}{3} && \text{(Divide both sides by 3 to solve for the variable)} \\
a &< 4
\end{aligned}$$



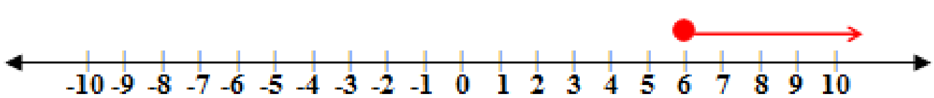
2.

$$\begin{aligned}
2b+4 &\geq 5b+19 \\
2b-5b+4 &\geq 5b-5b+19 && \text{(Subtract } 5b \text{ from both sides to get variables on same side)} \\
-3b+4 &\geq 19 && \text{(Simplify)} \\
-3b+4-4 &\geq 19-4 && \text{(Subtract 4 from both sides to isolate the variable)} \\
-3b &\geq 15 && \text{(Simplify)} \\
\frac{-3b}{-3} &\leq \frac{15}{-3} && \text{(Divide by -3 to solve for the variable, reverse sign of inequality)} \\
b &\leq -5
\end{aligned}$$



3.

$$\begin{aligned}
0.6c+2 &\geq 5.6 \\
0.6c+2-2 &\geq 5.6-2 && \text{(Subtract 2 from both sides to isolate the variable)} \\
0.6c &\geq 3.6 && \text{(Simplify)} \\
\frac{0.6c}{0.6} &\geq \frac{3.6}{0.6} && \text{(Divide both sides by 0.6 to solve for the variable)} \\
c &\geq 6
\end{aligned}$$



Practice

Solve each inequality and represent the solution set on a number line.

1. $-4v > 12$
2. $\frac{-2r}{3} > 4$
3. $4(t-2) \leq 24$
4. $\frac{1}{2}(x+5) > 6$
5. $\frac{1}{4}(g+2) \leq 2$
6. $0.4(b+2) \geq 2$

7. $0.5(r-1) < 4$
8. $\frac{1}{4}(x+16) > 2$
9. $2-k > 5(1-k)$
10. $2(1.5c+4) \leq -1$
11. $-\frac{1}{2}(3x-5) \geq 7$
12. $0.35+0.10(m-1) < 0.45$
13. $\frac{1}{4}+\frac{2}{3}(t+1) > \frac{1}{2}$
14. The prom committee is selling tickets for a fundraiser for the decorations. Each ticket costs \$3.50. What is the least number of tickets the committee needs to sell to make \$1000? Write an inequality and solve.
15. Brenda got 69%, 72%, 81%, and 88% on her last four major tests. How much does she need on her next test to have an average of at least 80%? Write an inequality and solve.

8.3 Linear Inequalities in Two Variables

Concept Problem

Yasmeen is selling handmade bracelets for \$5 each and necklaces for \$7 each. She would like to make at least \$100 worth of sales. She would like to know how many bracelets/necklaces she can sell in order to achieve this.

Can we model this scenario with a one-variable inequality? No. In this case Yasmeen has two separate quantities (bracelets and necklaces) that contribute to her revenue, so we need two variables to describe the situation.

Introduction

A **linear inequality** in two variables can be written in the form $y > mx+b$, $y < mx+b$, $y \geq mx+b$, or $y \leq mx+b$. Linear inequalities are closely related to graphs of straight lines; recall that a straight line has the equation $y = mx+b$.

The **solution** to a linear inequality is the set of *all* points whose coordinates satisfy the inequality when they are plugged in for x and y.

Example A

Consider the inequality $y \geq 2x-5$.

The point $(1,3)$ is part of the solution to the inequality $y \geq 2x-5$, because when those values are plugged in for x and y the resulting statement is true:

$$\begin{aligned} 3 &\geq 2(1)-5 \\ 3 &\geq 2-5 \\ 3 &\geq -3 \qquad \checkmark \end{aligned}$$

Keep in mind, though, that $(1,3)$ is just a *part* of the solution. The entire solution is comprised of **all** the points that satisfy the inequality in this way. In fact, there will be an infinite number of points in the solution.

How can we get a sense of the entire solution to a two-variable inequality? The best way is with a picture, and this is how we usually express such a solution. When we graph a line in the coordinate plane, we can see that it divides the plane in half:

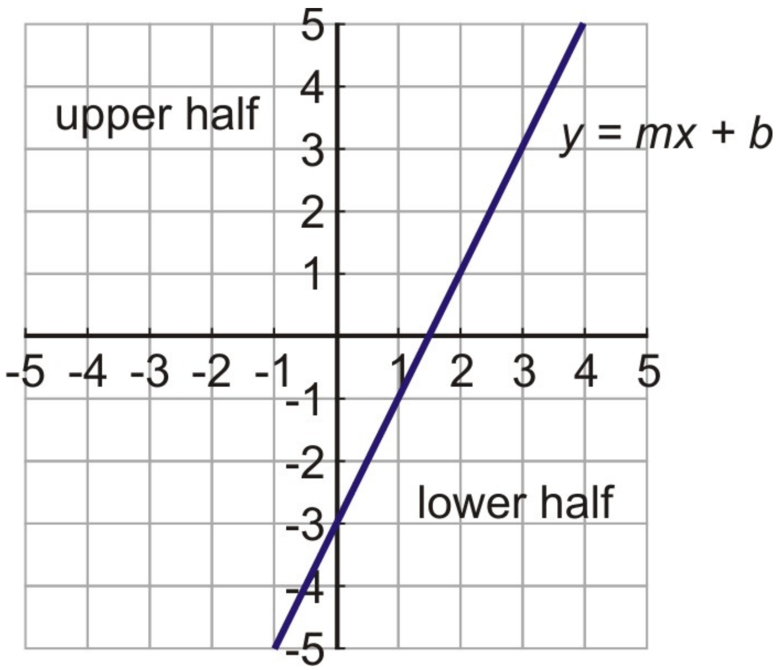


FIGURE 8.1

Visually, the solution to an inequality includes all the points on one side of the line or the other. We can tell which half by looking at the inequality sign:

>The solution set is the half plane above the line but not the line itself.

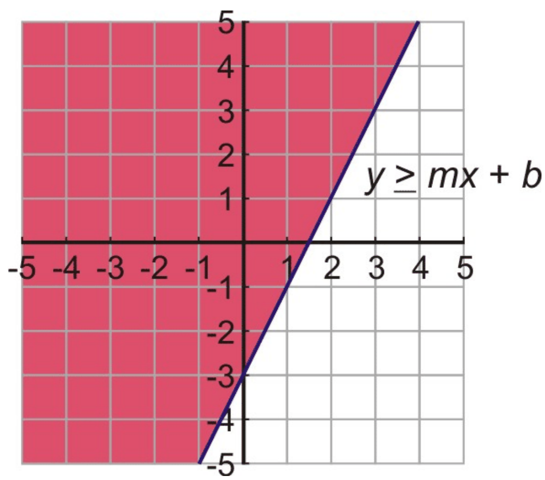
$\geq$ The solution set is the half plane above the line and also all the points on the line.

<The solution set is the half plane below the line but not the line itself.

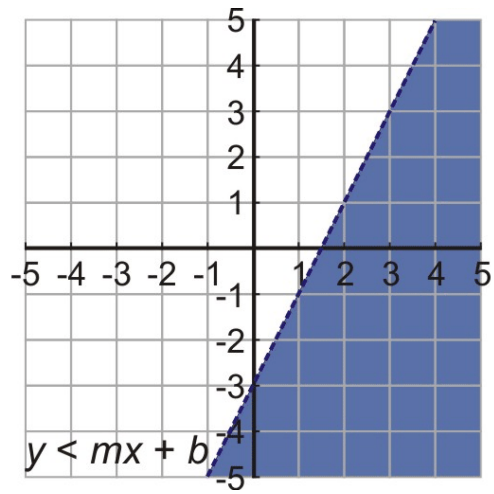
$\leq$ The solution set is the half plane below the line and also all the points on the line.

For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

Here are some examples of linear inequality graphs. This is a graph of $y \geq mx+b$; the solution set is the line and the half plane above the line.



This is a graph of $y < mx+b$; the solution set is the half plane above the line, not including the line itself.



Graph Linear Inequalities in One Variable in the Coordinate Plane

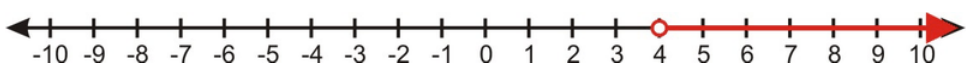
In the last few sections we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line, and when we graph an equation of the type $y = b$ we get a horizontal line.

Example B

Graph the inequality $x > 4$ *on the coordinate plane.*

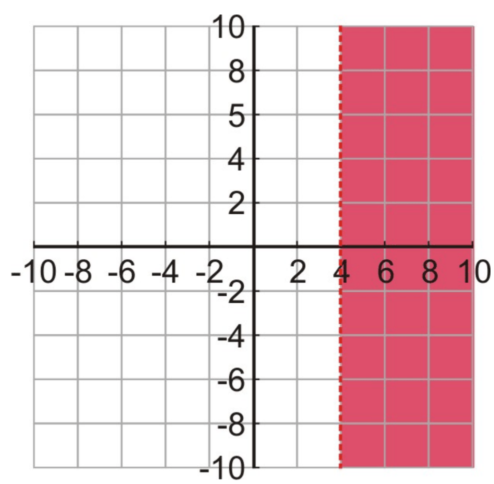
Solution

First let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than 4, not including 4. The solution is represented by a line.

In two dimensions, the solution still consists of all the points to the right of $x = 4$, but for all possible $y-$values as well. This solution is represented by the half plane to the right of $x = 4$. (You can think of it as being like the solution graphed on the number line, only stretched out vertically.)



The line $x = 4$ is dashed because the equals sign is not included in the inequality, meaning that points on the line are not included in the solution.

Graph Linear Inequalities in Two Variables

The general procedure for graphing inequalities in two variables is as follows:

1. Use algebra to isolate the y variable, so that the inequality 'looks like' the slope-intercept form ($y \le mx+b$, for example).
2. Graph the line of the equation $y = mx+b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade above the line if the inequality is "greater than." Shade under the line if the inequality is "less than."

Example C

Graph the inequality $y \ge 2x-3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x-3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.

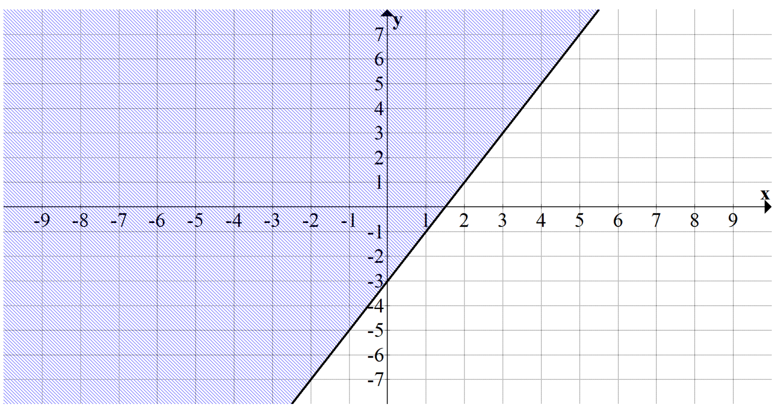


FIGURE 8.2

Example D

Graph the inequality $5x-2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x+4$$
$$y < \frac{5}{2}x-2$$

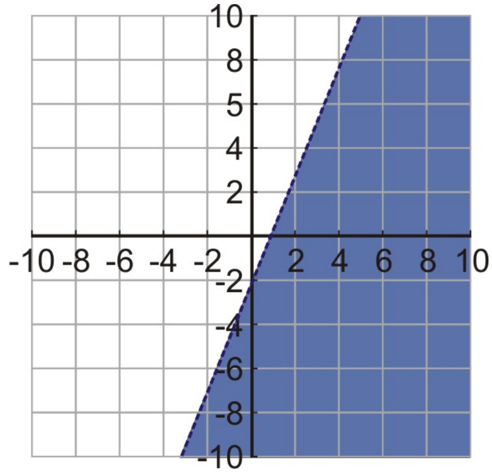
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

TABLE 8.16:

x	y
-2	$\frac{5}{2}(-2)-2 = -7$
0	$\frac{5}{2}(0)-2 = -2$
2	$\frac{5}{2}(2)-2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Concept Problem Revisited

Let's go back to Yasmeen's scenario described at the beginning of this section. She sells her bracelets for \$5 each, her necklaces for \$7 each, and she would like to achieve at least \$100 in sales. Let x represent the number of bracelets she sells and y represent the number of necklaces she sells. So 5x is her revenue from bracelets, and 7y is her revenue from necklaces. From these we make the inequality:

$$\underbrace{5x}_{\text{bracelet rev}} + \underbrace{7y}_{\text{necklace rev}} \geq \underbrace{100}_{goal}$$

$$5x + 7y \geq 100$$

$$7y \geq -5x + 100$$

$$y \geq -\frac{5}{7}x + \frac{100}{7}$$

Graphing this shows us the solution:

Example E

Your favorite coffee is a mix of Peruvian coffee beans and Ethiopian coffee beans, which cost \$7 and 9\$ per pound, respectively. Being budget-conscious, you would prefer to spend \$8.50 or less on this week's pound of beans. Which combinations

Solution

Let $x =$ weight of Ethiopian coffee beans in pounds.

Let $y =$ weight of Peruvian coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x + 7y$.

We are looking for the mixtures that cost \$8.50 or less, so we write the inequality $9x + 7y \leq 8.50$.

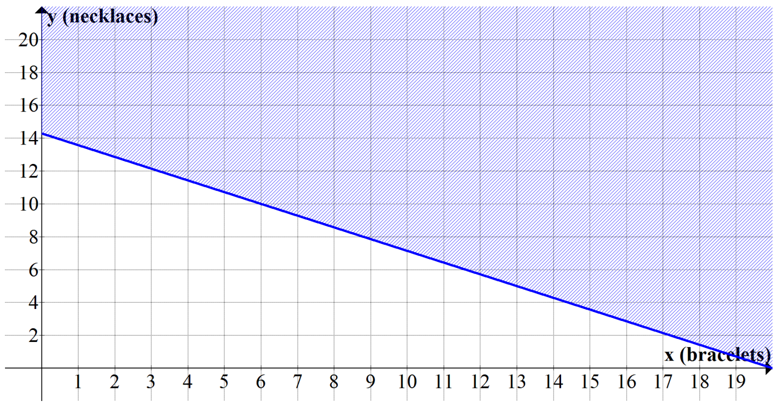


FIGURE 8.3

Next, isolate y to get the inequality into slope-intercept form: $y \leq -\frac{9}{7}x + \frac{8.5}{7}$

Now we are ready to graph and shade. In this case, the direction of the inequality indicates that we should shade **below** the line after we draw it.

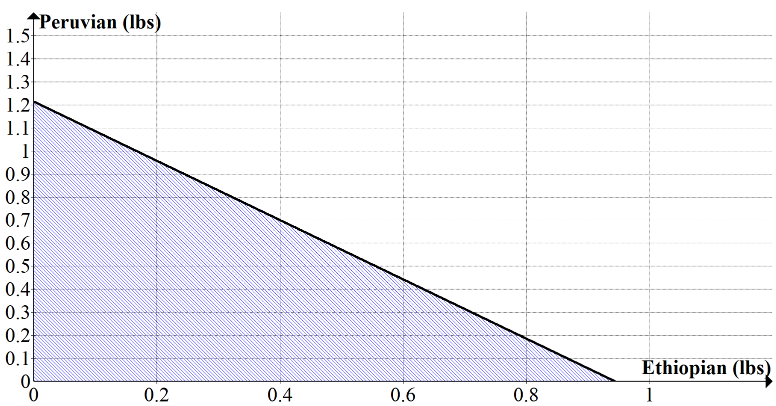


FIGURE 8.4

Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Example F

Julius has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make $1000 or more in commissions?

Solution

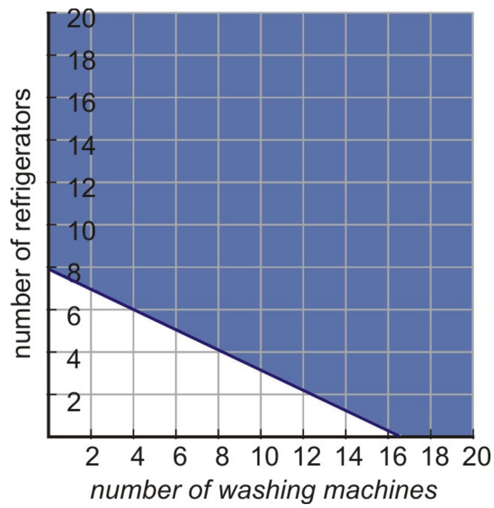
Let $x =$ number of washing machines Julius sells.

Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of $1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Next, isolate y to get the correct form: $y \geq -\frac{6}{13}x + \frac{100}{13}$. The inequality direction here indicates that we should shade **above** the line after we draw it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be nonnegative.

The video at http://www.youtube.com/watch?v=7629PsZLP1A&feature=related contains more examples of real-world problems using inequalities in two variables.

Review Questions

Graph the following inequalities on the coordinate plane.

1. $x < 20$
2. $y \geq -5$
3. $|x| > 10$
4. $|y| \leq 7$
5. $y \leq 4x+3$
6. $y > -\frac{x}{2} - 6$
7. $3x - 4y \geq 12$
8. $x + 7y < 5$
9. $6x + 5y > 1$
10. $y + 5 \leq -4x + 10$
11. $x - \frac{1}{2}y \geq 5$
12. $6x + y < 20$
13. $30x + 5y < 100$
14. Remember what you learned in the last chapter about families of lines.
    a. What do the graphs of $y > x+2$ and $y < x+5$ have in common?
    b. What do you think the graph of $x+2 < y < x+5$ would look like?
15. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?
16. How would the answer to problem 7 change if you added 12 to the right-hand side?
17. How would the answer to problem 8 change if you flipped the inequality sign?
18. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than $20?
19. Suppose you are graphing the inequality $y > 5x$.

a. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
b. What happens if you do plug it in?
c. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?

20. A theater wants to take in at least \$2000 for a certain matinee. Children's tickets cost \$5 each and adult tickets cost \$10 each.

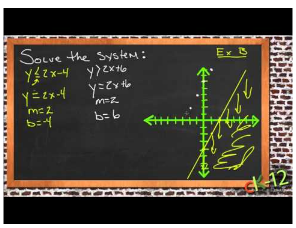
a. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
b. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
c. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

8.4 Systems of Linear Inequalities

Systems of Linear Inequalities

To graph a linear inequality in two variables, you graph the equation of the straight line on the coordinate plane. The line is solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line is dashed for <or >signs (where the equals sign is not included). Then, shade above the line (if the inequality begins with $y >$ or $y \geq$) or below the line (if it begins with $y <$ or $y \leq$).

Here you will see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two half-planes. A **system** of two or more linear inequalities can divide the plane into more complex shapes.



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Recall that for a system of equations, a solution is a pair of numbers that satisfy **all** the equations in the system. The same is true for systems of inequalities: a point is part of the solution to the system if and only if it satisfies **all** of the inequalities in the system. What does this mean graphically?

Consider the following system:

$$y < m_1x + b_1 \qquad (1)$$
$$y > m_2x + b_2 \qquad (2)$$

Suppose that the lines (before shading) have this graph:

If we only shaded the solution for inequality (1), then this would be the picture:

On the other hand, if we only shaded for inequality (2), then this would be the picture:

The solution to the system is supposed to be the points that satisfy **both** (1) and (2), so let's graph both solutions and see if we can identify it:

It's the overlap of the individual solutions! Any point in the overlap satisfies both (1) and (2) and thus is part of the solution to the system. If we take away the non-overlap shading we can see the solution precisely:

So, to solution to a system of inequalities is the intersection (i.e. overlap) of the solutions to the individual inequalities in the system.

Graph a System of Two Linear Inequalities

Solve the following system:

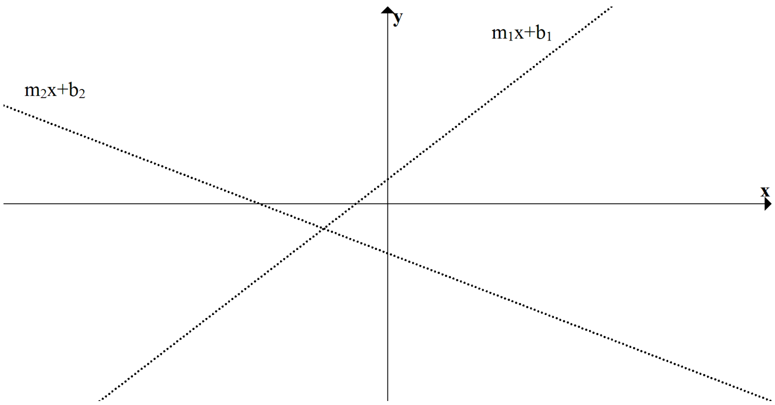


FIGURE 8.5

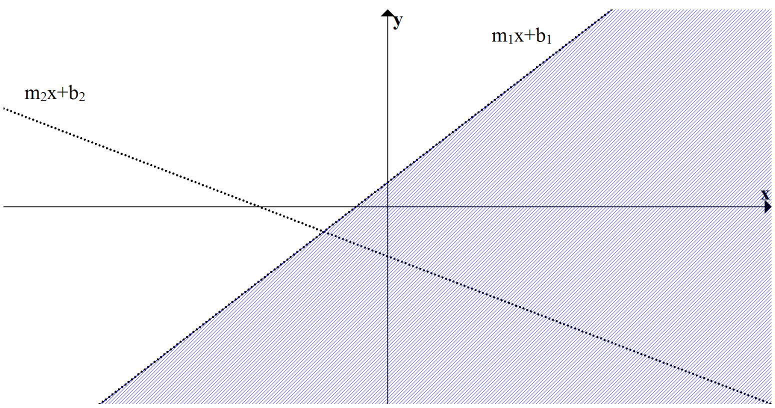


FIGURE 8.6

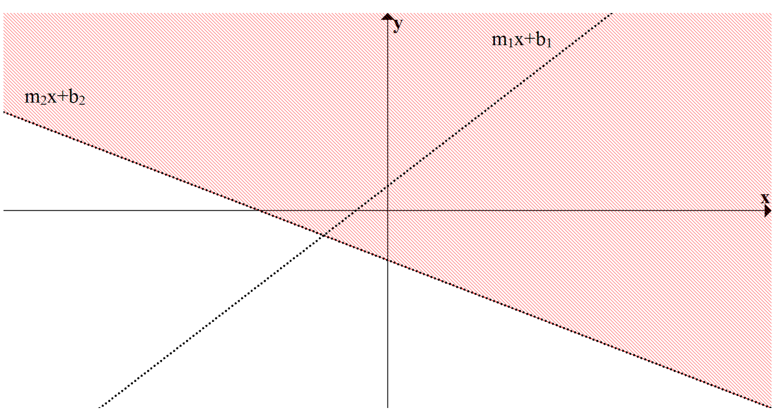


FIGURE 8.7

$$2x+3y \le 18$$
$$x-4y \le 12$$

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

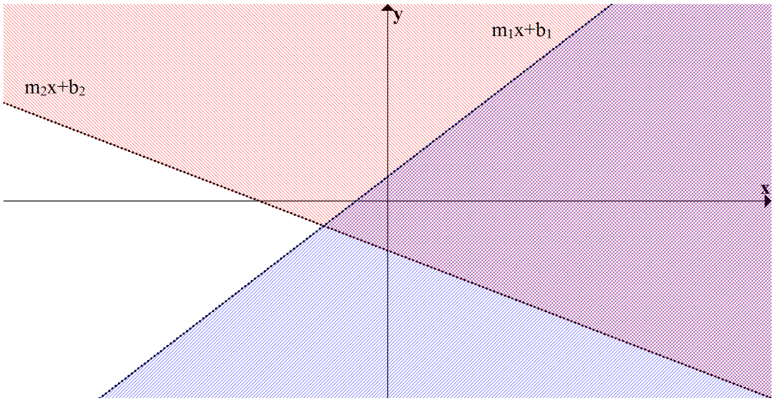


FIGURE 8.8

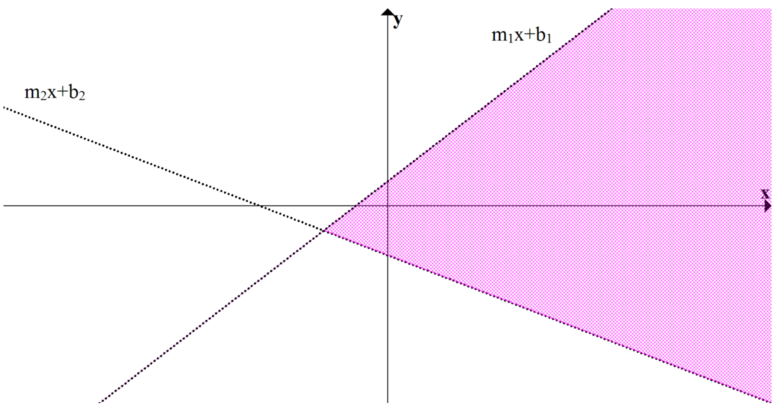


FIGURE 8.9

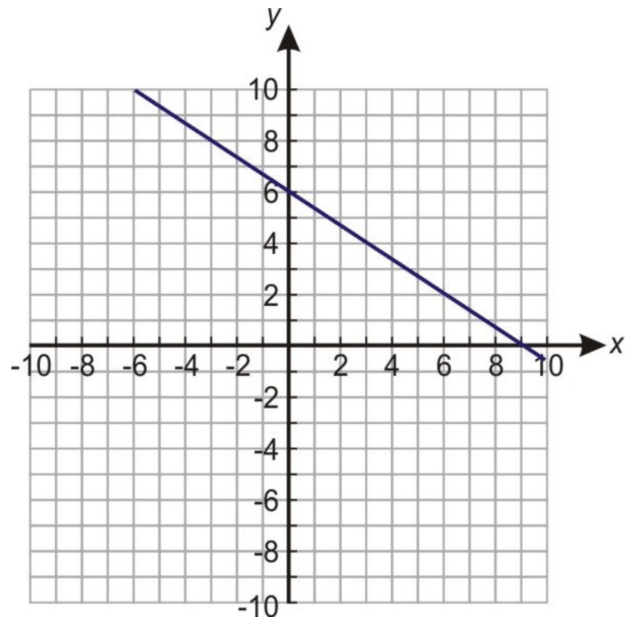
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$
\begin{aligned}
3y \le -2x+18 \quad \Rightarrow \qquad & amp;y \le -\frac{2}{3}x+6 \\
-4y \le -x+12 \quad \Rightarrow \qquad & amp;y \ge \frac{x}{4}-3
\end{aligned}
$$

