Fourier Analysis Test

Format: There will be 60 points necessary, 15 for the major theorems, and 10 apiece for the others. You can answer more than one major theorem, and get bonus points. You can answer as many as possible, but only the highest 5 will count. Thus two major theorems, and 3 minor problems, correctly presented will yield a perfect grade.

The Hard Ones: 15 pts

Theorem 0.1 (Fourier Isometry) *Let $f(t) \in L^2(\mathbf{R})$, and let $\hat{f}(s)$ be its continuous Fourier Transform as defined in* **??**. *Then we have*

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt. \tag{1}$$

In addition the Fourier Transform is an isometry from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.

Theorem 0.2 (Uncertainty Principle) *Let $f(t)$ and $\hat{f}(s)$ be in $L^2(R)$, and let $tf(t)$ and $s\hat{f}(s)$ also be in $L^2(R)$. Assume for simplicity that $tf^2(t) \to 0$. Then we have*

$$\int t^2 |f(t)|^2 dt \int s^2 |\hat{f}(s)|^2 ds \geq \frac{1}{4} \|f(t)\|^4. \tag{2}$$

If $\|f(t)\| = 1$, then we have the simpler form,

$$\int t^2 |f(t)|^2 dt \int s^2 |\hat{f}(s)|^2 ds \geq \frac{1}{4}. \tag{3}$$

In addition, equality is only obtained by scaled dilates of the Gaussian, or normal distribution function e^{-t^2}.

Theorem 0.3 (Partition of Unity) *Suppose that $\{\phi_n(t)\}$ is an orthonormal basis for $L^2[a,b]$. Then*

$$\frac{1}{b-a} \sum_n |\hat{\phi}_n(s)|^2 = 1.$$

Theorem 0.4 (The Shannon Sampling Theorem) *Let $f(t) \in L^2(-\infty, \infty)$ and suppose that $\hat{f}(s) = 0$ for $|s| > \pi$. Then*

$$f(t) = \sum_k f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

Theorem 0.5 (Fourier Transform of a Gaussian) *The Fourier transform of a Gaussian is once again a Gaussian.*

The Easy Ones, 10 pts:

Theorem 0.6 (Differentiation) *Let $f(t) \in L^2(\mathbf{R})$ and let $\hat{f}(s) \in L^2(\mathbf{R})$ be its Fourier transform. If $(-is)\hat{f}(s) \in L^2(\mathbf{R})$, then it follows that $f'(t)$ exists as an element of $L^2(\mathbf{R})$, and that its Fourier transform is*

$$\mathcal{F}(f'(t)) = (-is)\hat{f}(s).$$

Theorem 0.7 (Convolution Theorem) *If $f(t)$ and $g(t)$ are elements of $L^2(R)$, then the convolution $f * g(t)$ is also an element of $L^2(R)$. More importantly, the Fourier transform of $f * g$ is given simply by $\hat{f}(s)\hat{g}(s)$, or pointwise multiplication of the corresponding Fourier transforms. Mathematically we have*

$$\mathcal{F}(f * g) = \hat{f}(s)\hat{g}(s). \tag{4}$$

Theorem 0.8 (Translation) *If $f(t) \in L^2(R)$ and $\hat{f}(s)$ is its Fourier transform then*

$$\mathcal{F}(T_a(f(t)) = \mathcal{F}(f(t-a)) = \hat{f}(s)e^{isa}. \tag{5}$$

Theorem 0.9 (Dilation) *Let $f(t) \in L^2(R)$ and let $\hat{f}(s)$ be its Fourier transform. Then if the Fourier operator is denoted by $\mathcal{F}$ we have*

$$\mathcal{F}(D_a(f(t)) = \mathcal{F}(f(at)) = \frac{1}{a}\hat{f}(\frac{s}{a}).$$

1. Show that if f is even, $\hat{f}(s)$ is real and symmetric, and if f is odd, $\hat{f}(s)$ is purely imaginary and antisymmetric.
2. Show that if $f(t) \geq 0$, then the maximum value of $\hat{f}(s)$ is attained by $\hat{f}(0)$.
3. Show how the differential equation $ay'' + by' + cy = f(t)$ can be written as a convolution problem.