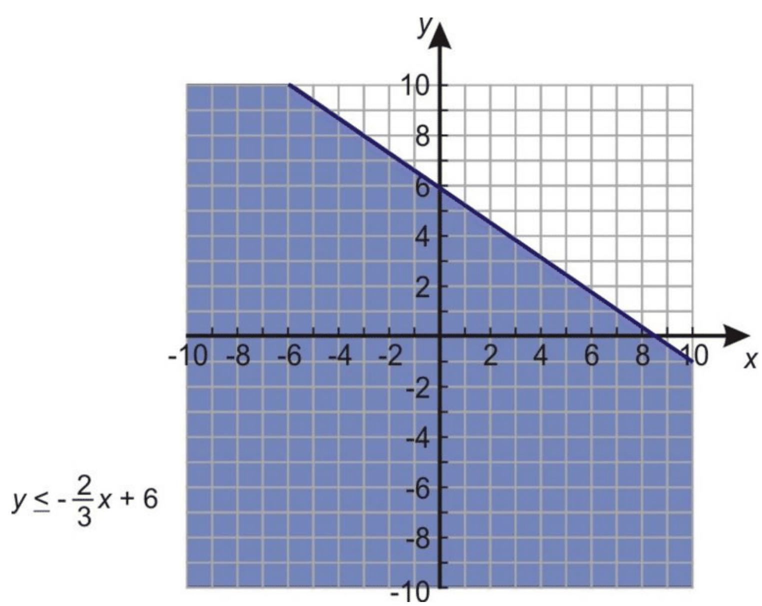
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

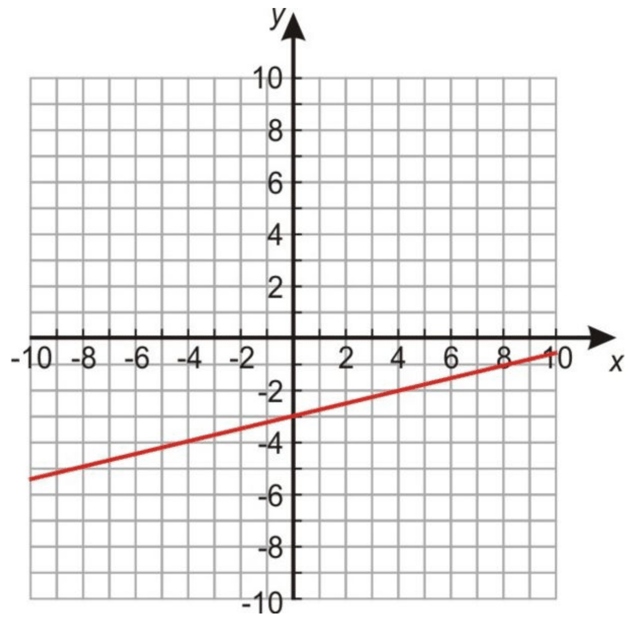
Here's the graph of the first inequality. First, graph the line:



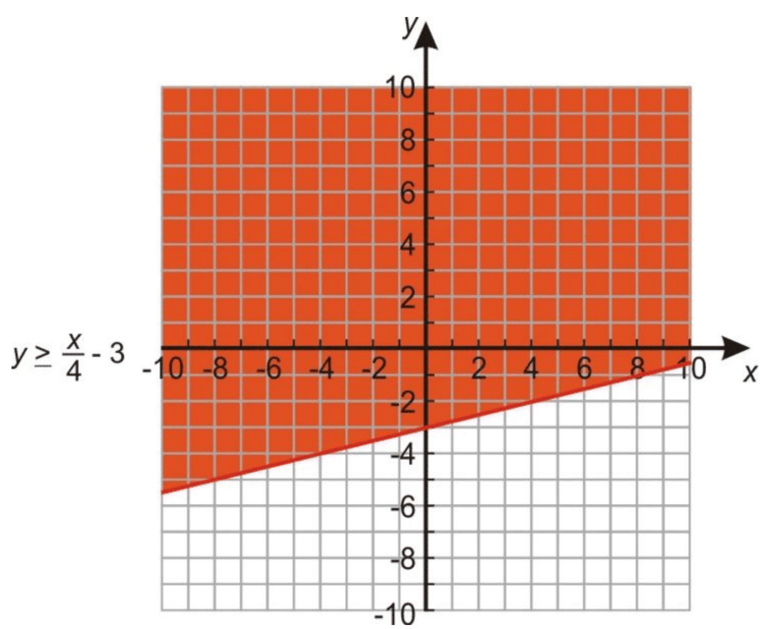
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



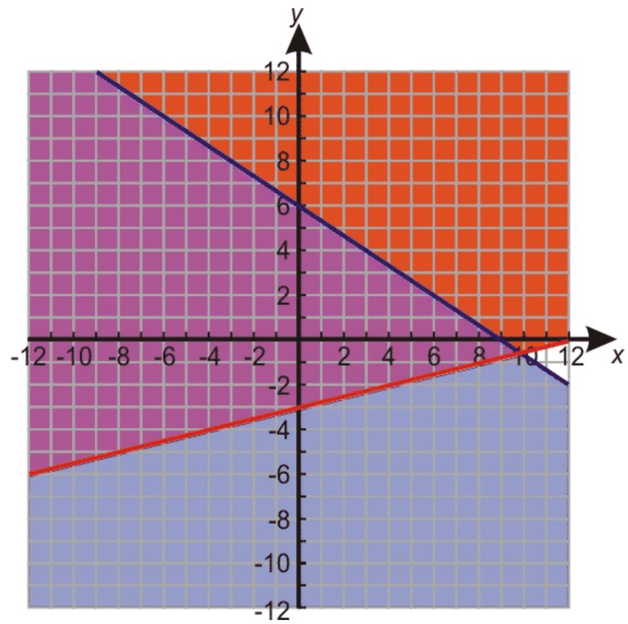
And here's the graph of the second inequality. First, graph the line:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



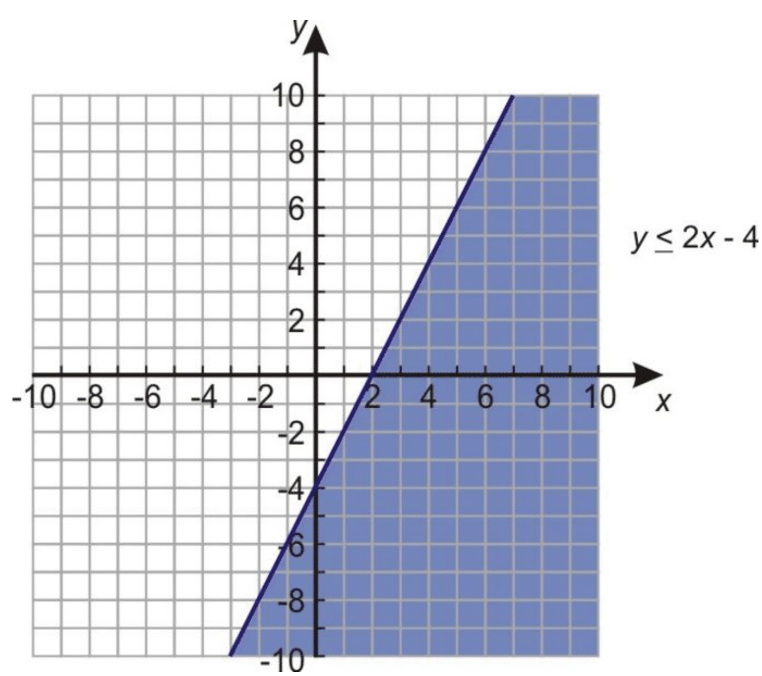
The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

Systems with No Solution

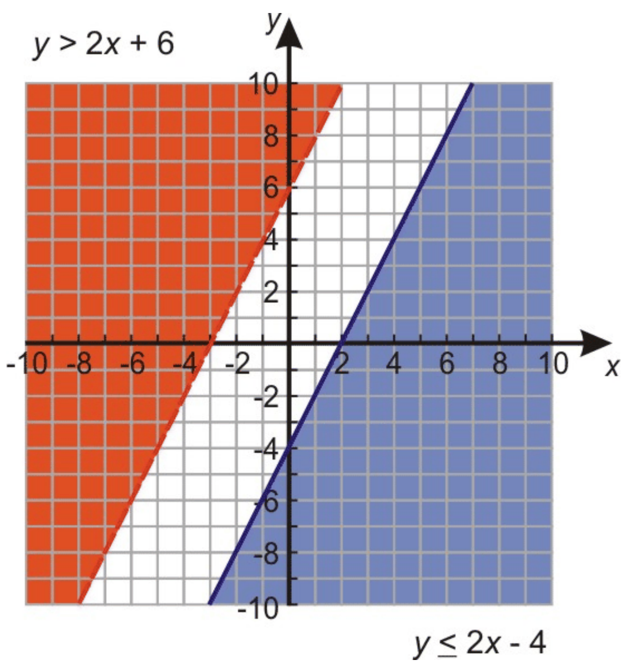
There are also situations where a system of inequalities has no solution. For example, let's solve this system.

$$y \leq 2x - 4$$
$$y > 2x + 6$$

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



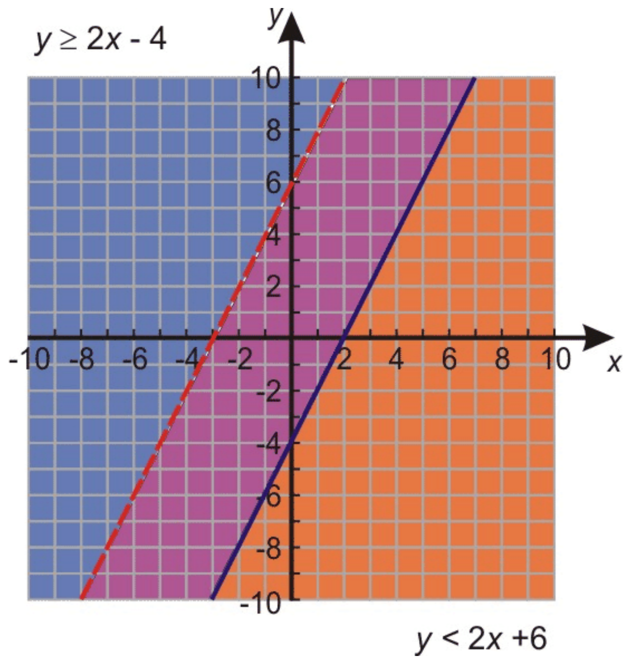
It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

Graph the system

$$y \geq 2x - 4$$
$$y < 2x + 6$$

Draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region (one that continues infinitely in at least one direction). If we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded** - a finite region with three or more sides.

Let's look at a simple example.

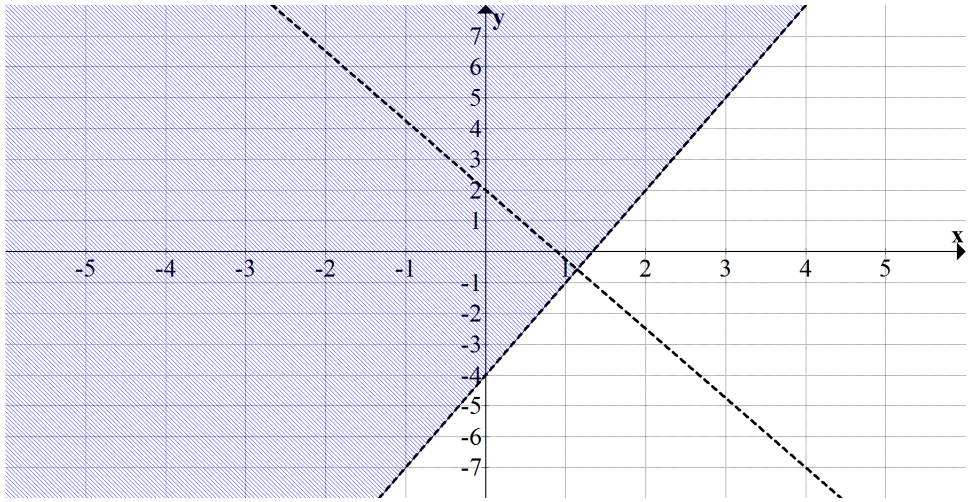
Find the solution to the following system of inequalities.

$$\begin{aligned}3x - y &< 4\\4y + 9x &< 8\\x &\geq 0\\y &\geq 0\end{aligned}$$

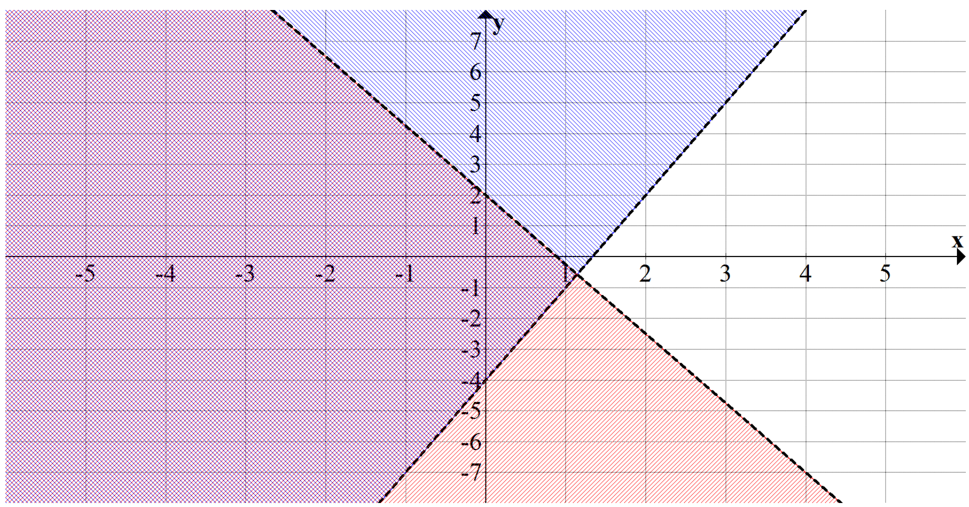
Let's start by writing our inequalities in slope-intercept form.

$$\begin{aligned}&y > 3x - 4\\&y < -\frac{9}{4}x + 2\\&x \geq 0\\&y \geq 0\end{aligned}$$

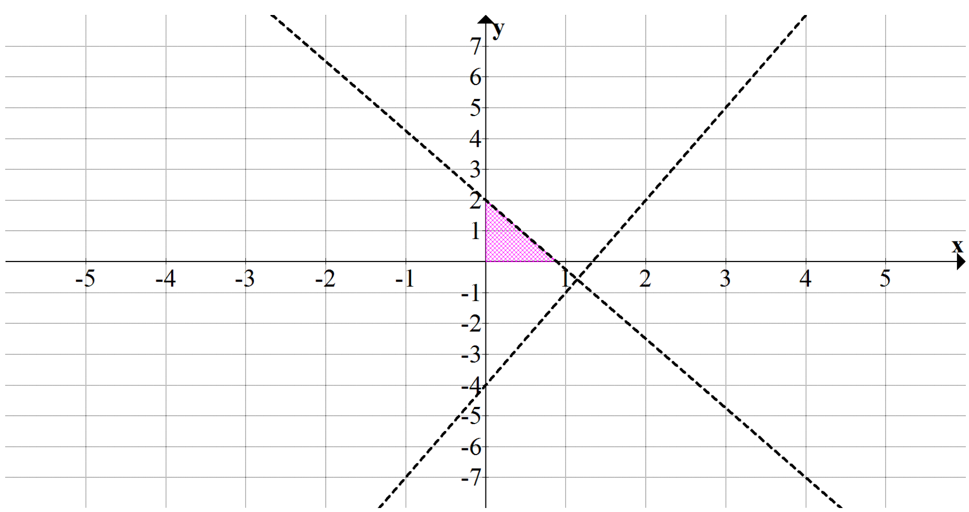
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



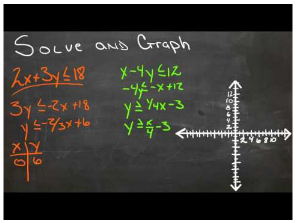
Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

That wasn't obvious until we actually drew the graph!



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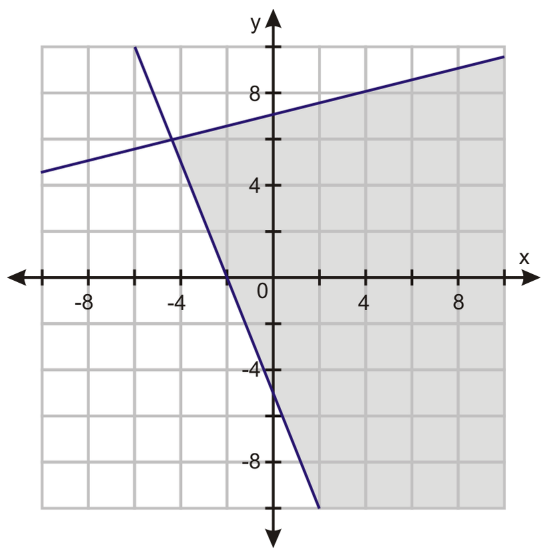
Write the system of inequalities shown below.



There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Review

1. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.
   a. What would happen if we changed that inequality to $y < 3x - 4$?

b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.
c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

2. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.

   a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
   b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6.

$$\begin{aligned} x - y &< -6 \\ 2y &\geq 3x + 17 \end{aligned}$$

7.

$$\begin{aligned} 4y - 5x &< 8 \\ -5x &\geq 16 - 8y \end{aligned}$$

8.

$$\begin{aligned} 5x - y &\geq 5 \\ 2y - x &\geq -10 \end{aligned}$$

9.

$$\begin{aligned} 5x + 2y &\geq -25 \\ 3x - 2y &\leq 17 \\ x - 6y &\geq 27 \end{aligned}$$

10.

$$\begin{aligned} 2x - 3y &\leq 21 \\ x + 4y &\leq 6 \\ 3x + y &\geq -4 \end{aligned}$$

11.

$$\begin{aligned} 12x - 7y &< 120 \\ 7x - 8y &\geq 36 \\ 5x + y &\geq 12 \end{aligned}$$

Summary

You learned about one variable inequalities. Remember that inequalities do not have an equals sign but rather use the $>, <, \geq$, and $\leq$ signs to relate the two mathematical expressions. The rules for solving inequalities remained the same as for equations with one exception. The exception was if you have to multiply or divide by a negative number. If this happens, you have to reverse the sign of the inequality.

You learned to graph inequalities on a real number line. You used an open circle for $>$and $<$. You used a closed circle for $\leq$ and $\geq$.

You also learned about two variable inequalities. The solution to these is expressed graphically using shading.

CHAPTER 9 Unit 9 - Roots and Radicals

Chapter Outline

So far we have seen roots and radicals in passing in the form of square roots. What else is there to see? The square root comes from the squaring operation (raising a number to the power 2), but there is no reason to limit ourselves to just that. We can define a root based on any exponent. There are third roots, fourth roots, fifth roots, etc.

In this chapter we will define these roots and talk about their properties. Then we will discover how to simplify them and combine them arithmetically (add, multiply, etc). Lastly we will investigate how to solve equations that involve radicals and see some applications.

9.1 Square Roots and Irrational Numbers

Learning Objectives

Here you'll learn how to decide whether a number is rational or irrational and how to take the square root of a number.

Concept Problem

Suppose an elementary school has a square playground with an area of 3000 square feet. Could you find the width of the playground? Would the width be a rational or irrational number?

Square Roots and Irrational Numbers

The **square root** of a number n is any number s such that $s^2 = n$.

Every positive number has two square roots, one positive and the other negative. The symbol used to represent the square root is $\sqrt{x}$. To avoid ambiguity, we assume that this is the positive square root of x (called the **principal root**). To show both the positive and negative values, you can use the symbol $\pm$, read "plus or minus."

For example:

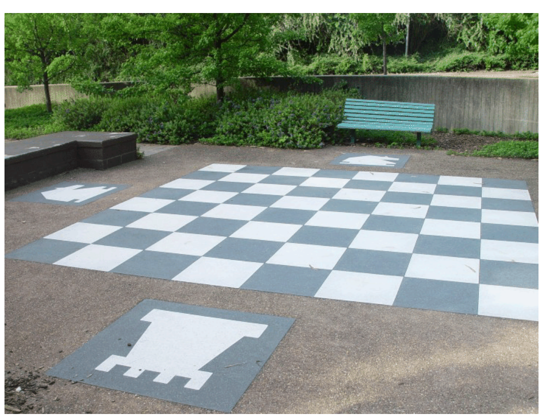
$\sqrt{81} = 9$ means the positive square root of 81.

$-\sqrt{81} = -9$ means the negative square root of 81.

$\pm\sqrt{81} = \pm 9$ refers to the positive and negative square roots of 81.

Let's solve the following problem by using square roots:

Human chess is a variation of chess, often played at Renaissance fairs, in which people take on the roles of the various pieces on a chessboard. The chessboard is played on a square plot of land that measures 324 square meters with the chess squares marked on the grass. How long is each side of the chessboard?



The human chessboard measures 324 square meters.

The area of a square is $s^2 = Area$. The value of Area can be replaced with 324.

$$s^2 = 324$$

The value of s represents the square root of 324.

$$s = \sqrt{324} = 18$$

The chessboard is 18 meters long by 18 meters wide.

Approximating Square Roots

When the square root of a number is a whole number, this number is called a **perfect square.** 9 is a perfect square because $\sqrt{9} = 3$.

Not all square roots are whole numbers. Many square roots are irrational numbers, meaning there is no rational number equivalent. For example, 2 is the square root of 4 because $2 \times 2 = 4$. The number 7 is the square root of 49 because $7 \times 7 = 49$. What is the square root of 5?

There is no whole number multiplied by itself that equals five, so $\sqrt{5}$ is not a whole number. To find the value of $\sqrt{5}$, we can use estimation.

To estimate the square root of a number, look for the perfect integers **less than** and **greater than** the value, and then estimate the decimal.

Let's estimate $\sqrt{5}$:

The perfect square below 5 is 4 and the perfect square above 5 is 9. Therefore, $4 < 5 < 9$. Therefore, $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, or $2 < \sqrt{5} < 3$. Because 5 is closer to 4 than 9, the decimal is a low value: $\sqrt{5} \approx 2.2$.

Simplifying Square Roots

Many positive integers are not perfect squares, but their square roots can still be partially simplified using the multiplicative property of square roots:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$
$$\text{if a>0 and b>0}$$

There are other properties of radicals that we will get into later, but for the moment we just need this one.

For example, suppose you do not have a calculator and you need to find $\sqrt{18}$. You know there is no whole number squared that equals 18, so $\sqrt{18}$ is an irrational number. The value is between $\sqrt{16} = 4$ and $\sqrt{25} = 5$. However, we need to find the exact value of $\sqrt{18}$.

Begin by writing the **prime factorization** of $\sqrt{18}$. $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$. $\sqrt{9} = 3$ but $\sqrt{2}$ does not have a whole number value. Therefore, the exact value of $\sqrt{18}$ is $3\sqrt{2}$.

You can check your answer on a calculator by finding the decimal approximation for each square root.

Let's find the exact value of $\sqrt{75}$:

$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$$

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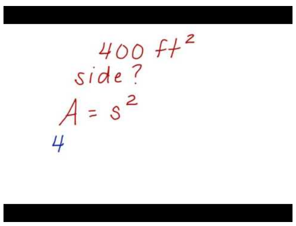

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Concept Problem Revisited

Earlier, you were told that an elementary school has a square playground with an area of 3000 square feet. What is the width of the playground? Is the width a rational or irrational number?

Since the playground is a square, we can use the formula for the area of a square: $a = s^2$.

$$\begin{aligned}
a &= s^2 \\
3000 &= s^2 \\
\sqrt{3000} &= \sqrt{s^2} \\
\sqrt{3000} &= s
\end{aligned}$$

Now, simplify:

$$\begin{aligned}
\sqrt{100 \times 30} &= s \\
\sqrt{100} \times \sqrt{30} &= s \\
10\sqrt{30} &= s
\end{aligned}$$

The width of the playground is $10\sqrt{30}$ feet. Since 30 is not a perfect square, the width is irrational.

Example A

The area of a square is 50 square feet. What are the lengths of its sides?

Using the area of a square formula, $a = s^2$:

$$\begin{aligned}
a &= s^2 \\
50 &= s^2 \\
\sqrt{50} &= \sqrt{s^2} \\
\sqrt{50} &= s
\end{aligned}$$

Now we will simplify:

$\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

The length of each side of the square is $5\sqrt{2}$ feet.

Practice Problems

Find the following square roots exactly without using a calculator. Give your answer in the simplest form.

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{20}$
4. $\sqrt{200}$
5. $\sqrt{2000}$
6. $\sqrt{\frac{1}{4}}$
7. $\sqrt{\frac{9}{4}}$
8. $\sqrt{0.16}$
9. $\sqrt{0.1}$
10. $\sqrt{0.01}$

Use a calculator to find the following square roots. Round to two decimal places.

11. $\sqrt{13}$
12. $\sqrt{99}$
13. $\sqrt{123}$
14. $\sqrt{2}$
15. $\sqrt{2000}$
16. $\sqrt{0.25}$
17. $\sqrt{1.35}$
18. $\sqrt{0.37}$
19. $\sqrt{0.7}$
20. $\sqrt{0.01}$

9.2 Defining nth Roots

Learning Objectives

Here you'll learn how to define and use n^{th} roots.

Concept Problem

The volume of a cube is found to be 512. What is the length of each side of the cube?

nth Roots

So far, we have seen exponents with integers and the square root. In this concept, we will link roots and exponents. For example, we know that $\sqrt{25} = 5$ because $5^2 = 25$. There is a fundamental relationship between the square root and squaring operations.

We can expand on this idea to include powers other than 2.

An **nth** root of a number a is another number b such that $b^n = a$. We use a radical notation based on the square root to discuss nth roots:

$$b = \sqrt[n]{a}$$

In this situation n is called the **index** of the radical, and a is called the **radicand**.

And, just like simplifying square roots, we can simplify n^{th} roots.

Example A

Find $\sqrt[6]{729}$.

To simplify a number to the sixth root, there must be 6 of the same factor to pull out of the root.

$729 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$

Therefore, $\sqrt[6]{729} = \sqrt[6]{3^6} = 3$. The sixth root and the sixth power cancel each other out. We say that 3 is the sixth root of 729.

From this problem, we can see that it does not matter where the exponent is placed, it will always cancel out with the root.

$$\sqrt[6]{3^6} = \sqrt[6]{3}^6 \ or \ \left(\sqrt[6]{3}\right)^6$$
$$\sqrt[6]{729} = (1.2009\ldots)^6$$
$$3 = 3$$

So, it does not matter if you evaluate the root first or the exponent.

Now, let's evaluate the following expressions without a calculator.

1. $\sqrt[5]{32^3}$

If you solve this problem as written, you would first find 32^3 and then apply the 5^{th} root.

$\sqrt[5]{32^3} = \sqrt[5]{38768} = 8$

However, this would be very difficult to do without a calculator. This is an example where it would be easier to apply the root and then the exponent. Let's rewrite the expression and solve.

$\sqrt[5]{32}^3 = 2^3 = 8$

2. $\sqrt{16}^3$

This problem does not need to be rewritten. $\sqrt{16} = 4$ and then $4^3 = 64$.

Rules for Radicals

We can see that radicals and exponents are closely related. In fact, there are rules for radicals that correspond to the rules for exponents we saw earlier.

If $a > 0$ and $b > 0$, then...

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

We can use these rules to evaluate and simplify radicals. They allow us to break radical expressions into smaller, more manageable pieces.

Concept Problem Revisited

Earlier, we were asked to find the length of each side of the cube.

Recall that the volume of a cube is $V = s^3$, where s is the length of each side. Since the volume is 512 we have $512 = s^3$. This means that $s = \sqrt[3]{512}$.

We need to evaluate $\sqrt[3]{512}$. We can do this by factoring 512 and finding groups of 3 numbers to pull out.

$$\sqrt[3]{512} = \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

We have three groups of 3 in this case. Each one will pull a single 2 out of the radical:

$$\begin{aligned}
\sqrt[3]{512} &= \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} \\
&= \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} \\
&= \sqrt[3]{2^3} \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{2^3} \\
&= 2 \cdot 2 \cdot 2 \\
\sqrt[3]{512} &= 8
\end{aligned}$$

So the cube must have a side length of 8.

Variables and Radicals

Numbers are not the only things that can appear under radicals. Is is possible to simplify a radical that contains a variable, but first we have to have a discussion about signs.

As we will see in detail in the next section, negative numbers and radicals have a complicated relationship. To keep things straightforward, we will assume that variables only represent positive numbers in this section. With that in mind, let's continue.

How could we simplify $\sqrt[4]{x^{12}}$? We can use the same strategy as in the concept problem: break the radicand into pieces that will simplify individually. In this case, our index is 4, so we want to split x^{12} into groups of 4 x's:

$$\begin{aligned}\sqrt[4]{x^{12}} &= \sqrt[4]{x^4 \cdot x^4 \cdot x^4} \\ &= \sqrt[4]{x^4} \cdot \sqrt[4]{x^4} \cdot \sqrt[4]{x^4} \\ &= x \cdot x \cdot x \\ &= x^3\end{aligned}$$

You may notice a shortcut here: if the power is divisible by the index, you can divide to find out the remaining power on x.

Example B

$\sqrt[5]{x^{20}} = x^4$

$\sqrt[7]{k^{14}} = k^2$

$\sqrt{x^{16}} = x^8$

Simplify each expression below, without a calculator.

Example C

$\sqrt[4]{625z^8}$

First, you can separate this number into two different roots, $\sqrt[4]{625} \cdot \sqrt[4]{z^8}$. Now, simplify each root.

$\sqrt[4]{625} \cdot \sqrt[4]{z^8} = \sqrt[4]{5^4} \cdot \sqrt[4]{z^4 \cdot z^4} = 5z^2$

When looking at the z^8, think about how many z^4 you can even pull out of the fourth root. The answer is 2, or a z^2, outside of the radical.

Example D

$\sqrt[7]{32x^5y}$

$32 = 2^5$, which means there are not 7 2's that can be pulled out of the radical. Same with the x^5 and the y. Therefore, you cannot simplify the expression any further.

9.3 Simplification of Radical Expressions and Rational Exponents

Learning Objectives

Here you'll learn how to simplify radical expressions.

Concept Problem

A cube has a volume of 1080. Can we determine the measure of the cube's side length?

Simplifying Radical Expressions

Some radical expressions simplify completely. For instance, $\sqrt[3]{125} = 5$. This works because the prime factorization of 125 is $5 \cdot 5 \cdot 5 = 5^3$. So $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$.

Other expressions, though, like $\sqrt[3]{250}$, don't simplify completely. They can, however, be partially simplified by using the prime factorization of the radicand and the multiplication rule for radicals. We can split the radical into two parts, one of which simplifies, and the other is leftover:

$$\begin{aligned}\sqrt[3]{250} &= \sqrt[3]{5 \cdot 5 \cdot 5 \cdot 2}\\ &= \sqrt[3]{5 \cdot 5 \cdot 5} \cdot \sqrt[3]{2}\\ &= \sqrt[3]{5^3} \cdot \sqrt[3]{2}\\ &= 5\sqrt[3]{2}\end{aligned}$$

Simplifying Radicals

To simplify a radical expression the basic strategy is to use the rules of radicals to split the expression into parts that simplify and parts that are leftover. As in the last section, in this section we will assume that all variables represent positive numbers only.

Example A

Simplify $\sqrt[3]{72}$

Solution: First, make the prime factorization of 72:

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

The index of this radical is 3, so we need to find groups of 3 numbers to pull anything out of the radical. In this case we have enough 2's to pull something out and then simplify:

$$\begin{aligned}\sqrt[3]{72} &= \sqrt[3]{2\cdot 2\cdot 2\cdot 3\cdot 3}\\ &= \sqrt[3]{2^3}\cdot\sqrt[3]{3^2}\\ &= 2\cdot\sqrt[3]{9}\\ &= 2\sqrt[3]{9}\end{aligned}$$

Notice that we did not have enough 3's to pull anything out of the radical. They are the leftovers that stay inside.

Example B

Simplify $\sqrt[4]{x^{14}}$.

Solution: We can't use the division shortcut here, because 14 is not divisible by 4. What we can do, though, is split our x's into two groups where one simplifies and the other is leftover. In this case, if we group 12 x's together, then it will work out:

$$\begin{aligned}\sqrt[4]{x^{14}} &= \sqrt[4]{x^{12}\cdot x^2}\\ &= \sqrt[4]{x^{12}}\cdot\sqrt[4]{x^2}\\ &= x^3\cdot\sqrt[4]{x^2}\\ &= x^3\sqrt[4]{x^2}\end{aligned}$$

Why did we choose 12? Because it is the largest integer that is less than 14 (the number of x's we have) but still divisible by 4.

Example C

Simplify $\sqrt[3]{3024x^{19}y^{32}}$

Solution: Sweet honey bee of infinity, this looks tough! Actually, there is good news here: the multiplication rule for radicals will let us solve each piece (number, x, and y) individually and then put them the together at the end to get the final answer.

First, simplify $\sqrt[3]{3024}$ using its prime factorization:

$$\begin{aligned}\sqrt[3]{3024} &= \sqrt[3]{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 7}\\ &= \sqrt[3]{2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdot 2\cdot 7}\\ &= \sqrt[3]{2^3}\cdot\sqrt[3]{3^3}\cdot\sqrt[3]{2\cdot 7}\\ &= 2\cdot 3\cdot\sqrt[3]{2\cdot 7}\\ &= 6\sqrt[3]{14}\end{aligned}$$

Next, simplify $\sqrt[3]{x^{19}}$ by breaking the x's into two groups:

$$\begin{aligned}\sqrt[3]{x^{19}} &= \sqrt[3]{x^{18}} \cdot \sqrt[3]{x^{1}} \\ &= x^{6}\sqrt[3]{x}\end{aligned}$$

Now, simplify $\sqrt[3]{y^{32}}$: by breaking the y's into two groups:

$$\begin{aligned}\sqrt[3]{y^{32}} &= \sqrt[3]{y^{30} \cdot y^{2}} \\ &= \sqrt[3]{y^{30}} \cdot \sqrt[3]{y^{2}} \\ &= y^{10} \cdot \sqrt[3]{y^{2}} \\ &= y^{10}\sqrt[3]{y^{2}}\end{aligned}$$

We have simplified our three pieces. The last step is to combine them to produce the final answer:

$$6\sqrt[3]{14} \cdot x^{6}\sqrt[3]{x} \cdot y^{10}\sqrt[3]{y^{2}} =$$

$$\boxed{6x^{6}y^{10}\sqrt[3]{14xy^{2}}}$$

Unreal Radicals

Some roots do not have real values. For instance, take $\sqrt[4]{-16}$. To evaluate this we would need to find a real number whose fourth power is -16... but there isn't one! We could try -2, but $(-2)^{4} = 16$. So we can say that $\sqrt[4]{-16}$ is not a real number.

Even-numbered roots of negative numbers are not real numbers, since any real number raised to an even power will produce a positive result.

Negative numbers are not always a problem, though. For example, $\sqrt[3]{-64} = -4$ since $(-4)^{3} = -64$. We only run into issues with negative numbers under even roots.

This leads us to the general statement: "$\sqrt[n]{x}$ is not a real number when n is an even whole number and $x < 0$."

Let's evaluate the following radicals:

1. $\sqrt[3]{64}$

   $\sqrt[3]{64} = 4$ because $4^{3} = 64$

2. $\sqrt[4]{-81}$

   $\sqrt[4]{-81}$ is not a real number because n is an even whole number and $-81 < 0$.

Rational Exponents

For integer values of x and whole values of y:

$$a^{\frac{x}{y}} = \sqrt[y]{a^{x}}$$

Example D

Rewrite $x^{\frac{5}{6}}$ using radical notation:

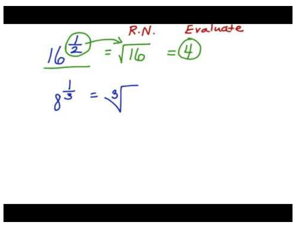
Solution: This is correctly read as the sixth root of x to the fifth power. Writing in radical notation:

$$x^{\frac{5}{6}} = \sqrt[6]{x^5}$$

Example E

Evaluate $81^{\frac{5}{4}}$.

Solution: By the rule above we have $81^{\frac{5}{4}} = \sqrt[4]{81}^5 = 3^5 = 243$.



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Concept Problem Revisited

We know that our cube has a volume of 1080, so its side length is $\sqrt[3]{1080}$. We can simplify this by factoring:

$$\begin{aligned}\sqrt[3]{1080} &= \sqrt[3]{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5} \\ &= 2 \cdot 3 \cdot \sqrt[3]{5} \\ &= 6\sqrt[3]{5}\end{aligned}$$

Practice Problems

1. For which values of n is $\sqrt[n]{-16}$ undefined?

Evaluate each radical expression.

2. $\sqrt{169}$
3. $\sqrt[4]{81}$
4. $\sqrt[3]{-125}$
5. $\sqrt[5]{1024}$

Write each expression as a rational exponent.

6. $\sqrt[3]{14}$
7. $\sqrt[4]{zw}$
8. $\sqrt{a}$

9. $\sqrt[9]{y^3}$

Write the following expressions in simplest radical form.

10. $\sqrt{24}$
11. $\sqrt{300}$
12. $\sqrt[5]{96}$
13. $\sqrt{\frac{240}{567}}$
14. $\sqrt[3]{500}$
15. $\sqrt[6]{64x^8}$

9.4 Multiplication and Division of Radicals

Learning Objectives

Here you'll learn how to multiply and divide by radicals, as well as how to rationalize denominators.

What if you knew that the area of a rectangular mirror is $12\sqrt{6}$ square feet and that the width of the mirror is $2\sqrt{2}$ feet? Could you find the exact length of the mirror?

Multiplying and Dividing Radicals

Multiplying Radicals

In the first section of this chapter, we introduced the multiplication rule for radicals:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \qquad \text{if a>0 and b>0}$$

It is important to note that this rule only works when the radicals have the same index.

Example A

Multiply $\sqrt{3} \cdot \sqrt{12}$

Solution: $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = \boxed{6}$

Sometimes the individuals radicals cannot be simplified, but their product can, as with the next example.

Example B

Multiply and simplify the result: $\sqrt[4]{12} \cdot \sqrt[4]{756}$

Solution: We could multiply the two radicands, but in this case it will be easier to factor them and then put the them into a single radical.

$$\sqrt[4]{12} = \sqrt[4]{2 \cdot 2 \cdot 3}$$

$$\sqrt[4]{756} = \sqrt[4]{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7}$$

Notice that neither of these can be simplified on their own. When we put them together, though, we will be able to pull some factors out:

$$\begin{aligned}\sqrt[4]{12}\cdot\sqrt[4]{756} &= \sqrt[4]{2\cdot2\cdot3}\cdot\sqrt[4]{2\cdot2\cdot3\cdot3\cdot3\cdot7}\\ &= \sqrt[4]{2\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot3\cdot7}\\ &= 2\cdot3\cdot\sqrt[4]{7}\\ &= \boxed{6\sqrt[4]{7}}\end{aligned}$$

Dividing Radicals

Dividing radicals is more complicated; however, we can often utilize the division rule for radicals to simplify. The radical rule for division says:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \qquad \text{if a>0 and b>0}$$

Again, it is important to note that the indexes of the radicals must be the same in order for this rule to work.

Example C

Simplify the expression: $\frac{\sqrt{7350}}{\sqrt{6}}$.

Solution: Notice that the denominator will not simplify on its own; however we will be able to simplify once we combine into a single radical:

$$\begin{aligned}\frac{\sqrt{7350}}{\sqrt{6}} &= \sqrt{\frac{7350}{6}}\\ &= \sqrt{1225}\\ &= \sqrt{5\cdot5\cdot7\cdot7}\\ &= 5\cdot7\\ &= \boxed{35}\end{aligned}$$

Example D

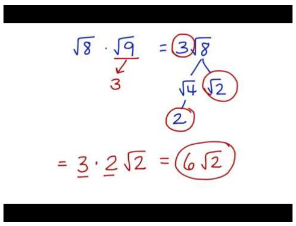
Simplify the expression: $\frac{\sqrt{3x^9y^5}}{\sqrt{48x^4y^{11}}}$

Solution: In this problem, we could simplify each radical separately and then see what can be cancelled, but it will be easier to combine the radicals first and then simplify:

$$\begin{aligned}\frac{\sqrt{3x^9y^5}}{\sqrt{48x^4y^{11}}} &= \sqrt{\frac{3x^9y^5}{48x^4y^{11}}}\\ &= \sqrt{\frac{x^5}{16y^6}}\end{aligned}$$

Now that we have cancelled terms inside the fraction, we can split the radical back into two and simplify each one:

$$\sqrt{\frac{x^5}{16y^6}} = \frac{\sqrt{x^5}}{\sqrt{16y^6}} = \boxed{\frac{x\sqrt{x}}{4y^3}}$$



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Concept Problem Revisited

We were asked to find the length of a mirror with an area of $12\sqrt{6}$ square feet and a width of $2\sqrt{2}$ feet. We know that for a rectangle Area = Length · Width which means that $\text{Length} = \frac{\text{Area}}{\text{Width}}$.

You can find the length by plugging in the known values into the formula for area and width and solving.

$$\text{Length} = \frac{\text{Area}}{\text{Width}} = \frac{12\sqrt{6}}{2\sqrt{2}} = \boxed{6\sqrt{3}}$$

The length of the mirror is $6\sqrt{3}$.

9.5 Addition and Subtraction of Radicals

Concept Problem

Suppose that you're taking a trip, and you'll be making two stops. This distance from your starting point to your first stop is $14\sqrt{2}$ miles, and the distance from your first stop to your second stop is $9\sqrt{2}$ miles. How far will you travel in total? What operation would you have to perform to find the answer to this question?

Adding and Subtracting Radicals

If you see an expression like $\sqrt{2}+\sqrt{3}$ you may be tempted to say that it is equal to $\sqrt{5}$; however, treating radicals like this lead to contradictions. For instance:

$$\sqrt{4}+\sqrt{4} = \qquad \sqrt{4}+\sqrt{4} =$$
$$\sqrt{8} = \qquad 2+2 =$$
$$2\sqrt{2} \qquad 4$$

So radical expressions cannot be added/subtracted in the same way as integers. In fact, radicals follow the rules of combining like terms that we have used with variables before. With a variable x we have:

$$2x+5x=7x$$

The same would hold if we had $\sqrt[3]{7}$ instead of x:

$$2\sqrt[3]{7}+5\sqrt[3]{7}=7\sqrt[3]{7}$$

Notice that the $\sqrt[3]{7}$ term did not change; only the coefficient in front of it.

To add or subtract radicals, they must have the same root and radicand, and in that case they follow the rules of like terms.

$$a\sqrt[n]{x}+b\sqrt[n]{x}=(a+b)\sqrt[n]{x}$$

Example A

$3\sqrt{5}+6\sqrt{5}$

The value "$\sqrt{5}$" is considered a like term. Using the rule above:

$$3\sqrt{5}+6\sqrt{5}=(3+6)\sqrt{5}=9\sqrt{5}$$

Example B

$2\sqrt[3]{13}+6\sqrt[3]{12}$

The cube roots are not like terms, so there can be no further simplification.

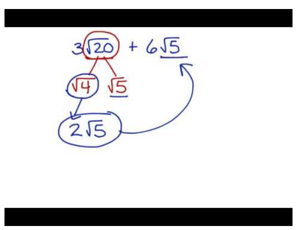
Example C

$4\sqrt{3}+2\sqrt{12}$

In some cases, the radical may need to be reduced before addition/subtraction is possible.

$\sqrt{12}$ simplifies to $2\sqrt{3}$.

$$4\sqrt{3}+2\sqrt{12} \rightarrow 4\sqrt{3}+2\left(2\sqrt{3}\right)$$
$$4\sqrt{3}+4\sqrt{3} = 8\sqrt{3}$$



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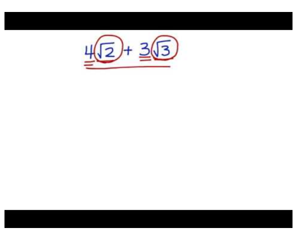

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Concept Problem Revisited

Earlier, you were asked to find the total distance you travel in a trip. You know that the distance from your starting point to your first stop is $14\sqrt{2}$ miles, and the distance from your first stop to your second stop is $9\sqrt{2}$ miles.

To calculate the total distance you traveled, you need to add the radicals.

$$Total\ distance\ traveled = 14\sqrt{2}+9\sqrt{2}$$
$$Total\ distance\ traveled = 23\sqrt{2}$$

Example D

$3\sqrt[3]{2}+5\sqrt[3]{16}$.

Begin by factoring the second radical.	$3\sqrt[3]{2}+5\sqrt[3]{16}=3\sqrt[3]{2}+5\sqrt[3]{2\cdot 8}=3\sqrt[3]{2}+5\sqrt[3]{2\cdot 2^3}$
Simplify the second radical using properties of roots.	$=3\sqrt[3]{2}+5\sqrt[3]{2^3}\cdot\sqrt[3]{2}=3\sqrt[3]{2}+5\cdot 2\sqrt[3]{2}=3\sqrt[3]{2}+10\sqrt[3]{2}$
The terms are now alike and can be added.	$=(3+10)\sqrt[3]{2}=13\sqrt[3]{2}$

Practice Problems

Write the following expressions in simplest radical form.

1. $\sqrt[3]{48a^3b^7}$
2. $\sqrt[3]{\frac{16x^5}{135y^4}}$
3. True or false? $\sqrt[7]{5}\cdot\sqrt[6]{6}=\sqrt[42]{30}$

Simplify the following expressions as much as possible.

4. $3\sqrt{8}-6\sqrt{32}$
5. $\sqrt{180}+6\sqrt{405}$
6. $\sqrt{6}-\sqrt{27}+2\sqrt{54}+3\sqrt{48}$
7. $\sqrt{8x^3}-4x\sqrt{98x}$
8. $\sqrt{48a}+\sqrt{27a}$
9. $\sqrt[3]{4x^3}+x\sqrt[3]{256}$

9.6 Radical Equations

Learning Objectives

Here you'll learn how to find the solutions to radical equations.

Concept Problem

Suppose your teacher has instructed the members of your math class to work in pairs, and she has asked you to find the length of a line segment. You get $\sqrt{2x+6}$ units for the length, and your partner gets 10 units. You ask your teacher who is right, and she says both of you are right! Can you set up an equation and solve for x in this case? How would you do it?

Radical Equations

We saw in sections on simplifying radicals that radicals and exponents are opposites of each other in some senses. A radical can 'undo' an exponent and vice versa. We can utilize this to solve equations that involve radicals.

Solving radical equations is no different from solving linear or quadratic equations. Before you can begin to solve a radical equation, you must know how to cancel the radical. To do that, you must know its **inverse**.

TABLE 9.1:

Original Operation	Inverse Operation
Square Root	Squaring (to the second power)
Cube Root	Cubing (to the third power)
Fourth Root	Fourth power
"nth" Root	"nth" power

To solve a radical equation, you apply the solving equation steps you learned in previous sections, including the inverse operations for roots.

Steps for Solving

1. Isolate the radical
2. Raise to an appropriate power to remove the radical
3. If there are multiple radicals, repeats steps 1 and 2 until they are all removed.
4. Solve the resulting equation. This will give you the potential solutions to the equation.
5. Check your solutions in the **original** equation to see which ones actually work.

Example A

Solve $\sqrt{2x-1}=5$.

In this case the radical is already isolated, so the next step is to remove the square root. Square both sides.

$$\begin{aligned}\left(\sqrt{2x-1}\right)^2 &= 5^2\\ 2x-1 &= 25\\ 2x &= 26\\ x &= 13\end{aligned}$$

Remember to check your answer by substituting it into the original problem to see if it makes sense.

Extraneous Solutions

Not every potential solution of a radical equation will check in the original problem. This is called an **extraneous solution**. This means you can find a solution using algebra, but it will not work when checked. This is because of the rule in a previous section:

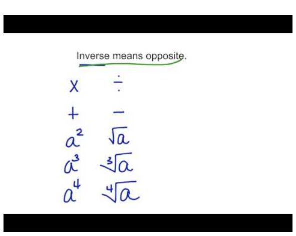
$\sqrt[n]{x}$ is undefined when n is an even whole number and $x < 0$, or, in words, even roots of negative numbers are undefined.

Example B

$\sqrt{x-3} - \sqrt{x} = 1$

$$\begin{aligned}\text{Isolate one of the radical expressions.} &\qquad \sqrt{x-3} = \sqrt{x}+1\\ \text{Square both sides.} &\qquad \left(\sqrt{x-3}\right)^2 = \left(\sqrt{x}+1\right)^2\\ \text{Remove parentheses.} &\qquad x-3 = \left(\sqrt{x}\right)^2 + 2\sqrt{x}+1\\ \text{Simplify.} &\qquad x-3 = x+2\sqrt{x}+1\\ \text{Now isolate the remaining radical.} &\qquad -4 = 2\sqrt{x}\\ \text{Divide all terms by 2.} &\qquad -2 = \sqrt{x}\\ \text{Square both sides.} &\qquad x = 4\end{aligned}$$

Check: $\sqrt{4-3} \stackrel{?}{=} \sqrt{4}+1 \Rightarrow \sqrt{1} \stackrel{?}{=} 2+1 \Rightarrow 1 \neq 3$. The solution does not check out. Therefore, $x = 4$ is an extraneous solution. The equation has no real solutions.



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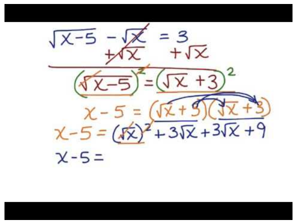

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Concept Problem Revisited

Earlier, you and a class partner are asked to find the length of a line segment. You get $\sqrt{2x+6}$ units for the length, and your partner gets 10 units. You ask your teacher who is right, and she says both of you are right! Can you set up an equation and solve for x in this case? How would you do it?

Because both answers are right, you can set them equal to each other and solve for x.

$$\begin{aligned}\sqrt{2x+6} &= 10\\ \sqrt{2x+6}^2 &= 10^2\\ 2x+6 &= 100\\ 2x &= 94\\ x &= 47\end{aligned}$$

If you set $x = 47$, then both you and your partner have the same answer.

Example C

Solve $\sqrt{x+15} = \sqrt{3x-3}$.

Begin by canceling the square roots by squaring both sides.

$$\begin{aligned}\left(\sqrt{x+15}\right)^2 &= \left(\sqrt{3x-3}\right)^2\\ x+15 &= 3x-3\\ \text{Isolate the } x-\text{variable}: \quad 18 &= 2x\\ x &= 9\end{aligned}$$

Check the solution: $\sqrt{9+15} = \sqrt{3(9)-3} \rightarrow \sqrt{24} = \sqrt{24}$. The solution checks out.

Practice Problems

In 1-16, find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{x+2}-2=0$

2. $\sqrt{3x-1} = 5$
3. $2\sqrt{4-3x}+3 = 0$
4. $\sqrt[3]{x-3} = 1$
5. $\sqrt[4]{x^2-9} = 2$
6. $\sqrt[3]{-2-5x}+3 = 0$
7. $\sqrt{x} = x-6$
8. $\sqrt{x^2-5x}-6 = 0$
9. $\sqrt{(x+1)(x-3)} = x$
10. $\sqrt{x+6} = x+4$
11. $\sqrt{x} = \sqrt{x-9}+1$
12. $\sqrt{3x+4} = -6$
13. $\sqrt{10-5x}+\sqrt{1-x} = 7$
14. $\sqrt{2x-2}-2\sqrt{x}+2 = 0$
15. $\sqrt{2x+5}-3\sqrt{2x-3} = \sqrt{2-x}$
16. $3\sqrt{x}-9 = \sqrt{2x-14}$
17. The area of a triangle is 24 in^2 and the height of the triangle is twice as long as the base. What are the base and the height of the triangle?
18. The volume of a square pyramid is given by the formula $V = \frac{A(h)}{3}$, where $A =$ area of the base and $h =$ height of the pyramid. The volume of a square pyramid is 1,600 cubic meters. If its height is 10 meters, find the area of its base.
19. The volume of a cylinder is 245 cm^3 and the height of the cylinder is one-third the diameter of the cylinder's base. The diameter of the cylinder is kept the same, but the height of the cylinder is increased by two centimeters. What is the volume of the new cylinder? $(\text{Volume} = \pi r^2 \cdot h)$
20. The height of a golf ball as it travels through the air is given by the equation $h = -16t^2+256$. Find the time when the ball is at a height of 120 feet.

In this chapter you learned that there are different kinds of radicals beyond square roots. You saw how an nth root can be simplified by pulling out perfect nth powers. On the arithmetic side, you learned that radicals can usually only be combined (added or multiplied) if they have the same index.

You also learned how to solve equations by isolating a radical and then raising both sides to an appropriate power. For example, if you isolated a square root you would then square both sides in order to remove it.

CHAPTER 10 Unit 10 - Quadratic Equations and Quadratic Functions

Chapter Outline

Introduction

Here you'll learn more about quadratic equations and quadratic functions. You will learn new methods for solving quadratic equations and discover the connections between a quadratic equation and its corresponding quadratic function. You will discover a new set of numbers called complex numbers and see how complex numbers are related to quadratic functions with no x-intercepts.

10.1 Solving Equations Using Square Roots

Concept Problem

Joe has two square garden plots in his backyard where he grows eggplants and watermelons. They each have a side length of 5 feet.

Currently the gardens have a total area of $5^2+5^2=50$ sq feet. Joe realizes that this is too small. He would like to double the total area to 100 sq feet; however, he also wants to maintain the identical square shapes of the plots, because that's just how Joe is. How much larger does he need to make each plot in order to achieve his goal?

Solving Equations with Square Roots

You may already know that squaring a number and taking the square root of a number are opposite operations. If you know one, you can find the other. We can use this aspect of their relationship to solve equations.

Consider the equation below:

$$x^2=36$$

If you stare at it for a little while, you can come up with two solutions: $x=6$ and $x=-6$. Staring doesn't always work, however. What if we wanted to solve it algebraically? As we noted above, the square and square root are opposite operations, so taking the square root on both sides of the equation is the natural thing to try:

$$\begin{aligned} x^2 &= 36 \\ \sqrt{x^2} &= \sqrt{36} \\ x &= 6 \end{aligned}$$

Hmmm, something's wrong... where is the other solution?

Equations with squared terms will often have two solutions; however, the nature of the square root function does not allow it to produce two results. We have to account for that when we use it. We do this by inserting a $\pm$ symbol into the problem and then splitting into two separate equations. Here is how it would work in the problem above

$$\begin{aligned} x^2 &= 36 \\ \sqrt{x^2} &= \sqrt{36} \\ x &= \pm 6 \\ x = 6 \ \text{ or } \ x &= -6 \end{aligned}$$

Before we list the steps for solving, it is important to note that this technique will **not** work for every equation that has a squared term. It will only work for equations that can be written in the form variable$^2 = \#$ or (variable expression)$^2 = \#$. This is easy for equations like $k^2 - 12 = 37$ or $2(n+4)^2 - 5 = 13$, but not so easy to do for an equation like $x^2 = x + 1$.

Steps for Solving

1. Isolate the squared term
2. Take the square root of both sides. Insert a $\pm$ sign.
3. Split into two separate equations and solve each individually.

Examples

Example A

Solve.

$x^2 + 3 = 12$

1. Isolate the squared term.

$$\begin{aligned} x^2 + 3 - 3 &= 12 - 3 \\ x^2 &= 9 \end{aligned}$$

2. Take the square root on both sides.

$$\begin{aligned} \sqrt{x^2} &= \sqrt{9} \\ x &= \pm\sqrt{9} \end{aligned}$$

3. Split into two separate equations and solve each individually.

$$x = +\sqrt{9}$$

$$amp; x = -\sqrt{9}$$
$$x = 3$$

$$amp; x = -3$$

So the solutions are $x = 3$ and $x = -3$.

Example B

Solve for y:

$$(y-4)^2 + 1 = 26$$

1. Isolate the squared term

$$(y-4)^2 + 1 = 26$$
$$(y-4)^2 + 1 - 1 = 26 - 1$$
$$(y-4)^2 = 25$$

2. Take the square root on both sides and insert $\pm$ sign:

$$(y-4)^2 = 25$$
$$\sqrt{(y-4)^2} = \sqrt{25}$$
$$y-4 = \pm\sqrt{25}$$
$$y-4 = \pm 5$$

3. Separate into two equations and solve individually:

$$y-4 = 5$$

$$amp; y-4 = -5$$
$$y = 9$$

$$amp; y = -1$$

So the solutions are $y = 9$ and $y = -1$.

Example C

Solve for x:

$$2(2x-1)^2+6=16$$

1. Isolate the squared term:

$$\begin{aligned} 2(2x-1)^2+6&=16 \\ 2(2x-1)^2+6-6&=16-6 \\ 2(2x-1)^2&=10 \\ \frac{2(2x-1)^2}{2}&=\frac{10}{2} \\ (2x-1)^2&=5 \end{aligned}$$

2. Take the square root of both sides and insert $\pm$ sign:

$$\begin{aligned} (2x-1)^2&=5 \\ \sqrt{(2x-1)^2}&=\sqrt{5} \\ 2x-1&=\pm\sqrt{5} \end{aligned}$$

3. Separate into two equations and solve individually:

$$2x-1=\sqrt{5}$$

$$\begin{aligned} 2x-1&=-\sqrt{5} \\ 2x-1+1&=\sqrt{5}+1 \end{aligned}$$

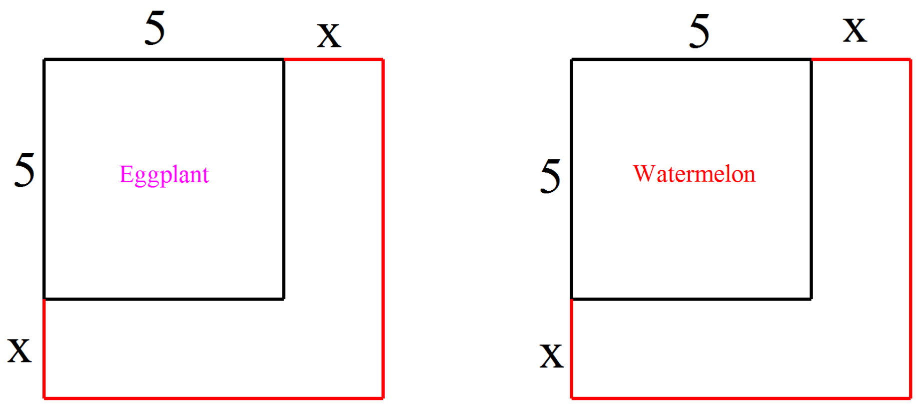
$$\begin{aligned} 2x-1+1&=\sqrt{5}+1 \\ 2x&=1+\sqrt{5} \end{aligned}$$

$$\begin{aligned} 2x&=1-\sqrt{5} \\ x&=\tfrac{1+\sqrt{5}}{2} \end{aligned}$$

$$x=\tfrac{1-\sqrt{5}}{2}$$

Concept Problem Revisited

Let x be the number of feet Joe is going to increase the sides of his garden plots. We can draw a sketch of what the final plots will be:



The side length of each expanded garden plot is $x+5$ so the area of each is $(x+5)^2$. Since Joe has two plots, the total area is $(x+5)^2+(x+5)^2=2(x+5)^2$. Joe wants the new plots to have a total area of 100 sq feet, so we set this expression equal to 100 and solve:

$$2(x+5)^2=100$$

First, isolate the squared term:

$$\frac{2(x+5)^2}{2}=\frac{100}{2}$$
$$(x+5)^2=50$$

Next, take the square root on both sides and insert $\pm$ sign:

$$(x+5)^2=50$$
$$\sqrt{(x+5)^2}=\sqrt{50}$$
$$x+5=\pm\sqrt{50}$$
$$x+5=\pm 5\sqrt{2}$$

Now separate into two equations and solve each one:

$$x+5=+5\sqrt{2}$$

$$amp;x+5=-5\sqrt{2}$$
$$x=-5+5\sqrt{2}$$

$$amp;x=-5-5\sqrt{2}$$

At this point we have two possible answers:

$$x = -5 + 5\sqrt{2} \approx 2.07$$
$$x = -5 - 5\sqrt{2} \approx -12.07$$

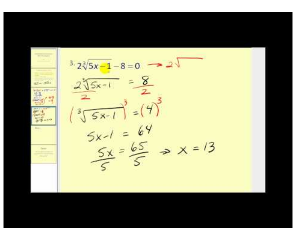
In this situation x represents a length, so a negative value does not make sense. So Joe should increase each side of his garden plots by 2.07 feet in order to double their total area.

Practice Problems

Solve each equation.

1. $x^2 = 9$
2. $x^2 = 49$
3. $x^2 = 100$
4. $x^2 = 64$
5. $x^2 = 225$
6. $x^2 = 256$
7. $x^2 + 3 = 12$
8. $x^2 - 5 = 20$
9. $x^2 + 3 = 39$
10. $x^2 - 4 = 60$
11. $x^2 + 11 = 92$
12. $\sqrt{x+1} = 10$
13. $x^2 + 5 = 41$
14. $x^3 = 8$
15. $x^3 + 4 = 31$

Resources



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10.2 More Solving Equations Using Square Roots

Learning Objectives

Here you will learn how to solve a quadratic equation by using the properties of the square root. This concept introduces solving quadratic equations using the square root method

Concept Problem

Consider the equation $(x-3)^2 = 25$. We can solve it using factoring:

$$
\begin{aligned}
(x-3)^2 &= 25 \\
x^2-6x+9 &= 25 && \text{(FOIL)} \\
x^2-6x+9-25 &= 25-25 \\
x^2-6x-16 &= 0 && \text{(Get 0 on one side)} \\
(x-8)(x+2) &= 0 && \text{(Factor)} \\
x-8 = 0 & \qquad x+2=0 \\
x = 8,\ & x = -2
\end{aligned}
$$

This seems like a lot of work for a problem that started off so nice and compact. Is there an easier way that we could have done that?

Guidance

The last section introduced the technique of solving certain quadratic equation using square roots. In this section we do several more examples using that technique.

Steps for Solving with Square Roots

Recall that you can use square roots to solve equations of the form $\text{variable}^2 = \#$ or $(\text{variable expression})^2 = \#$. Here are the steps:

1. Isolate the squared term

2. Take the square root on both sides and insert $\pm$ symbol.

3. Separate into two equations and solve each one.

Example A

Solve for $x:\ x^2-25=11$

Solution:

1. Isolate the x^2 term:

$$\begin{aligned} x^2 - 25 &= 11 \\ x^2 - 25 + 25 &= 11 + 25 \\ x^2 &= 36 \end{aligned}$$

2. Take the square root of both sides and insert $\pm$ symbol:

$$\begin{aligned} x^2 &= 36 \\ \sqrt{x^2} &= \sqrt{36} \\ x &= \pm 6 \end{aligned}$$

3. Separate into two equations and solve each one:

$$x = \pm 6$$

$$x = 6 \qquad x = -6$$

The solutions to the equation are $\boxed{x = 6 \text{ and } x = -6}$

Example B

Solve the following quadratic equation using a square root:$(x+1)^2 = 36$

Solution:

1. The squared term is already isolated, so we can go to the second step

2. Take the square root on both sides and insert $\pm$ symbol:

$$\begin{aligned} (x+1)^2 &= 36 \\ \sqrt{(x+1)^2} &= \sqrt{36} \\ x+1 &= \pm 6 \end{aligned}$$

3. Separate into two equations and solve each one:

$$\begin{array}{l|r} x+1 = 6 & x+1 = -6 \\ x+1-1 = 6-1 & x+1-1 = -6-1 \\ x = 5 & x = -7 \end{array}$$

The solutions to the equation are $\boxed{x = 5 \text{ and } x = -7}$

Example C

Solve for x: $3(2x-5)^2+6=15$

Solution:

1. Isolate the squared term:

$$\begin{aligned} 3(2x-5)^2+6&=15 \\ 3(2x-5)^2+6-6&=15-6 \\ 3(2x-5)^2&=9 \\ \frac{3(2x-5)^2}{3}&=\frac{9}{3} \\ (2x-5)^2&=3 \end{aligned}$$

2. Take the square root of both sides and insert $\pm$ symbol:

$$\begin{aligned} (2x-5)^2&=3 \\ \sqrt{(2x-5)^2}&=\sqrt{3} \\ 2x-5&=\pm\sqrt{3} \end{aligned}$$

3. Separate into two equations and solve each one:

$$\begin{array}{r|r} 2x-5=\sqrt{3} & 2x-5=-\sqrt{3} \\ 2x-5+5=\sqrt{3}+5 & 2x-5+5=-\sqrt{3}+5 \\ 2x=5+\sqrt{3} & 2x=5-\sqrt{3} \\ \frac{2x}{2}=\frac{5+\sqrt{3}}{2} & \frac{2x}{2}=\frac{5-\sqrt{3}}{2} \\ x=\frac{5+\sqrt{3}}{2} & x=\frac{5-\sqrt{3}}{2} \end{array}$$

So the solutions are $\boxed{x=\frac{5+\sqrt{3}}{2} \quad \text{and} \quad x=\frac{5-\sqrt{3}}{2}}$

Concept Problem Revisited

To solve $(x-3)^2=25$, take the square root of both sides. We get that $x-3=5$ or $x-3=-5$. The two solutions are therefore $x=8,-2$. These are the same solutions that we obtained from factoring, which is reassuring.

10.3 Introduction to Quadratic Functions

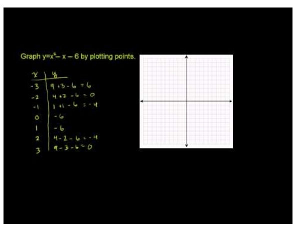
Concept Question

If you toss a ball into the air, it goes up for a little while, and then turns around and comes down. What kind of a graph would demonstrate this behavior?

What kind of a function can make such a graph?

Watch This

Khan Academy Quadratic Functions1



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Guidance

Until now we have been dealing primarily with linear functions. Linear functions are very useful, but they can't do everything.

The defining characteristic of linear functions is their constant rate of change; they always go in the same direction at the same rate. But in the real world things change directions: a ball will go up and then come down; the population of a city may decrease and then increase. A linear function will not work to describe things that change direction. We need something else.

In this chapter we will be learning about quadratic functions. A **quadratic function** looks like this:

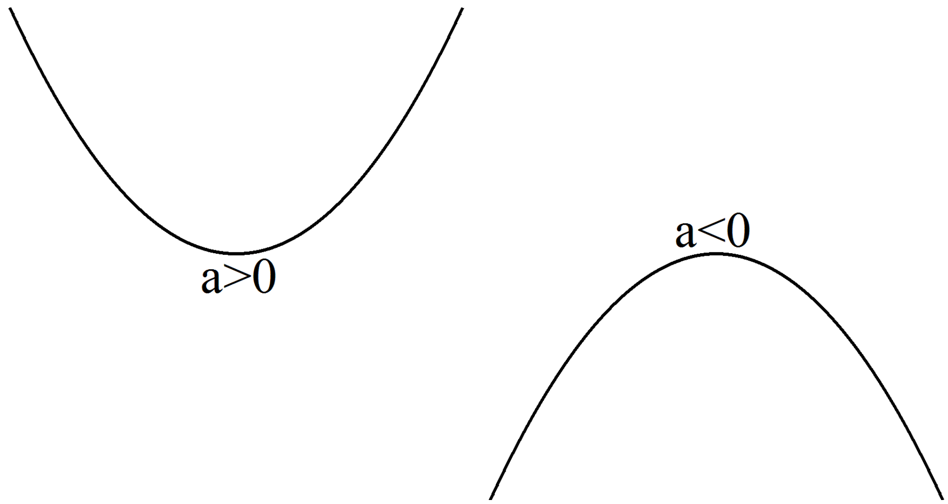
$$f(x) = ax^2 + bx + c$$

Here, x is the variable, while a, b and c are real numbers with $a \neq 0$. Notice in the first term that the variable has an exponent of 2 on it. This is what makes the difference between a quadratic function and a linear function. The functions listed below are quadratic functions:

$$f(x) = x^2$$
$$g(x) = 2x^2 - 9$$
$$h(x) = -3x^2 + \frac{3}{2}x + 16$$

Graphs of Quadratic Functions

The graph of a quadratic function is called a **parabola**. There are two basic shapes that a parabola can have:



The basic shape of the parabola is determined by the value of a. The exact shape of a parabola will depend on the values of $a, b,$ and c.

The simplest quadratic function is $f(x) = x^2$. Let's take a closer look at it.

Example A - Values

To get an idea of the values the function $f(x) = x^2$ takes, let's make a table with values from -3 to 3.

Solution:

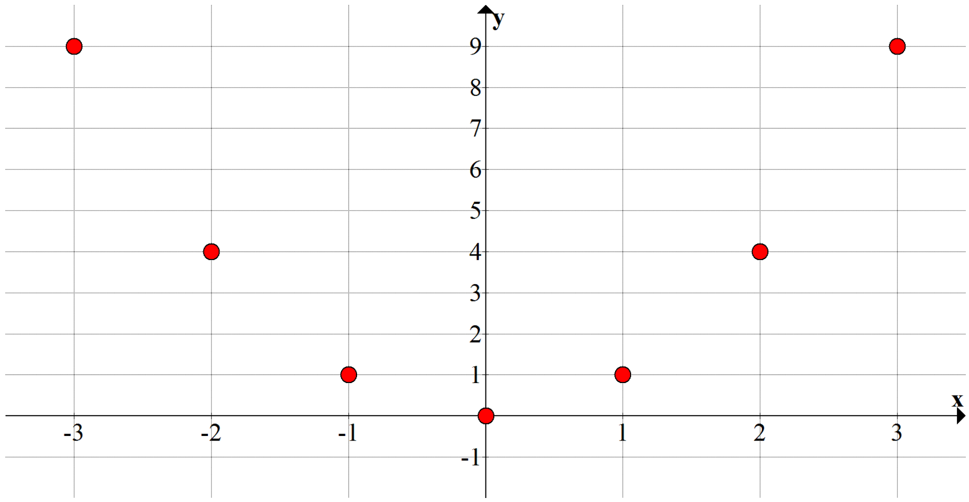
To complete the table of values, substitute the given x-values into the function $f(x) = x^2$.

Example B - The Graph

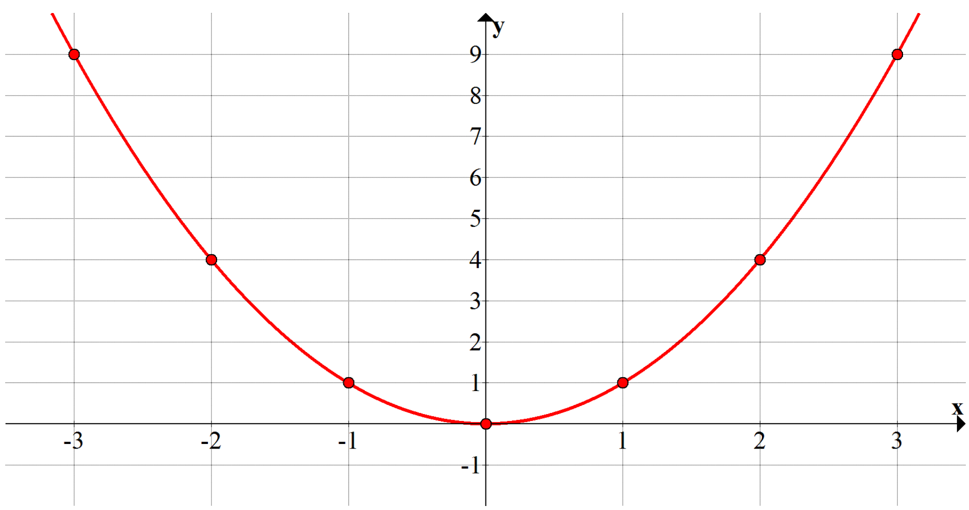
On a Cartesian plane, graph the function $f(x) = x^2$.

Solution:

We can start with the points from the table above. Plotting them creates this graph:



We can already see the basic shape. We can connect the dots with a smooth curve to produce the rest of the graph.

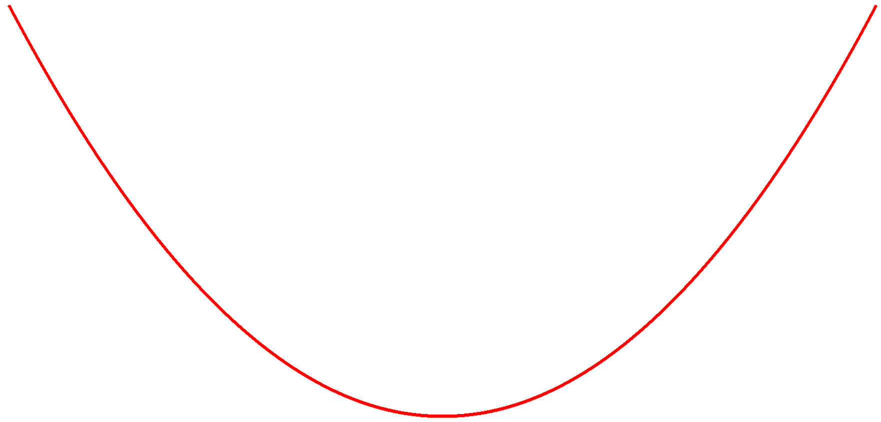


Characteristics of Parabolas

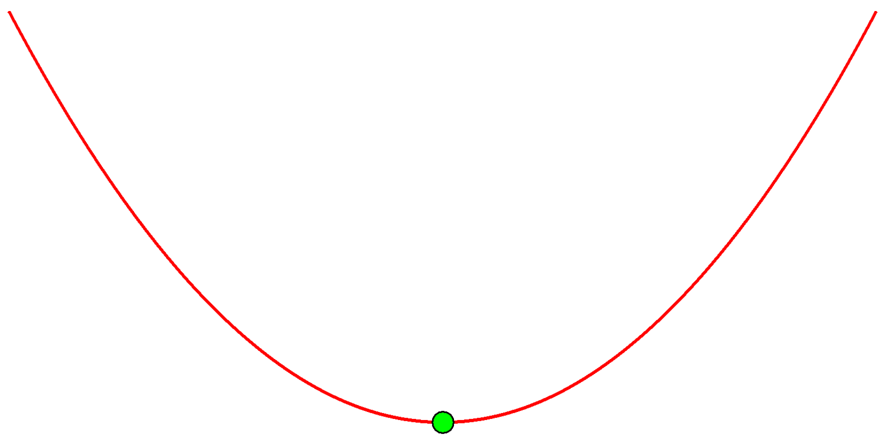
There are several important features that all parabolas have. We will illustrate using upward-facing parabolas, but all of these apply to downward-facing parabolas as well.

Vertex

Quadratic functions are different from linear functions, because they change direction. Look at the graph below and see if you can identify the place where the graph changes direction:



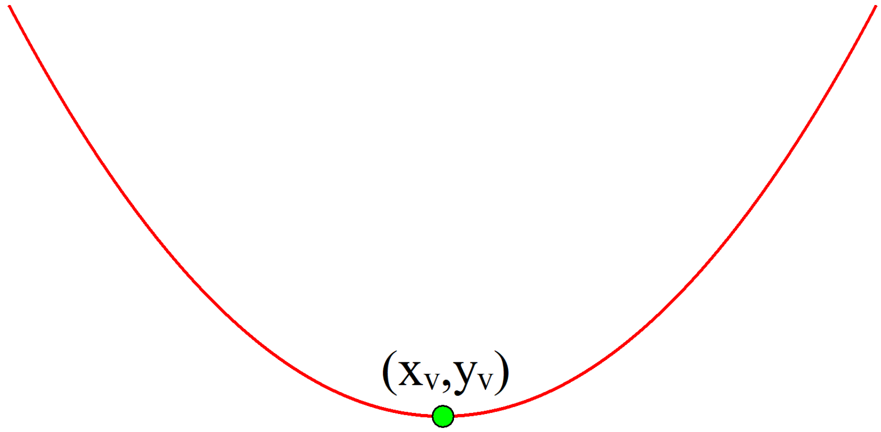
It's at the bottom of the graph! On the left side, the graph is going down, while on the right side the graph is going up. The change in direction happens at the bottom. As you might expect, the point where this change happens is important. In many ways, it is the central point of a parabola.



Every parabola has a turning point known as the **vertex**. It is either the highest or lowest point on the graph. The vertex is the lowest point if the parabola opens upward and the highest point if the parabola opens downward. This makes the vertex very important in applications. For instance, it may represent the maximum height of a ball or the minimum possible cost for a business.

Finding the Vertex

If the vertex is so important, how do we figure out where it is? Recall that the general quadratic function has the form $f(x) = ax^2 + bx + c$.



The vertex is a point (x_v, y_v) .There is a simple formula that will tell you the x-coordinate:

$$x_v = \frac{-b}{2a}$$

Once you determine the x-coordinate, you can treat it like any other point and plug the value into $f(x)$ to find the y-coordinate.

$$y_v = f\left(\frac{-b}{2a}\right)$$

Example C

Find the coordinates of the vertex of the function $h(x) = -x^2 + 6x - 2$. Is it the highest or lowest point on the graph?

Solution: It is a good idea to write down the values of $a, b,$ and c since we will be using them:

$$a = -1 \qquad b = 6 \qquad c = -2$$

Now, use the formula:

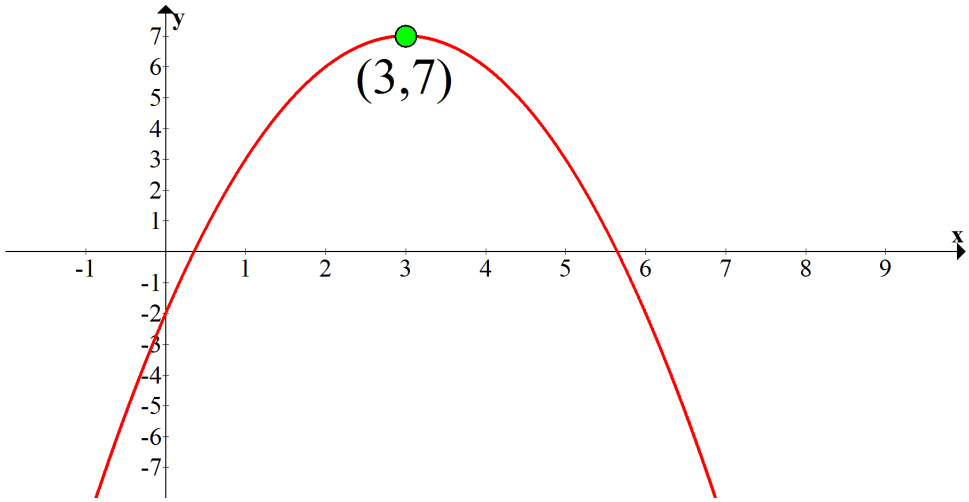
$$x_v = \frac{-b}{2a} = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3$$

Next, plug the value you just found into $h(x)$:

$$h(3) = -3^2 + 6 \cdot (3) - 2 = 7$$

So the coordinates of the vertex are $(3, 7)$.

To determine if it is the highest or lowest point, we need to know the basic shape of the parabola. In this case we have $a = -1$ so the parabola will be downward facing with the vertex at the top. It is the highest point on the graph.



Example D

Find the vertex of the parabola $h(x) = 3x^2 - 18x + 31$ and state the domain and range of $h(x)$.

Solution: First, write down the values of $a, b,$ and c. In this case we have

$$a = 3 \qquad b = -18 \qquad c = 31$$

Now use the formula to find the x-coordinate of the vertex:

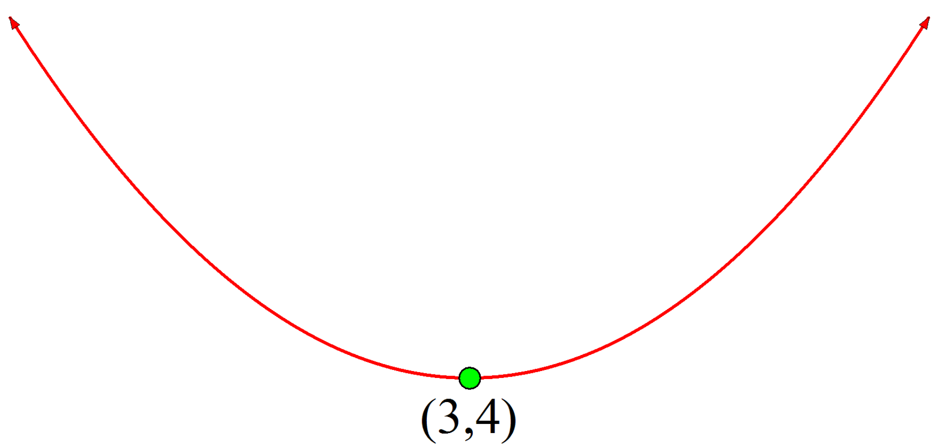
$$x_v = \frac{-b}{2a} = \frac{-(-18)}{2(3)} = \frac{18}{6} = 3$$

Next, we find y_v by plugging the value we just found into $h(x)$:

$$y_v = h(3) = 3(3)^2 - 18(3) + 31 = 4$$

So this parabola's vertex is at the point $(3,4)$.

What about the domain and range of $h(x)$? We can use what we just learned to draw a sketch of its graph:

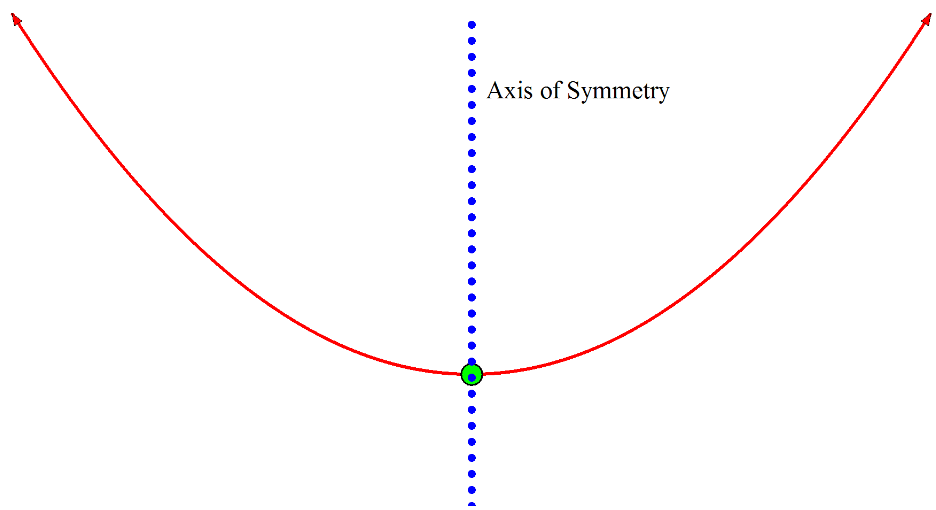


Horizontally, the graph extends indefinitely to the left and to the right, so its domain is all real numbers.

Vertically, the graph extends upwards indefinitely; however, it never goes lower than the vertex, which has y-coordinate 4. So the range is $y \geq 4$.

Axis of Symmetry

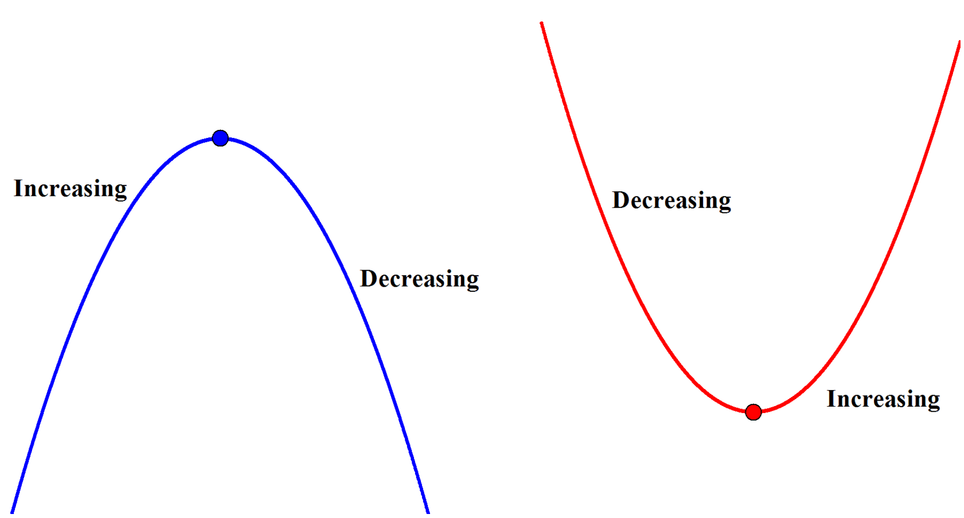
The **axis of symmetry** is the vertical line that passes through the vertex of the parabola. It cuts the parabola into two mirror-image pieces.



The equation for the axis of symmetry is always $x = a$, where a is the the $x-$coordinate of the vertex.

Increasing/Decreasing

A parabola changes direction at its vertex, which means that one side of its graph is *decreasing* and the other is *increasing*. When evaluating increasing vs decreasing on a graph, we always look from left to right. A graph is increasing when its goes up from left to right. A graph is decreasing when it goes down from left to right. The picture below illustrates the portions of each of the basic parabola shapes that are increasing and decreasing.



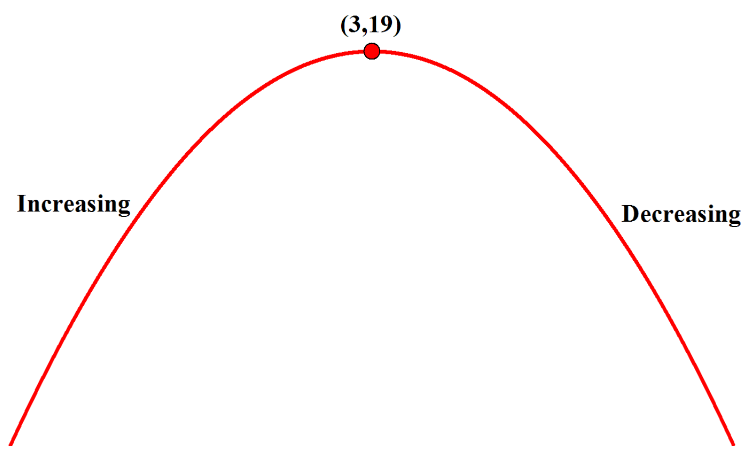
Example E

Determine the values of x for where the function $g(x) = -x^2 + 6x + 10$ is increasing and the values of x for where it is decreasing.

Solution: We know that the graph changes direction at its vertex, so we start by identifying that point. In this case we have $a = -1, b = 6,$ and, $c = 10$.

$$x_v = \frac{-b}{2a} = \frac{-6}{2(-1)} = 3$$
$$y_v = g(x_v) = g(3) = -(3)^2 + 6(3) + 10 = 19$$

So the vertex is at the point $(3, 19)$. We also know the graph is downward facing since a is negative. We can draw a rough sketch of our parabola and label the increasing and decreasing portions:

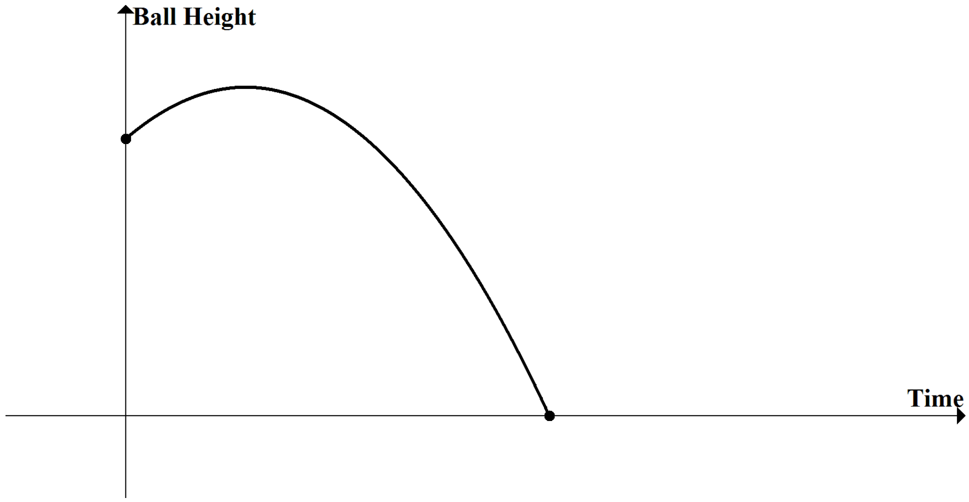


The graph is increasing on its left side, which corresponds to values of x less than 3. We would say that $g(x)$ is increasing when $x < 3$.

The graph is decreasing on its right side, which corresponds to values of x greater than 3. We would say that $g(x)$ is decreasing when $x > 3$.

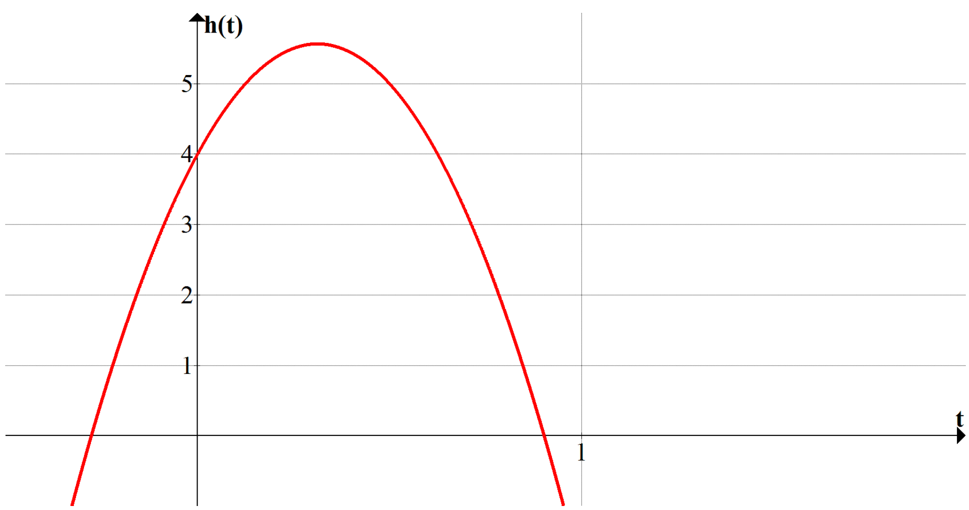
Concept Problem Revisited

A thrown ball should go up for a little and then come down. A sketch of the graph would look like this:



We can see that the graph looks like a parabola. In fact, we can use quadratic functions to describe the height of an object that is subject to gravity (if we ignore wind resistance and other small factors). In the case of a ball, it might be thrown at a speed of 10 feet per second from a height of 4 feet. In this case, it's height t seconds after it is thrown is given by the function

$$h(t) = -16t^2 + 10t + 4$$



Now that we have a function to use, we can answer questions like 'How high does the ball go?' or 'How long will it take to hit the ground?'

Guided Practice

1. If the graph of $y = x^2$ opened downward, what changes would exist in the base table of values?

2. How can we change the formula $y = x^2$ so that its graph opens downward?

3. Draw the image of the basic quadratic function that opens downward. State the domain and range for this parabola.

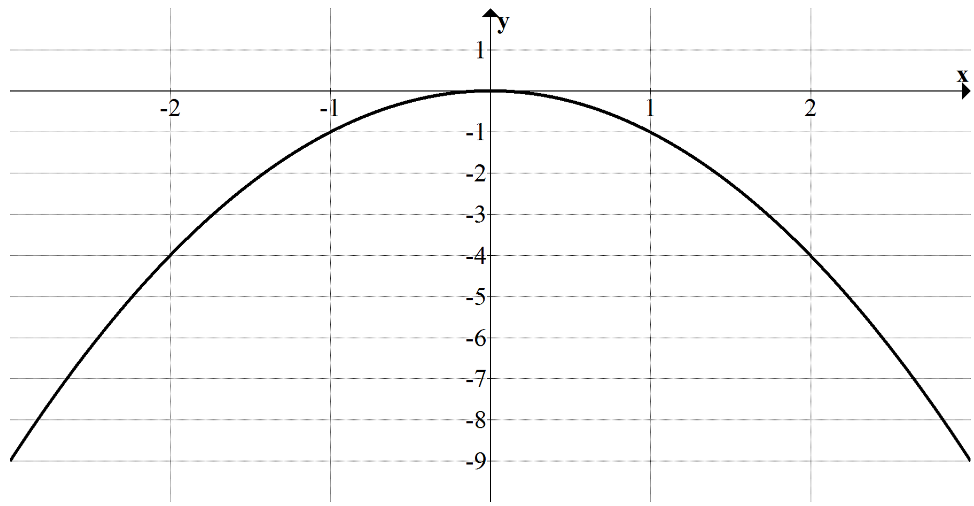
Answers:

1. If the parabola were to open downward, the x-values would not change. The y-values would become negative values. The points would be plotted from the vertex as: right and left one and down one; right and left two and down four; right and left three and down nine. The table of values would be

x	-3	-2	-1	0	1	2	3
y	-9	-4	-1	0	-1	-4	-9

2. To match the table of values, the basic quadratic function would have to be written as $y = -x^2$.

3.



The domain is all real numbers. The range for this parabola is $y \leq 0$.

Practice Problems

Complete the following statements in the space provided.

1. The name given to the graph of $y = x^2$ is _____________________.
2. The domain of the graph of $y = x^2$ is _____________________.
3. If the vertex of a parabola was (-3, 5), the equation of the axis of symmetry would be _____________________.
4. A parabola has a maximum value when it opens _____________________.
5. The point (-2, 4) on the graph of $y = x^2$ has a corresponding point on the right side of the vertex at _________-_____________.
6. The range of the graph of $y = -x^2$ is _____________________.
7. If the table of values for the basic quadratic function included 4 and -4 as x-values, the y-value(s) would be _____________________.
8. The vertical line that passes through the vertex of a parabola is called _____________________.
9. A minimum value exists when a parabola opens _____________________.
10. The turning point of the graph of $y = x^2$ is called the _____________________.

11. Make a sketch of the function $y = x^2 - 2x + 1$ by first making a table and then plotting the points. What is the vertex of this parabola?
12. Make a sketch of the function $y = x^2 + 2x - 3$ by first making a table and then plotting the points. What is the vertex of this parabola?

13. Make a sketch of the function $y = x^2 - 4x + 5$ by first making a table and then plotting the points. What is the vertex of this parabola?
14. Make a sketch of the function $y = -x^2 + 2x + 1$ by first making a table and then plotting the points. What is the vertex of this parabola?
15. Make a sketch of the function $y = -x^2 - 4x$ by first making a table and then plotting the points. What is the vertex of this parabola?

10.4 Graphs to Solve Quadratic Equations

Learning Objectives

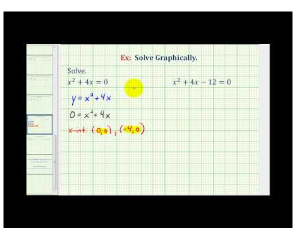
Here you will learn how to solve a quadratic equation by graphing.

Concept Problem

One way to solve the equation $x^2-2x-3=0$ is to use factoring and the zero product property. Is it possible to use a graph to solve $x^2-2x-3=0$?

Watch This

James Sousa: Solve a Quadratic Equation Graphically on the Calculator



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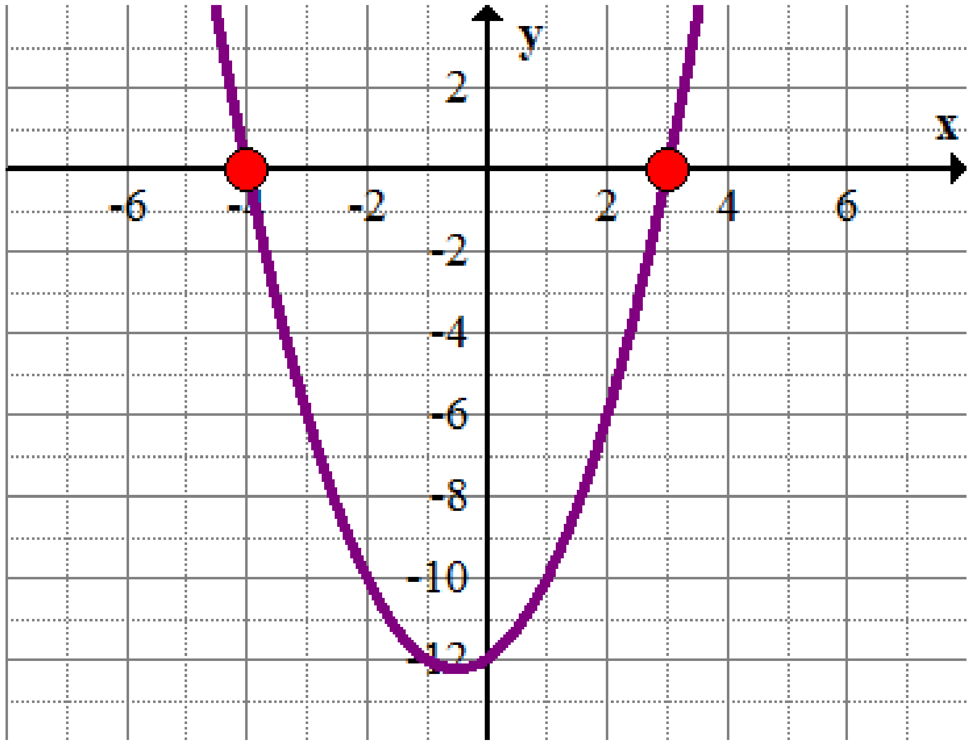
Guidance

We have discussed both quadratic functions and quadratic equations. Are they the same thing? (Hint: No they aren't)

If they are not the same, then how are they related to each other?

A quadratic *function* has the form $y=ax^2+bx+c$, while a quadratic *equation* can be written in the form $ax^2+bx+c=0$. Every quadratic function has a corresponding quadratic equation that you get by changing the y to a 0. In other words, so solutions to the equations will tell you the values of x that will produce a y-value of 0. This tells us something about the graph of the function.

How do the solutions to the equation $x^2+x-12=0$ show up on the graph of $y=x^2+x-12$? On the graph you are looking for the points that have a y-coordinate that is equal to 0. Therefore, the solutions to the equation will show up as the x-intercepts on the graph of the function. These are also known as the ***roots*** or ***zeros*** of the function. Here is the graph of $y=x^2+x-12$:



You can see the x-intercepts are at $(-4,0)$ and $(3,0)$. This means that the solutions to the equation $x^2+x-12=0$ are $x=-4$ and $x=3$. You can verify these solutions by substituting them back into the equation:

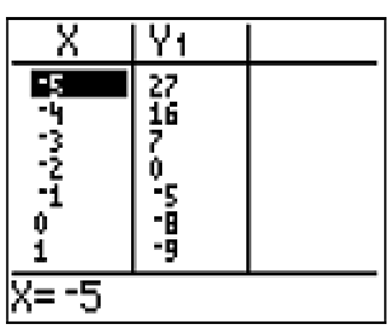
- $(-4)^2+(-4)-12=16-4-12=0$
- $(3)^2+(3)-12=9+3-12=0$

Graphing is a great way to solve some quadratic equations. Keep in mind that you can also solve many quadratic equations by factoring or using other algebraic methods such as the quadratic formula. To get exact solutions you often must use algebraic methods. Sometimes graphing will produce exact solutions, and other times it will produce decimal approximations of solutions.

Example A

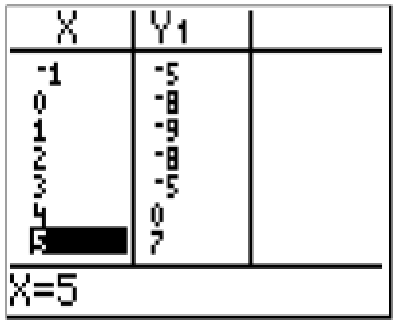
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2-2x-8=0$

Solution: The corresponding function is $y=x^2-2x-8$. Use your graphing calculator to make a table and a graph for this function.



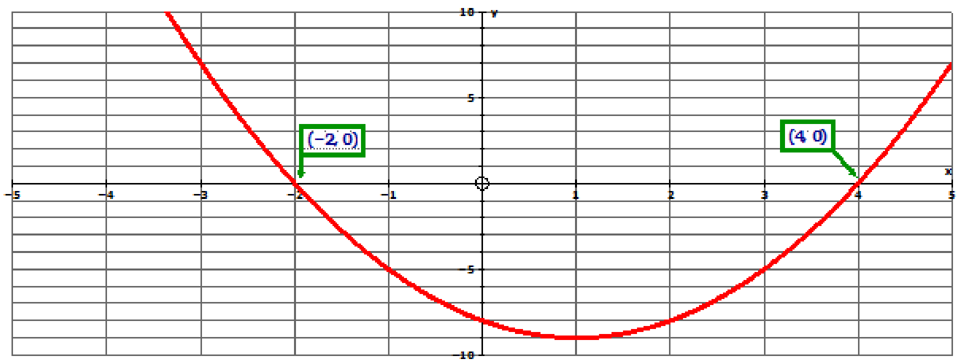
X	Y1	
-5	27	
-4	16	
-3	7	
-2	0	
-1	-5	
0	-8	
1	-9	

X= -5



X	Y1
-1	-5
0	-8
1	-9
2	-8
3	-5
4	0
5	7

X=5

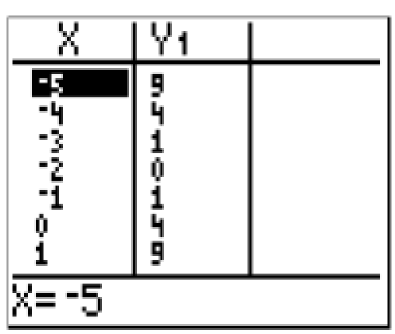


The x-intercepts are (-2, 0) and (4, 0). The x-intercepts are the values for 'x' that result in $y = 0$ and are therefore the solutions to your equation. The solutions for the quadratic are $x = -2$ and $x = 4$.

Example B

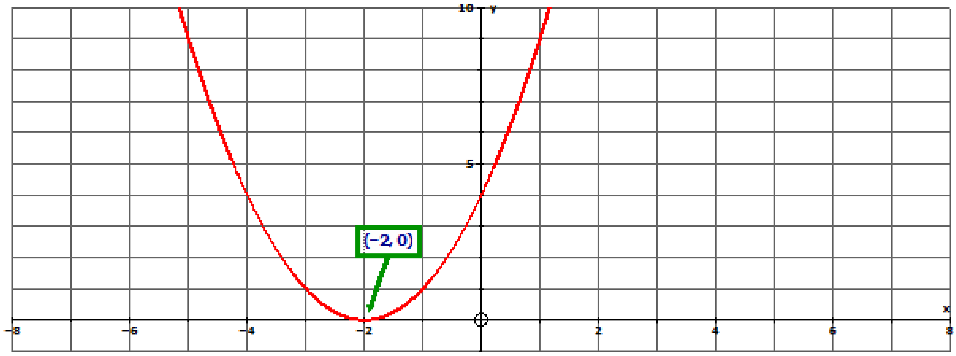
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2 + 4x + 4 = 0$

Solution: The corresponding function is $y = x^2 + 4x + 4$. Use your graphing calculator to make a table and a graph for this function.



X	Y1
-5	9
-4	4
-3	1
-2	0
-1	1
0	4
1	9

X=-5

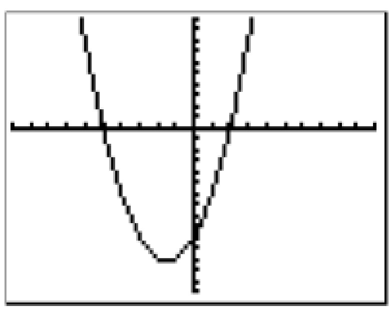


The only x-intercept is (-2, 0). There is only one solution to the equation: $x = -2$. Keep in mind that quadratic equations can have 0, 1, or 2 real solutions. If you were to factor the quadratic x^2+4x+4, you would get $(x+2)(x+2)$–two of the same factors. The root of -2 for this function is said to have a ***multiplicity*** of 2, because 2 factors produce the same solution. You will learn more about ***multiplicity*** when you study polynomials in future courses.

Example C

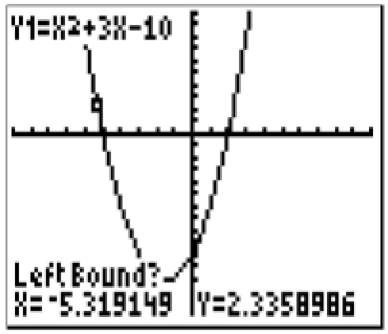
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2+3x=10$

Solution: First rewrite the equation so it is set equal to zero to get $x^2+3x-10=0$. Now, the corresponding function is $y=x^2+3x-10$. Use your graphing calculator to make a graph for this function. You will see that there are two x-intercepts.



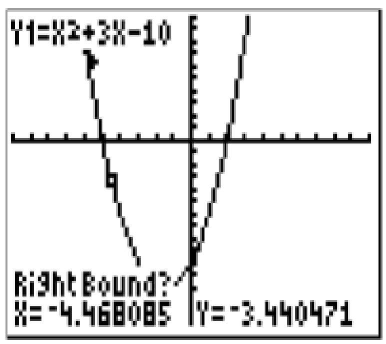
For this example you will see how the calculator can calculate the zeros of a function on a graph. *This technique is particularly useful when the intercepts are not at whole numbers.* Have the calculator find the x-intercept on the left first. Press

The calculator will display "Left Bound?" Use the arrow to position the cursor so that it is to the left and above the x-axis.



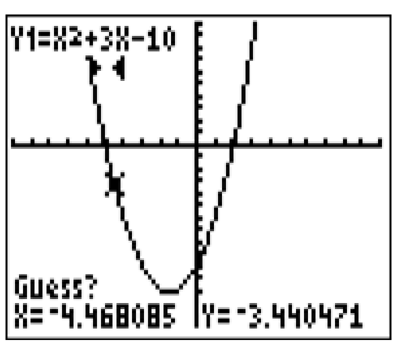
When the cursor has been positioned, press

The calculator will now display "Right Bound?" Use the arrows to position the cursor so that it is to the right and below the x-axis.

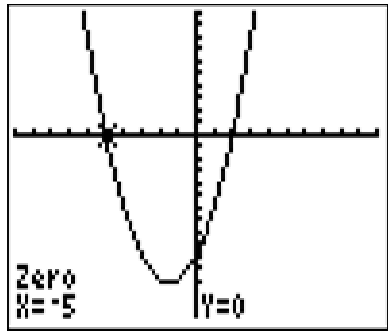


When the cursor has been positioned, press

The calculator will now display "Guess?"

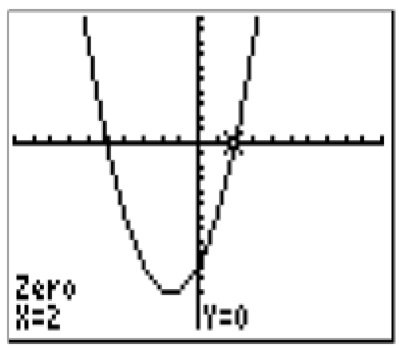


Position the cursor close to the point you are trying to find and press



At the bottom of the screen you can see it says "Zero" and the x and y coordinates. You are interested in the x-coordinate because that is one of the solutions to the original equation. The x-intercept is (-5, 0) which means that one of the solutions is $x = -5$.

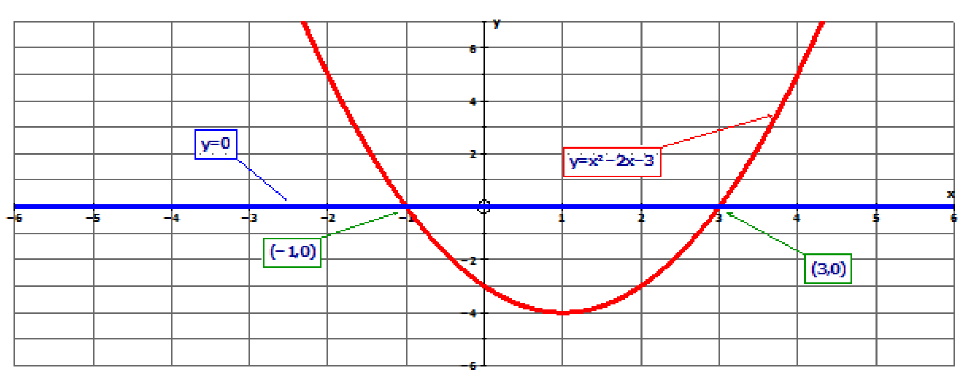
Repeat the same process to determine the value of the x-intercept on the right, this time choosing points to the bottom left and top right as your bounds.



The x-intercept is (2, 0) which means that the second solution is $x = 2$.

Concept Problem Revisited

To solve the equation $x^2 - 2x - 3 = 0$ using a graph, use a calculator to graph the corresponding function $y = x^2 - 2x - 3$. Then, look for the values on the graph where $y = 0$, which will be the x-intercepts.



The x-intercepts are $(-1,0)$ and $(3,0)$. The solutions to the original equation are $x=-1$ and $x=3$.

Vocabulary

Zeros of a Quadratic Function

The ***zeros of a quadratic function*** are the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Roots of a Quadratic Function

The ***roots of a quadratic function*** are also the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Guided Practice

Solve each quadratic equation using a graph.

1. $x^2-3x-10=0$

2. $2x^2-5x+2=0$

3. $2x^2-5x=3$

Answers:

1. To begin, create a table of values for the corresponding function $y=x^2-3x-10$ by using your graphing calculator:

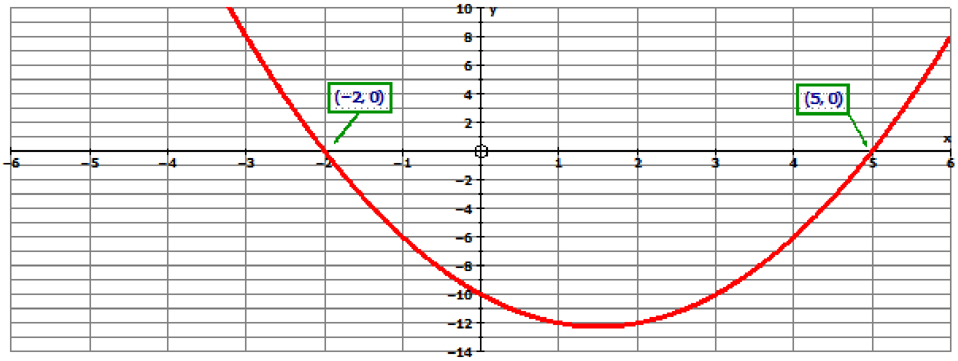
X	Y1	
-5	30	
-4	18	
-3	8	
-2	0	
-1	-6	
0	-10	
1	-12	

X= -5

X	Y1	
2	-12	
3	-10	
4	-6	
5	0	
6	8	
7	18	
8	30	

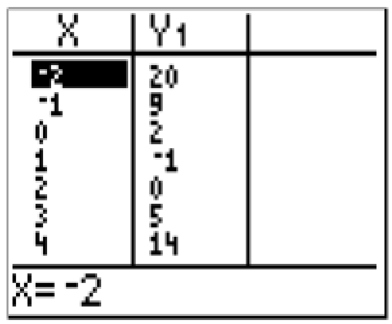
X=8

From the table, the x-intercepts are (-2, 0) and (5, 0). The x-intercepts are the values for 'x' that result in $y=0$ and are therefore the solutions to the equation.



The solutions to the equation are are $x = -2$ and $x = 5$.

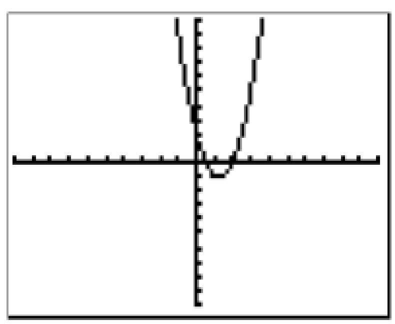
2. To begin, create a table of values for the corresponding function $y = 2x^2 - 5x + 2$ by using your graphing calculator:



X	Y1	
-2	20	
-1	9	
0	2	
1	-1	
2	0	
3	5	
4	14	

X= -2

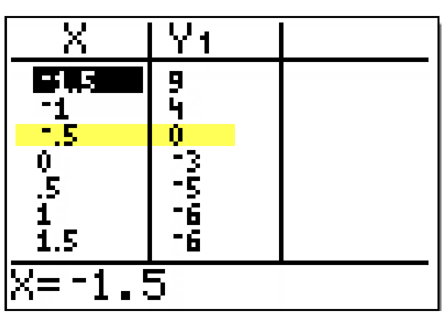
Press



Now you can use the steps laid out above to find the x-intercepts.

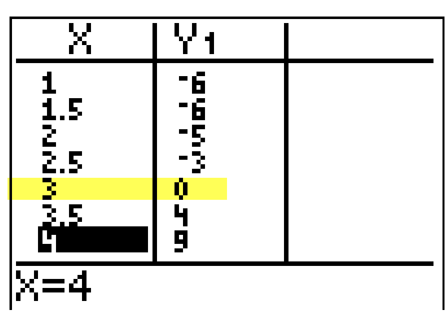
The x-intercepts of the function are (0.5, 0) and (2, 0). The solutions to the equation are, therefore, $x = 0.5$ and $x = 2$.

3. First rewrite the equation so it is set equal to zero: $2x^2 - 5x - 3 = 0$. Next, create a table of values for the corresponding function $y = 2x^2 - 5x - 3$ by using your graphing calculator:



X	Y1	
-1.5	9	
-1	4	
-.5	0	
0	-3	
.5	-5	
1	-6	
1.5	-6	

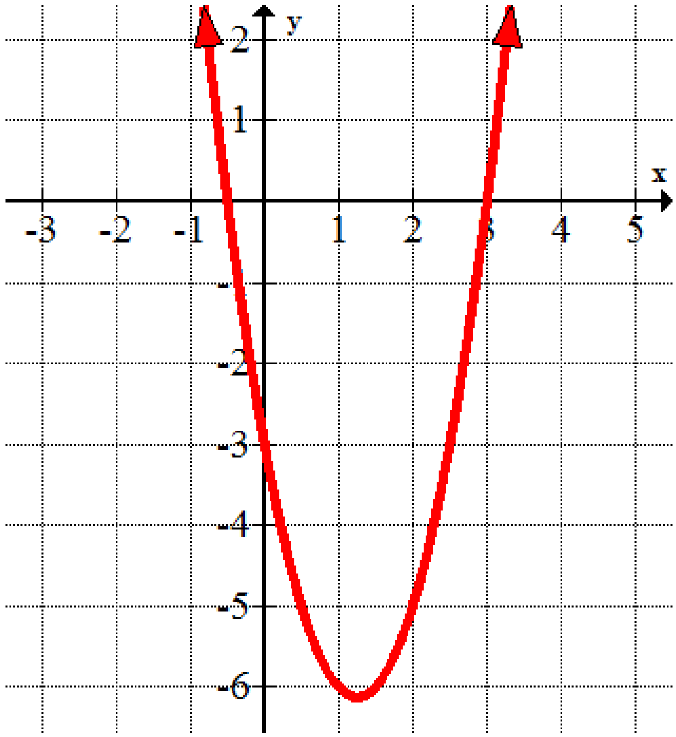
X= -1.5



X	Y1	
1	-6	
1.5	-6	
2	-5	
2.5	-3	
3	0	
3.5	4	
4	9	

X=4

Now sketch the graph of the function.



The zeros of the function are (-0.5, 0) and (3, 0). Therefore, the solutions to the equation are $x = -0.5$ and $x = 3$.

Practice Problems

Use your graphing calculator to solve each of the following quadratic equations by graphing:

1. $2x^2 + 9x - 18 = 0$
2. $3x^2 + 8x - 3 = 0$
3. $-5x^2 + 13x + 6 = 0$
4. $2x^2 - 11x + 5 = 0$
5. $3x^2 + 8x - 3 = 0$
6. $x^2 - x - 20 = 0$
7. $2x^2 - 7x + 5 = 0$
8. $3x^2 + 7x = -2$
9. $2x^2 - 15 = -x$
10. $3x^2 - 10x = 8$
11. How could you use the graphs of a system of equations to solve $3x^2 - 10x = 8$?
12. What's the difference between a quadratic equation and a quadratic function?
13. Will a quadratic equation always have 2 solutions? Explain.
14. The quadratic equation $x^2 + 4 = 0$ has no real solutions. How does the graph of $y = x^2 + 4$ verify this fact?
15. When does it make sense to use the graphing method for solving a quadratic equation?

10.5 The Quadratic Formula

Concept Problem

Solve the following quadratic equation:

$$x^2 = x + 1$$

To solve this equation by factoring, we would first need to get 0 on one side of the equation. We can do this by moving all the terms to the left-hand side:

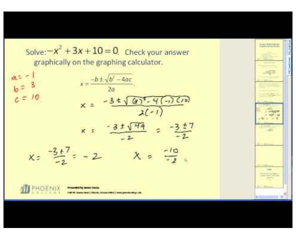
$$\begin{aligned} x^2 &= x + 1 \\ x^2 - x - 1 &= x + 1 - x - 1 \\ x^2 - x - 1 &= 0 \end{aligned}$$

The next step would be to factor. In this case we have $a = 1, b = -1, c = -1$ so to factor we would need to find two numbers that multiply to -1 (c) and add to -1 (b). You can think about it all day, but you won't be able to come up with integers that meet those conditions.

Does this mean that the equation has no solution? No. It means that we need a new tool in order to solve it.

Watch This

James Sousa: Using the Quadratic Formula



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The Quadratic Formula

A quadratic equation can be written in the form

$$ax^2 + bx + c = 0$$

In fact, there is a formula that gives the solution to this equation in terms of $a, b,$ and c. Here is the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is known as the **quadratic formula**. You can use the quadratic formula to solve ANY quadratic equation as long as it has 0 on one side. All you need to know are the values of a, b, and c. Keep in mind that while the factoring method for solving a quadratic equation will only sometimes work, the quadratic formula will ALWAYS work. You should memorize the quadratic formula because you will use it in algebra and future math courses.

There's a lot going on in that formula! It's not completely unfamiliar, however. You should be able to see the $\frac{-b}{2a}$ of the vertex formula embedded inside it. This comes from the fact that the x-intercepts of a parabola are always equally spaced on either side of its vertex. We wil look more closely at the $\sqrt{b^2-4ac}$ part of the formula later.

Example A

Find the exact solutions of the following quadratic equation:

$$5x^2+2x-2=0$$

Solution: For this quadratic equation,

$$a=5, b=2, c=-2$$

Factoring won't work here, so we use the formula. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
x &= \frac{-(2) \pm \sqrt{(2)^2-4(5)(-2)}}{2(5)} \\
x &= \frac{-2 \pm \sqrt{4+40}}{10} \\
x &= \frac{-2 \pm \sqrt{44}}{10} \\
x &= \frac{-2 \pm 2\sqrt{11}}{10} \\
x &= \frac{-1 \pm \sqrt{11}}{5}
\end{aligned}$$

The exact solutions to the quadratic equation are:

$$\frac{-1+\sqrt{11}}{5} \; or \; \frac{-1-\sqrt{11}}{5}$$

.

Example B

Use the quadratic formula to determine the approximate solutions of the equation:

$2x^2-3x=3$

Solution: Start by rewriting the equation in standard form so that it is set equal to zero. $2x^2 - 3x = 3$ becomes $2x^2 - 3x - 3 = 0$. For this quadratic equation, $a = 2, b = -3, c = -3$. Substitute these values into the quadratic formula and simplify.

$$
\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
m &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)} \\
x &= \frac{3 \pm \sqrt{9 + 24}}{4} \\
x &= \frac{3 \pm \sqrt{33}}{4} \qquad \sqrt{33} = 5.74 \\
x &= \frac{3 \pm 5.74}{4} \\
x &= \frac{3 + 5.74}{4} \text{ or } x = \frac{3 - 5.74}{4} \\
x &= \frac{8.74}{4} \text{ or } x = \frac{-2.74}{4} \\
x &= 2.2 \text{ or } x = -0.7
\end{aligned}
$$

The approximate solutions to the quadratic equation to the nearest tenth are $x = 2.2$ or $x = -0.7$.

Example C

Solve the following equation using the quadratic formula:

$$\boxed{\frac{2}{y} - \frac{3}{y+1} = 1}$$

Solution: While this does not look like a quadratic equation (it is actually a rational equation because it contains rational expressions), you can rewrite it as a quadratic equation by multiplying by $(y)(y+1)$ to get rid of the fractions. *Note that y and $y+1$ are the denominators you want to eliminate. This is why you want to multiply by* $(y)(y+1)$. After multiplying, simplify and put the equation in standard quadratic form set equal to 0.

$$
\begin{aligned}
&\frac{2}{y} - \frac{3}{y+1} = 1 \\
&\frac{2}{y}(y)(y+1) - \frac{3}{y+1}(y)(y+1) = 1(y)(y+1) \\
&\frac{2}{\cancel{y}}(\cancel{y})(y+1) - \frac{3}{\cancel{y+1}}(y)(\cancel{y+1}) = 1(y)(y+1) \\
&2(y+1) - 3(y) = 1(y^2 + y) \\
&2y + 2 - 3y = y^2 + y \\
&2 - y = y^2 + y \\
&y^2 + 2y - 2 = 0
\end{aligned}
$$

For this quadratic equation, $a = 1, b = 2, c = -2$. Substitute these values into the quadratic formula and simplify.

$$
\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
y &= \frac{-(2) \pm \sqrt{(2)^2-4(1)(-2)}}{2(1)} \\
y &= \frac{-2 \pm \sqrt{4+8}}{2} \\
y &= \frac{-2 \pm \sqrt{12}}{2} \\
y &= \frac{-2 \pm 2\sqrt{3}}{2} \\
y &= -1 \pm \sqrt{3}
\end{aligned}
$$

The exact solutions to the equation are $-1+\sqrt{3}$ or $-1-\sqrt{3}$. Note that neither of these solutions will cause the original equation to have a zero in the denominator, so they both work.

Concept Problem Revisited

To solve the equation $x^2-x-1=0$ algebraically, you can use the quadratic formula. For this quadratic equation, $a=1, b=-1, c=-1$. Substitute these values into the quadratic formula and simplify.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
x &= \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2(1)} \\
x &= \frac{1 \pm \sqrt{1+4}}{2} \\
x &= \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

The solutions are

$$x = \frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}$$

Guided Practice

1. For the following equation, rewrite as a quadratic equation and state the values for a, b and c:

$$\frac{2}{x-1} + \frac{3}{x+2} = 1$$

2. Solve the following quadratic equation using the quadratic formula:

$$6x^2 - 8x = 0$$

3. Find the approximate solutions to the following equation:

$$\frac{x+3}{2x-1}=\frac{2x+3}{x+5}$$

Answers:

1. Multiply by $(x-1)(x+2)$ to clear the fractions.

$$\begin{aligned}\frac{2}{x-1}+\frac{3}{x+2}&=1\\ \frac{2}{x-1}(x-1)(x+2)+\frac{3}{x+2}(x-1)(x+2)&=1(x-1)(x+2)\\ \frac{2}{\cancel{x-1}}(\cancel{x-1})(x+2)+\frac{3}{\cancel{x+2}}(x-1)(\cancel{x+2})&=1(x^2+2x-1x-2)\\ 2(x+2)+3(x-1)&=1(x^2+x-2)\\ 2x+4+3x-3&=x^2+x-2\\ 5x+1&=x^2+x-2\\ 0&=x^2-4x-3\end{aligned}$$

For this equation, $a=1, b=-4, c=-3$.

2. This equation does not have a 'c' term, so the value of 'c' is 0. For this equation, $a=6, b=-8, c=0$.

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(6)(0)}}{2(6)}$$

$$\begin{aligned}x&=\frac{8\pm\sqrt{64-0}}{12}\\ x&=\frac{8\pm\sqrt{64}}{12}\\ x&=\frac{8\pm 8}{12}\\ x&=\frac{8+8}{12}\ or\ x=\frac{8-8}{12}\\ x&=\frac{4}{3}\ or\ x=0\end{aligned}$$

3. Multiply by $(2x-1)(x+5)$ to clear the fractions.

$$\begin{aligned}\frac{x+3}{2x-1}&=\frac{2x+3}{x+5}\\ \frac{x+3}{2x-1}(2x-1)(x+5)&=\frac{2x+3}{x+5}(2x-1)(x+5)\\ \frac{x+3}{\cancel{2x-1}}(\cancel{2x-1})(x+5)&=\frac{2x+3}{\cancel{x+5}}(2x-1)(\cancel{x+5})\\ (x+3)(x+5)&=(2x+3)(2x-1)\end{aligned}$$

$$x^2+5x+3x+15=4x^2-2x+6x-3$$
$$x^2+8x+15=4x^2+4x-3$$
$$-3x^2+4x+18=0$$

For this equation, $a=-3, b=4, c=18$.

$$x=\frac{-(4)\pm\sqrt{(4)^2-4(-3)(18)}}{2(-3)}$$

$$x=\frac{-4\pm\sqrt{16+216}}{-6}$$
$$x=\frac{-4\pm\sqrt{232}}{-6}$$
$$x=3.21 \text{ and } x=-1.87$$

The solutions to the quadratic equation to the nearest tenth are $x=3.2$ or $x=-1.9$.

Practice Problems

State the value of a, b and c for each of the following quadratic equations.

1. $2x^2+7x-1=0$
2. $3x^2+2x=7$
3. $9x^2-7=4x$
4. $2x^2-7=0$
5. $4-2x^2=11x$

Determine the exact roots of the following quadratic equations using the quadratic formula.

6. $2x^2=8x-7$
7. $6y=2-y^2$
8. $1=8x+3x^2$
9. $2(n-2)(n+1)-(n+3)=0$
10. $\frac{2e}{e+1}-\frac{3}{e-1}=\frac{4}{e^2-1}$
11. $x^2-2x-5=0$
12. $\frac{m}{4}-\frac{m^2}{2}=-1$
13. $\frac{3}{y}-\frac{4}{y+2}=2$
14. $\frac{1}{2}x^2-\frac{x}{4}-1=0$
15. $3x^2+8x=1$

10.6 Applications of Quadratic Functions

Learning Objectives

Here you will consider real-world applications of quadratic functions.

Concept Problem

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the maximum height of the rocket?

b) How long was the rocket in the air before hitting the ground?

c) At what time(s) will the rocket be at a height of 22 yd?

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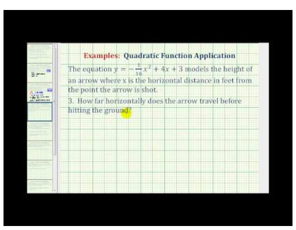
James Sousa: Quadratic Function Application- Horizontal Distance and Vertical Height



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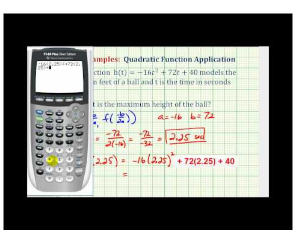
James Sousa: Quadratic Function Application- Time and Vertical Height



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Guidance

There are many real-world situations that deal with quadratics and parabolas. Throwing a ball, shooting a cannon, diving from a platform and hitting a golf ball are all examples of situations that can be modeled by quadratic functions.

In many of these situations you will want to know the highest or lowest point of the parabola, which is known as the vertex. For example, consider that when you throw a football, the path it takes through the air is a parabola. Natural questions to ask are:

- "When does the football reach its maximum height?"
- "How high does the football get?"
- "When does the football hit the ground?"

We have figured out how to find important points on a parabola like the vertex and x-intercepts. Often one of these points will represent the answer to a question. For instance, in the scenario laid out above:

- The x-coordinate of the vertex will give you the time when the football is at its maximum height.
- The y-coordinate will give you the maximum height.

To answer questions like these, you must be able to find the important points **and** interpret them correctly.

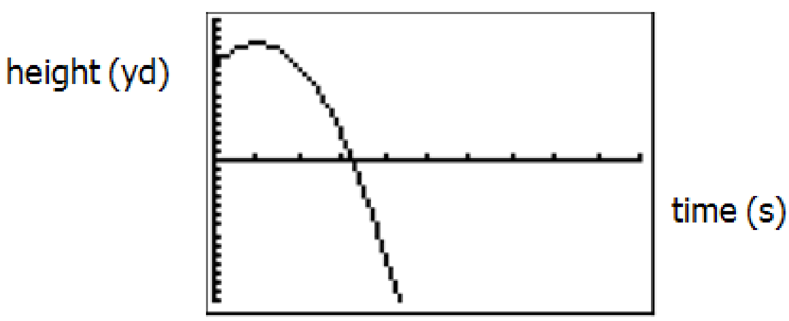
Example A

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the initial height of the rocket?

b) When did the rocket reach its maximum height?

Solution: Sketch a graph of the function. Your graphing calculator can be used to produce the graph.



a) The initial height of the rocket is the height from which it was fired. The time is zero.

$$h(t) = -5t^2 + 10t + 20$$
$$h(t) = -5(0)^2 + 10(0) + 20$$
$$\boxed{h(t) = 20\ yd}$$

The initial height of the toy rocket is 20 yards. This is the y-intercept of the graph. The y-intercept of a quadratic function written in general form is the value of 'c'.

b) The time at which the rocket reaches its maximum height is the x-coordinate of the vertex.

$$t = -\frac{b}{2a}$$
$$t = -\frac{10}{2(-5)}$$
$$\boxed{t = 1\ sec}$$

It takes the toy rocket 1 second to reach its maximum height.

Example B

The sum of a number and its square is 272. Find the number.

Solution: Let n represent the number. Write an equation to represent the problem.

$n^2 + n = 272$

You can solve this equation using the quadratic formula if you move everything to the left-hand side of the equation:

$$n^2 + n - 272 = 272 - 272$$
$$n^2 + n - 272 = 0$$

So we have $a = 1, b = 1$ and $c = -272$. Now plug into the formula:

$$n = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-272)}}{2(1)}$$
$$n = \frac{-1 \pm \sqrt{1089}}{2}$$
$$n = \frac{-1 \pm 33}{2}$$
$$n = \frac{-1+33}{2} = 16 \text{ or } n = \frac{-1-33}{2} = -17$$

These are both solutions to the problem. There are no restrictions listed in the problem regarding the solution.

Example C

The product of two consecutive positive odd integers is 195. Find the integers.

Solution: Let n represent the first positive odd integer. Let $n+2$ represent the second positive odd integer. Write an equation to represent the problem.

$$n(n+2) = 195$$
$$n^2 + 2n = 195$$

You can solve this equation with a few different methods. Here, use the quadratic formula.

$$n^2 + 2n - 195 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-195)}}{2(1)}$$

$$n = \frac{-2 \pm \sqrt{784}}{2}$$
$$n = \frac{-2 \pm 28}{2}$$

$$n = \frac{-2+28}{2} \ or \ n = \frac{-2-28}{2}$$

$$n = 13 \ or \ n = -15$$

There was a restriction on the solution presented in the problem. The solution must be an odd positive integer. Therefore, 13 is the solution you can use. The two positive odd integers are 13 and 15.

Concept Problem Revisited

a) The maximum height was reached by the rocket at one second as you found in Example A.

$$h(t) = -5t^2 + 10t + 20$$
$$h(t) = -5(1)^2 + 10(1) + 20$$
$$\boxed{h(t) = 25 \ yd}$$

The maximum height reached by the rocket was 25 yd.

b) When the rocket hits the ground, its height will be zero.

$$h(t) = -5t^2 + 10t + 20$$
$$0 = -5t^2 + 10t + 20$$

Use the quadratic formula to solve for 't'. You have $a = -5, b = 10, c = 20$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-5)(20)}}{2(-5)}$$

$$t = \frac{-10 \pm 10\sqrt{5}}{-10}$$
$$t = 1 \pm \sqrt{5}$$

$$t = 1 + \sqrt{5} \text{ or } t = 1 - \sqrt{5}$$

$$t = 3.24 \ s \text{ or } t = -1.24 \ s$$

$$\boxed{t = 3.24 \ s}$$

Accept this solution

$$\boxed{t = -1.24 \ s}$$

Reject this solution. Time cannot be a negative quantity.

The toy rocket stayed in the air for approximately 3.24 seconds.

c) The rocket reached a maximum height of 25 yd at a time of 1 second. The rocket must reach a height of 22yd before and after one second. Remember the old saying: "What goes up must come down."

Use the quadratic formula to determine these times.

$$h(t) = -5t^2 + 10t + 20$$
$$22 = -5t^2 + 10t + 20$$

$$0 = -5t^2 + 10t - 2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-5)(-2)}}{2(-5)}$$

$$t = \frac{5+\sqrt{15}}{5} \text{ or } t = \frac{5-\sqrt{15}}{5}$$

$$t = 1.77 \text{ or } t = 0.23$$

$$\boxed{t = 1.77\ sec}$$

Accept this solution

$$\boxed{t = 0.23\ sec}$$

Accept this solution.

The rocket reached a height of 22 yd at 0.23 seconds on its way up and again at 1.77 seconds on its way down.

Guided Practice

1. A rectangular piece of cardboard measuring 40 in. by 30 in. is to be made into an open box with a base (bottom) of 900 in^2 by cutting equal squares from the four corners and then bending up the sides. Find, to the nearest tenth of an inch, the length of the side of the square that must be cut from each corner.

2. The local park has a rectangular flower bed that measures 10 feet by 15 feet. The caretaker plans on doubling its area by adding a strip of uniform width around the flower bed. Determine the width of the strip.

3. $h(t) = -4.9t^2 + 8t + 5$ represents Jeremiah's height (h) in meters above the water t seconds after he leaves the diving board.

i) What is the initial height of the diving board?

ii) At what time did Jeremiah reach his maximum height?

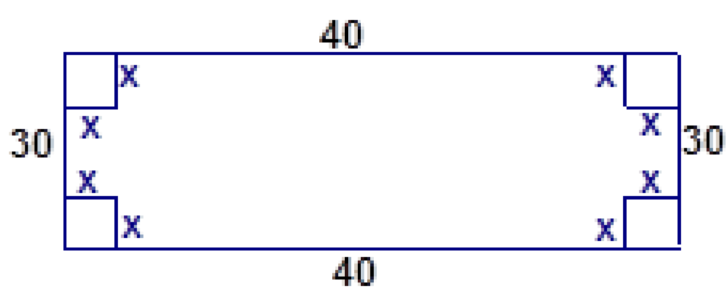
iii) What was Jeremiah's maximum height?

iv) How long was Jeremiah in the air?

4. Bob wants to build a rectangular pen for his pet rhinoceros. He has 3200 feet of fencing to use. Since he loves his rhino very much, he wants to give it the most area possible to roam around in. What dimensions should he make the pen in order to maximize its area?

Answers:

1. Sketch a diagram to represent the problem.



Let the variable x represent the side length of the square.

- $L = 40 - 2x$
- $W = 30 - 2x$

The area of a rectangle is the product of its length and its width. The area of the base of the rectangle must be 900 in^2, after the squares have been removed.

$$L \cdot W = Area$$
$$(40 - 2x)(30 - 2x) = 900$$

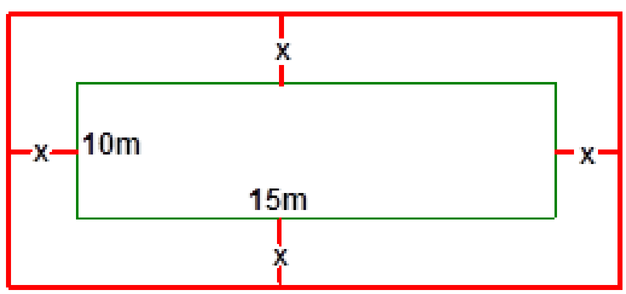
$$1200 - 80x - 60x + 4x^2 = 900$$
$$4x^2 - 140x + 1200 = 900$$

You can now solve using the quadratic formula.

$$x = 32.7 \text{ or } x = 2.3$$

The solution of 32.7 in must be rejected since it would cause the length and the width of the rectangle to result in negative values. The length of the side of the square that was cut from the cardboard was 2.3 inches.

2. Sketch a diagram to represent the problem.



Let the variable x, represent the side length of the uniform strip.

$$L = 15 + 2x \qquad L \cdot W = Area$$
$$W = 10 + 2x \qquad (15)(10) = 150\ ft^2$$

The area of a rectangle is the product of its length and its width. The area of the original flower bed is 150 ft^2. The new flower bed must be twice this area which is 300 ft^2.

$$L \cdot W = Area$$
$$(15 + 2x)(10 + 2x) = 300$$

$$\begin{aligned} 150+30x+20x+4x^2 &= 300 \\ 4x^2+50x+150 &= 300 \end{aligned}$$

You can now solve using the quadratic formula.

$$x = 2.5\ ft \text{ and } x = -15\ ft$$

$$\boxed{x = 2.5\ ft}$$

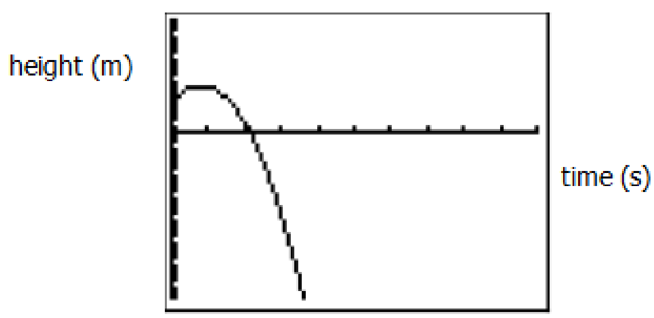
Accept this solution.

$$\boxed{x = -15\ ft}$$

Reject this solution. The width of the strip cannot be a negative value.

The width of the strip that is to be added to the flower bed is 2.5 feet.

3. Sketch a graph of the function.



i) The initial height of the diving board is when the time is zero.

$$\begin{aligned} &h(t) = -4.9t^2+8t+5 \\ &h(t) = -4.9(0)^2+8(0)+5 \\ &h(t) = 0+0+5 \\ &\boxed{h(t) = 5\ m} \end{aligned}$$

The initial height of the diving board is 5 meters.

ii) The time at which Jeremiah reaches his maximum height is the x-coordinate of the vertex.

$$t = -\frac{b}{2a}$$
$$a = -4.9$$
$$b = 8$$
$$t = -\frac{8}{2(-4.9)}$$
$$t = \frac{-8}{-9.8}$$
$$\boxed{t = 0.82\ sec}$$

It took Jeremiah 0.82 seconds to reach his maximum height.

iii) The maximum height was reached by Jeremiah at 0.82 seconds.

$$h(t) = -4.9t^2 + 8t + 5$$
$$h(t) = -4.9(0.82)^2 + 8(0.82) + 5$$
$$h(t) = -3.29 + 6.56 + 5$$
$$\boxed{h(t) = 8.27\ m}$$

The maximum height reached by Jeremiah was 8.27 meters.

iv) When Jeremiah hits the water, his height will be zero.

$$h(t) = -4.9t^2 + 8t + 5$$
$$0 = -4.9t^2 + 8t + 5$$

Use the quadratic formula to solve for 't'.

$$t = -0.48 \text{ or } t = 2.12$$

$$\boxed{t = 2.12\ sec}$$

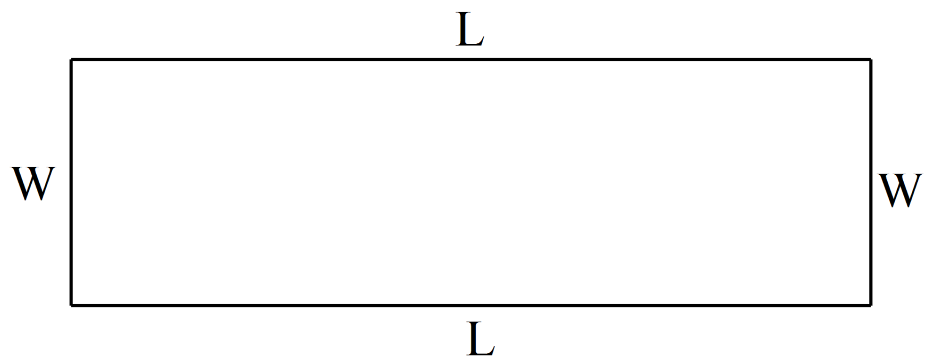
Accept this solution

$$\boxed{t = -0.48\ sec}$$

Reject this solution. Time cannot be a negative quantity.

Jeremiah stayed in the air for approximately 2.12 seconds.

4. Bob wants his pen to be rectangular, so let's draw a picture of it:



The pen has a length L and a width W. What do we know about them? The area of the pen is

$$A = L \cdot W \tag{1}$$

This is what we want to maximize, but that by itself isn't enough information. We also know that Bob has 3200 feet of fence. If he is going to make the biggest pen possible, then he needs to use all 3200 feet. This means that the perimeter of the pen will be 3200 i.e.

$$2L + 2W = 3200 \tag{2}$$

We can solve this equation for a variable (say, L), and then substitute into the area equation:

$$\begin{aligned} 2L + 2W &= 3200 \\ 2L + 2W - 2W &= 3200 - 2W \\ 2L &= 3200 - 2W \\ \frac{2L}{2} &= \frac{3200}{2} - \frac{2W}{2} \\ L &= 1600 - W \end{aligned} \tag{3}$$

So L is the same as $1600 - W$. Let's plug that into equation (1):

$$\begin{aligned} A &= L \cdot W \\ A &= (1600 - W) \cdot W \\ A &= 1600W - W^2 \\ A &= -W^2 + 1600W \end{aligned}$$

Notice that we now have A written as a quadratic function of W with $a = -1, b = 1600,$ and $c = 0$. The parabola will be downward-facing so... the maximum will be at the vertex! Let's find it:

$$x_v = \frac{-b}{2a} = \frac{-1600}{2(-1)} = \frac{-1600}{-2} = 800$$

$$y_v = -(800)^2 + 1600 * 800 = 640,000$$

$$(x_v, y_v) = (800\ ,\ 640,000)$$

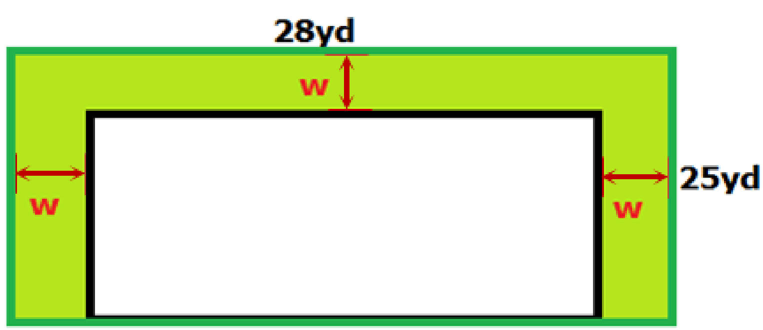
What do these values tell us? The 640,000 is a value for A so it tells us the maximum area for the pen. The 800 is a value for W, so it tells us the width required to obtain the maximum area. Since we were asked to find the dimensions of the pen, this is half of our answer. To get the other half (L), we can plug 800 in for W in equation (3): $L = 1600 - 800 = 800$.

So Bob should build an 800 by 800 pen to get the maximum area. The shape needed is a square, in fact.

Practice Problems

Solve the following problems using your knowledge of quadratic functions.

1. The product of two consecutive even integers is 224. Find the integers.
2. The hypotenuse of a right triangle is 26 inches. The sum of the legs is 34 inches. Find the length of the legs of the triangle.
3. The product of two consecutive integers is 812. What are the integers?
4. The width of a rectangle is 3 inches longer than the length. The area of the rectangle is 674.7904 square inches. What are the dimensions of the rectangle?
5. The product of two consecutive odd integers is 3135. What are the integers?
6. Josie wants to landscape her rectangular back garden by planting shrubs and flowers along a border of uniform width as shown in the diagram. Determine the width of the border if the outside fence has dimensions of 28 yd by 25 yd and the remaining garden is to be $\frac{3}{4}$ of the original size.



7. Gregory ran the 1800 yard race last year but knows that if he could run 0.5 yd/s faster, he could reduce his time by 30 seconds. What was Gregory's time when he ran the race last year?

During a high school baseball tournament, Lexie hits a pitch and the baseball stays in the air for 4.42 seconds. The function describes the height over time, where h is its height, in yards, and t is the time, in seconds, from the instant the ball is hit.

$h = -5t^2 + 22t + 0.5$

8. Algebraically determine the maximum height the ball reaches.

9. When will the ball reach its maximum height?
10. How long will the ball be at a height of less than 20 meters while it is in the air?

A rock is thrown off a 75 meter high cliff into some water. The height of the rock relative to the cliff after t seconds is given by $h(t) = -5t^2 + 20t$.

11. Where will the rock be after five seconds?
12. How long before the rock reaches its maximum height?
13. When will the rock hit the water?

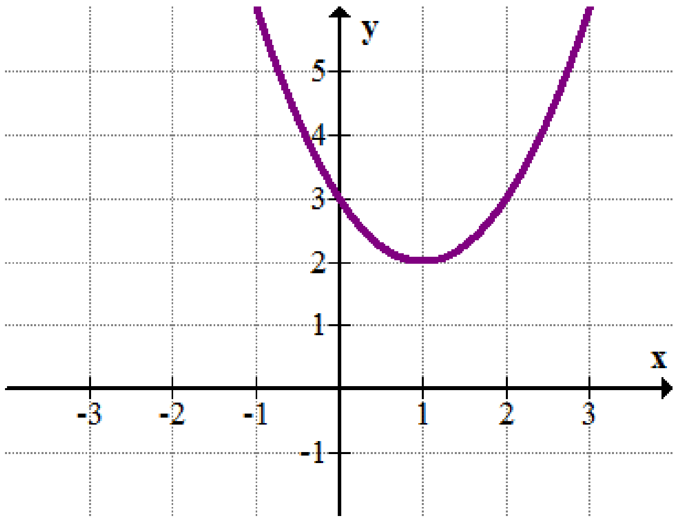
You jump off the end of a ski jump. Your height in meters relative to the height of the ski jump after t seconds is given by $h(t) = -5t^2 + 12t$.

14. How high will you be after 2 seconds? At this point are you going up or coming down?
15. If you spend 6.1 seconds in the air, how far below the end of the ski jump do you land? (What is the vertical distance?)

10.7 Imaginary and Complex Numbers

Concept Problem

The solutions to a quadratic equation show up as the x-intercepts of the corresponding quadratic function. The parabola $y = x^2 - 2x + 3$ is shown below.



What does this graph tell you about the solutions to $x^2 - 2x + 3 = 0$?

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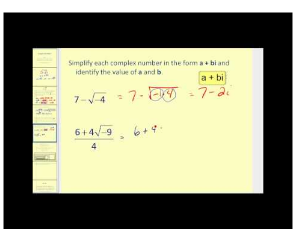
James Sousa: Complex Numbers



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Guidance

The first numbers used by humans were the counting numbers $\{1, 2, 3, 4 \ldots\}$ because numbers they needed to count physical objects. They were the only numbers people needed for a very long time. It took thousands of years before even the concept of the number zero was invented around the year 500 AD in India. Since then, humans have slowly been incorporating new things into our number system so that it can work for us. Fractions, decimals, and negative numbers have become an important part of our world.

For years, it was accepted that the square root of a negative number did not exist since the square of any nonzero real number is positive. In order for a solution to exist to the equation $x^2 = -1$, mathematicians invented a solution. This solution is called the imaginary unit and is noted by the letter i:

$\sqrt{-1} = i$ and $i^2 = -1$

Imaginary numbers were not commonly accepted in mathematics until the 1700s, but since then they have become an important part of our number system and are especially important in physics. They are called imaginary because they cannot be found on a traditional number line of real numbers. The square root of any negative number can be written in terms of the imaginary number i:

- $\sqrt{-4} = \sqrt{4 \cdot -1} = \sqrt{4}\sqrt{-1} = 2i$
- $\sqrt{-5} = \sqrt{-1 \cdot 5} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$
- $\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$

You can perform addition, subtraction, and multiplication with imaginary numbers just like regular numbers (you can also do division, but that is a bit more complicated and won't be considered here). When performing multiplication, remember that $i^2 = -1$. You should always express answers in such a way that the i does not have an exponent.

- $2i+1-3i+4$ simplifies to $-i+5$ or $5-i$
- $3i \cdot 2i$ can be multiplied to $6i^2 = 6(-1) = -6$

Imaginary numbers are not usually used by themselves; they are used along with real numbers. The combined real and imaginary numbers are called **complex numbers**. A complex number can be written in the form $a+bi$ where a and b are real numbers. a is called the *real part* of the number and b is called the *imaginary part*. These are examples of complex numbers:

$$\begin{aligned} 2+5i & \qquad (\text{Real part} = 2, \text{Imaginary Part} = 5) \\ -7+\frac{3}{2}i & \qquad (\text{Real part} = -7, \text{Imaginary Part} = \tfrac{3}{2}) \\ -i & \qquad (\text{Real part} = 0, \text{Imaginary Part} = -1) \\ 10 & \qquad (\text{Real part} = 10, \text{Imaginary Part} = 0) \end{aligned}$$

Any operation that can be done with real numbers can be done with complex numbers (addition/subtraction, multiplication/division). In fact, for the most part these operations are intuitive. Addition and subtraction follow the rules of combining like terms, while multiplication uses the same FOIL technique we are used to.

Example A

Express $\sqrt{-49}$ as a simplified imaginary number.

Solution: $\sqrt{-49} = \sqrt{49 \cdot -1} = \sqrt{49}\sqrt{-1} = 7i$

Example B

Express $\sqrt{-40}$ as a simplified imaginary number.

Solution: $\sqrt{-40} = \sqrt{4 \cdot 10 \cdot -1} = \sqrt{4}\sqrt{10}\sqrt{-1} = 2i\sqrt{10}$

It is customary to put the i on the left side of a radical expression to avoid confusion about whether it is inside or outside the radical.

Example C

Simplify the following expression: $(4+3i)+(6-5i)$

Solution: Simplify by combining like terms. The real parts are added and the imaginary parts are added:

$$(4+3i)+(6-5i) = 10-2i$$

Example D

Multiply $(4+3i)(6-5i)$.

Solution: At first, we can treat this like any other binomial multiplication and use FOIL:

$$\begin{aligned}(4+3i)(5-6i) &= \\ 20-24i+15i-18i^2 &= \\ 20-9i-18i^2\end{aligned}$$

So far, so good, but with complex numbers we can keep going. Remember that $i^2 = -1$ so we can replace the i^2 in the last term with -1 and simplify:

$$\begin{aligned}20-9i-18i^2 &= \\ 20-9i-18(-1) &= \\ 20-9i+18 &= \\ 38-9i\end{aligned}$$

Powers of i

The number i displays some interesting behavior when raised to powers. Let's calculate the first four powers of i and then see if we can establish a pattern:

$$\begin{aligned}
i^0 &= 1 && \text{(Any nonzero number raised to the 0 power is 1)} \\
i^1 &= i && \text{(Any number raised to the 1 power is itself)} \\
i^2 &= -1 && \text{(Definition of i)} \\
i^3 &= -i && (i^3 = i^2 \cdot i = -1 \cdot i = -i) \\
i^4 &= 1 && (i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1)
\end{aligned}$$

We came back to where we started! Real numbers don't display this kind of behavior. If we evaluate more powers we can see the pattern emerge:

$$\begin{array}{cccc}
i^0 = 1 & amp; i^4 = 1 & amp; i^8 = 1 & amp; i^{12} = 1 \\
i^1 = i & amp; i^5 = i & amp; i^9 = i & amp; i^{13} = i \\
i^2 = -1 & amp; i^6 = -1 & i^{10} = -1 & amp; i^{14} = -1 \\
i^3 = -i & amp; i^7 = -i & i^{11} = -i & amp; i^{15} = -i
\end{array}$$

We can evaluate any power of i we like by following the pattern established above. It's important to notice than when the power on i is a multiple of 4, then it will simplify to 1. For instance, $i^{44} = 1$.

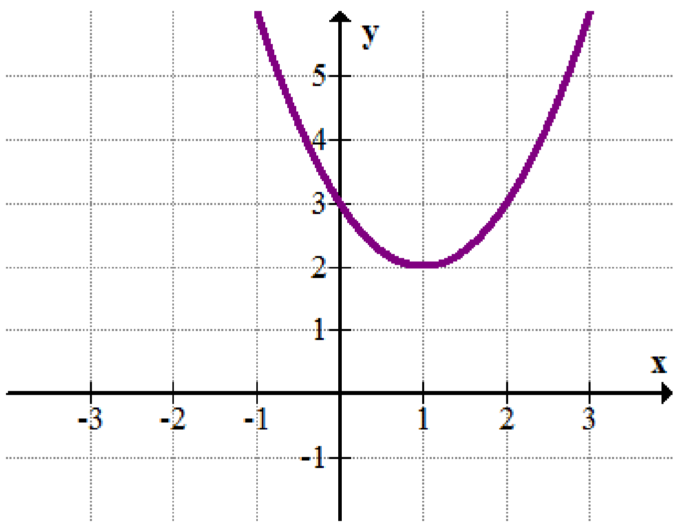
Example E

Simplify i^{23}.

Solution: $i^{23} = i^{20} \cdot i^3 = 1 \cdot -i = -i$

Concept Problem Revisited

The parabola $y = x^2 - 2x + 3$ has no x-intercepts, as shown below.



This means that the solutions to the equation $x^2 - 2x + 3 = 0$ are not real numbers. The solutions are complex numbers. You can still find the solutions using the quadratic formula, but your result will be 2 complex number solutions.

Guided Practice

Simplify each of the following:

1. $(5-3i)-(2+4i)$

2. $3i(4i^2-5i+3)$

3. $(7+2i)(3-i)$

4. $\sqrt{-12}$

5. Simplify i^7

Answers:

1. $(5-3i)-(2+4i) = 5-3i-2-4i = 3-7i$

2.

$$\begin{aligned}3i(4i^2-5i+3) &= 3i(4\cdot -1-5i+3)\\ &= 3i(-4-5i+3)\\ &= 3i(-1-5i)\\ &= -3i-15i^2\\ &= -3i-15(-1)\\ &= -3i+15\end{aligned}$$

3.

$$\begin{aligned}(7+2i)(3-i) &= 21-7i+6i-2i^2\\ &= 21-i-2i^2\\ &= 21-i-2(-1)\\ &= 21-i+2\\ &= 23-i\end{aligned}$$

4. $\sqrt{-12} = \sqrt{-1\cdot 4\cdot 3} = 2i\sqrt{3}$

5.

Practice

Express each as a simplified imaginary number.

1. $\sqrt{-300}$
2. $\sqrt{-32}$
3. $4\sqrt{-18}$
4. $\sqrt{-75}$
5. $\sqrt{-98}$

Simplify each of the following:

6. $(8+5i)-(12+8i)$
7. $(7+3i)(4-5i)$
8. $(2+i)(4-i)$
9. $3(5i-4)-2(6i-7)$
10. $5i(3i-2i^2+4)$
11. $(3+4i)+(11+6i)$
12. $(5+2i)(1-5i)$
13. $(1+i)(1-i)$
14. $2(6i-3)-4(2i+6)$
15. i^3
16. i^4
17. i^6

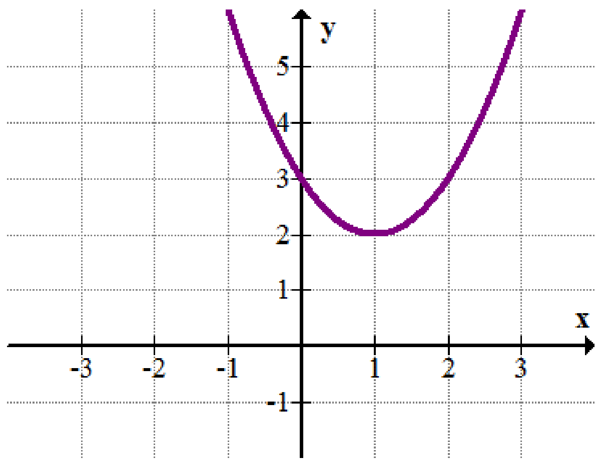
10.8 Complex Roots of Quadratic Functions

Learning Objectives

Here you'll learn how to find complex roots of a quadratic function and what it means when a function has complex roots.

Concept Problem

The quadratic function $f(x) = x^2 - 2x + 3$ (shown below) does not intersect the x-axis and therefore has no real x-intercepts. What does that mean about the solutions to the equation $0 = x^2 - 2x + 3$?



We know the equation cannot have real solutions because the graph does not intersect the x-axis; however, the quadratic formula is supposed to find solutions for **any** quadratic equation. What kind of solutions could it find in a case where there are no real number solutions?

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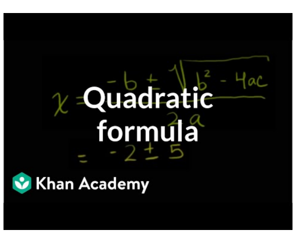
Khan Academy Using the Quadratic Formula



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Guidance

Recall that the imaginary unit, i, is a number whose square is -1:

$$i^2 = -1$$

This also means that $i = \sqrt{-1}$.

Recall also that the sum of a real number and an imaginary number is called a *complex number*. All complex numbers can be written in the form $a+bi$ where a and b are real numbers. The set of complex numbers includes real numbers, imaginary numbers, and combinations of real and imaginary numbers.

When a quadratic function does not intersect the x-axis, it has complex roots. When solving for the roots of a function algebraically using the quadratic formula, you will end up with a negative value under the square root. With your knowledge of complex numbers, you can still state the complex roots of a function just like you would state the real roots of a function.

Example A

Solve the following quadratic equation for m.

$$\boxed{m^2 - 2m + 5 = 0}$$

Solution: You can use the quadratic formula to solve. For this quadratic equation, $a = 1, b = -2, c = 5$.

$$\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
m &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \\
m &= \frac{2 \pm \sqrt{4-20}}{2} \\
m &= \frac{2 \pm \sqrt{-16}}{2} \qquad \sqrt{-16} = \sqrt{16} \cdot i = 4i \\
m &= \frac{2 \pm 4i}{2} \\
m &= 1 \pm 2i \\
m &= 1 + 2i \qquad \text{or} \qquad m = 1 - 2i
\end{aligned}$$

There are no real solutions to the equation. The solutions to the quadratic equation are $1+2i$ and $1-2i$.

Example B

Solve the following equation by rewriting it as a quadratic and using the quadratic formula.

$\frac{3}{k+3} - \frac{2}{k+2} = 1$

Solution: To rewrite as a quadratic equation, multiply each term by $(k+3)(k+2)$.

$$\frac{3}{k+3}(k+3)(k+2) - \frac{2}{k+2}(k+3)(k+2) = 1(k+3)(k+2)$$
$$3(k+2) - 2(k+3) = (k+3)(k+2)$$

Expand and simplify.

$$3k+6-2k-6 = k^2+2k+3k+6$$
$$k^2+4k+6 = 0$$

Solve using the quadratic formula. For this quadratic equation, $a = 1, b = 4, c = 6$.

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$k = \frac{-(4) \pm \sqrt{(4)^2-4(1)(6)}}{2(1)}$$
$$k = \frac{-4 \pm \sqrt{16-24}}{2}$$
$$k = \frac{-4 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \cdot \sqrt{-1} = \sqrt{4 \cdot 2} \cdot i = 2i\sqrt{2}$$
$$k = \frac{-4 \pm 2i\sqrt{2}}{2}$$
$$k = -2 \pm i\sqrt{2}$$
$$k = -2+i\sqrt{2} \quad \text{or} \quad k = -2-i\sqrt{2}$$

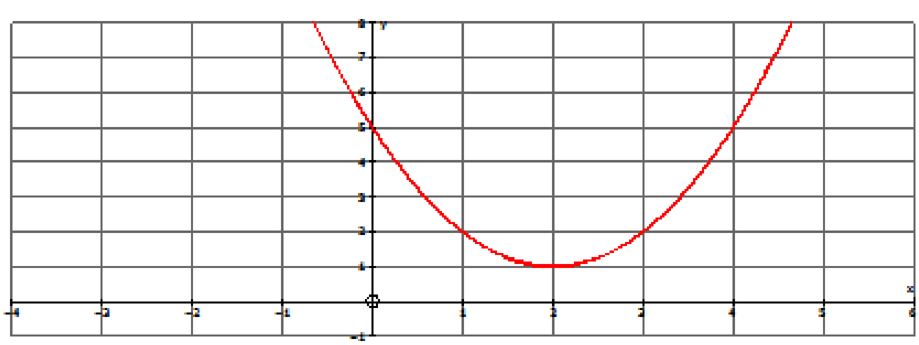
There are no real solutions to the equation. The solutions to the equation are $-2+i\sqrt{2}$ and $-2-i\sqrt{2}$

Example C

Sketch the graph of the following quadratic function. What are the roots of the function?

$$y = x^2 - 4x + 5$$

Solution: Use your calculator or a table to make a sketch of the function. You should get the following:



As you can see, the quadratic function has no x-intercepts; therefore, the function has no real roots. To find the roots (which will be complex), you must use the quadratic formula.

For this quadratic function, $a = 1, b = -4, c = 5$.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \\
x &= \frac{4 \pm \sqrt{16 - 20}}{2} \\
x &= \frac{4 \pm \sqrt{-4}}{2} \qquad \sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i \\
x &= \frac{4 \pm 2i}{2} \\
x &= 2 \pm i \\
x &= 2 + i \text{ or } x = 2 - i
\end{aligned}
$$

The complex roots of the quadratic function are $2+i$ and $2-i$.

Concept Problem Revisited

To find the solutions of the equation $0 = x^2 - 2x + 3$, we must use the quadratic formula.

For this quadratic equation,we have $a = 1, b = -2, c = 3$.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} \\
x &= \frac{2 \pm \sqrt{4 - 12}}{2} \\
x &= \frac{2 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \cdot \sqrt{-1} = 2i\sqrt{2} \\
x &= \frac{2 \pm 2\sqrt{2}i}{2} \\
x &= 1 \pm \sqrt{2}i
\end{aligned}
$$

The Discriminant

Consider the three examples shown below:

$x^2-4x+3=0$	$x^2-4x+4=0$	$x^2-4x+5=0$
$a=1, b=-4, c=3$	$a=1, b=-4, c=4$	$a=1, b=-4, c=5$
$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(3)}}{2(1)}$	$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(4)}}{2(1)}$	$amp;x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$
$x=\frac{4\pm\sqrt{4}}{2}$	$x=\frac{4\pm\sqrt{0}}{2}$	$amp;x=\frac{4\pm\sqrt{-4}}{2}$
$x=\frac{4\pm2}{2}$	$x=\frac{4\pm0}{2}$	$x=\frac{4\pm2i}{2}$
$x=3, x=1$	$x=2$	$x=2+i, x=2-i$
2 real solutions	1 real solution	2 complex solutions

The equations are very similar, but the nature of their solutions is very different. Why is this? The difference lies in the expression under the radical in the quadratic formula. This part of the formula is called the **discriminant** and it is given the name D:

$$D=b^2-4ac$$

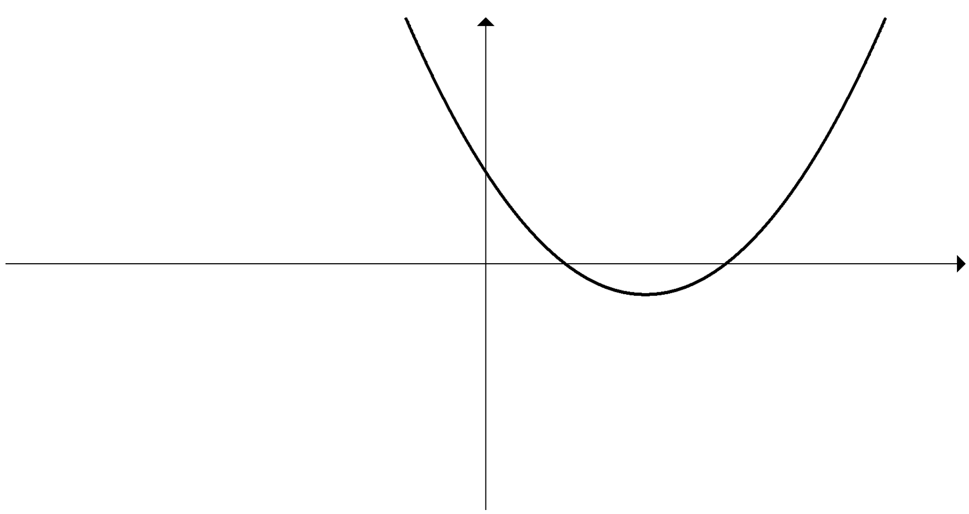
The value of D can tell you about the types of solutions an equation will have. In the examples above we had:

$D=4$	$D=0$	$D=-4$
Positive # under radical	Zero under radical	Negative # under radical
2 real solutions	1 real solution	2 complex solutions

This pattern will hold up more generally.

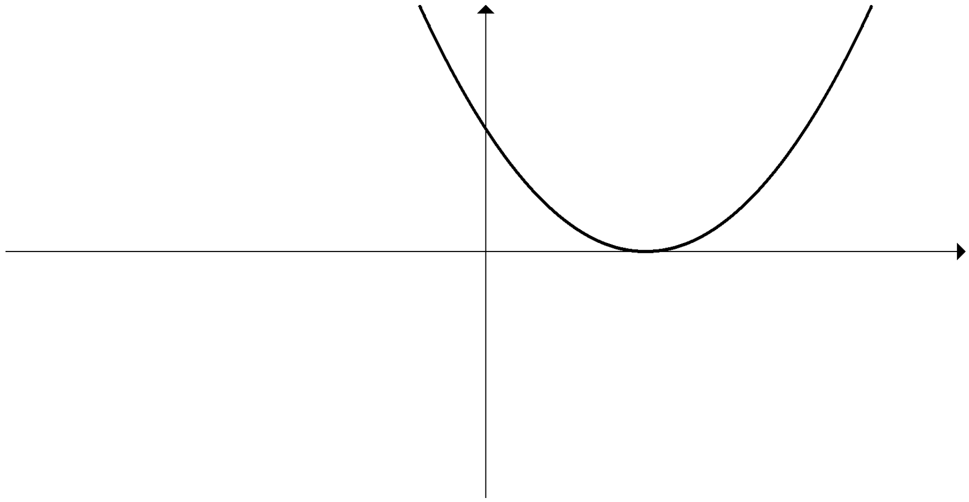
If $D>0$:

The equation will have two real number number solutions. The corresponding graph will have two x-intercepts



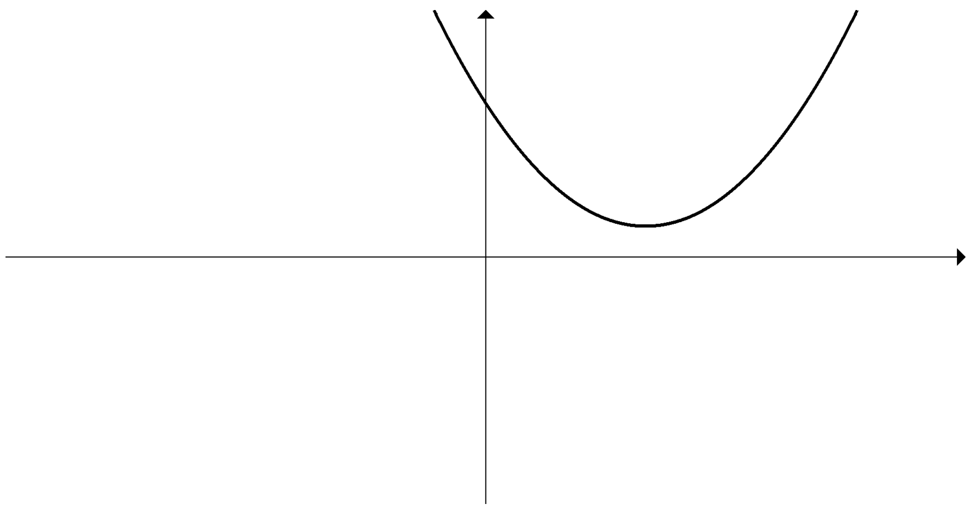
If $D=0$:

The equation will have one real number solution. The corresponding graph will have exactly one x-intercept.



If $D < 0$:

The equation will have two complex number solutions. The corresponding graph will have no x-intercepts.



Guided Practice

1. Solve the following quadratic equation. Express all solutions in simplest radical form.

$$2n^2 + n = -4$$

2. Solve the following quadratic equation. Express all solutions in simplest radical form.

$$m^2 + (m+1)^2 + (m+2)^2 = -1$$

3. Is it possible for a quadratic function to have exactly one complex root?

Answers:

1.

$$2n^2+n=-4$$

Set the equation equal to zero.

$$2n^2+n+4=0$$

Solve using the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$n=\frac{-(1)\pm\sqrt{(1)^2-4(2)(4)}}{2(2)}$$
$$n=\frac{-1\pm\sqrt{1-32}}{4}$$
$$n=\frac{-1\pm\sqrt{-31}}{4}$$
$$n=\frac{-1\pm i\sqrt{31}}{4}$$

2.

$$m^2+(m+1)^2+(m+2)^2=-1$$

Expand and simplify.

$$m^2+(m+1)(m+1)+(m+2)(m+2)=-1$$
$$m^2+m^2+m+m+1+m^2+2m+2m+4=-1$$
$$3m^2+6m+5=-1$$

Write the equation in general form.

$$3m^2+6m+6=0$$

Divide by 3 to simplify the equation.

$$\frac{3m^2}{3}+\frac{6m}{3}+\frac{6}{3}=\frac{0}{3}$$
$$m^2+2m+2=0$$

Solve using the quadratic formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$m = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$
$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$
$$m = \frac{-2 \pm \sqrt{-4}}{2}$$
$$m = \frac{-2 \pm 2i}{2}$$
$$m = -1 \pm i$$

3. No, even in higher degree polynomials, complex roots will always come in pairs. Consider when you use the quadratic formula– if you have a negative under the square root symbol, both the + version and the - version of the two answers will end up being complex.

Practice Problems

1. If a quadratic function has 2 x-intercepts, how many complex roots does it have? Explain.
2. If a quadratic function has no x-intercepts, how many complex roots does it have? Explain.
3. If a quadratic function has 1 x-intercept, how many complex roots does it have? Explain.
4. If you want to know whether a function has complex roots, which part of the quadratic formula is it important to focus on?
5. You solve a quadratic equation and get 2 complex solutions. How can you check your solutions?
6. In general, you can attempt to solve a quadratic equation by graphing, factoring, or using the quadratic formula. If a quadratic equation has complex solutions, what methods do you have for solving the equation?

Solve the following quadratic equations. Express all solutions in simplest radical form.

7. $x^2 + x + 1 = 0$
8. $5y^2 - 8y = -6$
9. $2m^2 - 12m + 19 = 0$
10. $-3x^2 - 2x = 2$
11. $2x^2 + 4x = -11$
12. $-x^2 + x - 23 = 0$
13. $-3x^2 + 2x = 14$
14. $x^2 + 5 = -x$
15. $\frac{1}{2}d^2 + 4d = -12$

Summary

You learned that all quadratic equations have a corresponding quadratic function. Real solutions to quadratic equations are the x-intercepts of the quadratic function. If a quadratic equation has only complex solutions, the quadratic function will not have x-intercepts.

You also learned that there are several methods for solving quadratic equations:

1. Factoring and the zero product property (learned previously)
2. Graphing and looking for x-intercepts
3. The quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The advantage of the quadratic formula is that it will always work to give you solutions, even if the solutions are not real numbers.

If you want to determine whether the roots of a given quadratic function are real or complex, but you don't need to know specifically what the roots are, you can use the discriminant. The discriminant is the part of the quadratic formula under the square root symbol ($b^2 - 4ac$). If the discriminant is negative, the roots will be complex. If the discriminant is equal to zero, there will only be one root (of multiplicity 2). If the discriminant is positive, the roots will be real.

10.9 References

1. . Parabola Increasing and Decreasing.
2. . Unit 10 Example E.