Precursors Explained: The approximate singular value decomposition for transmission through absorbing media Part 3: Signal Processing

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ABSTRACT

In Part 1 we derived an asymptotic singular value decomposition (SVD) for signal transmission through absorbing media under rather broad assumptions. We showed that under basic assumptions the singular vectors asymptotically approach the Legendre polynomials. We also showed that the dominant singular vector behaves very much like a Brillouin precursor, in that its decay would be at the rate $O(z^{-1/2\beta})$ if the absorption behaves like $O(\omega^{\beta})$ as $\omega \to 0$. This is in direct contrast to the expected exponential absorption of signals in an absorbing media, $O(e^{-kz})$. We were also able to derive the decay rates of the subdominant singular vectors or values, which were once again not exponential. Thus it followed that no causal signals decay exponentially in this broad class of media. These asymptotic results do not answer the question of how fast the convergence, or the decay rates agree with the predictions. These questions are addressed in Part 2.

In Part 2 we used discrete approximations to the continuous operators introduced in Part 1, following in the spirit of Slepian and Pollack.[6] These discrete approximations support the conclusions of Part 1 very closely. We utilize two different models for the dielectric material and show that in each case the results of previous Theorems are shown to come true. We show that the singular values decay in a precursor like manner, as predicted in Part 1. We also show that the singular vectors converge to the Legendre polynomials, as predicted in Part 1.

In Part 3 we will begin to describe the optimal way to use the results of Parts 1 and 2 to construct a viable radar system. This is the signal processing design of the system. There are a number of factors in this transition. We want to 1) maximize the signal to noise ratio of the system while 2) minimizing the demands and cost of the hardware used in the system. We also want to allow the system to 3) self-adapt to the changing dielectric factors of the medium, whether it be foliage or some other medium.

1. Introduction

Precursors were first explained in 1914 by Sommerfeld and Brillouin[1,2] where they examined the transmission of a square-windowed sinusoid through an absorbing and dispersive media. The most stunning result of this analysis was that the Brillouin precursor decays at a rate of $O(z^{-1/2})$, as opposed the to normal exponential attenuation $O(e^{-kz})$. Exponential attenuation is expected from the solution of the most basic of differential equation which models absorption, $y' = -ky$.

Brillouin concluded originally, due to incorrect asymptotics, that these pulses would not be useful. In his 1960 book, however, he restated that "The subject was a fascinating one, but it had, at the time, only academic importance. Experimental verifications were discovered much later, in connection with reflections of radio signals from Heaviside layers, and also for problems of radar systems."[3] The English translations of both papers[1,2] can be found in Brillouin's text.[3]

We examine this transmission operator in Part 1 as a compact operator and from this analysis the generated structure is very informative. The inputs and outputs to the operator are separated into orthogonal subspaces, with the power passing through each subspace clearly described by the singular vectors and values. We have shown in Part 1 that Brillouin precursors, or pulses remarkably similar to Brillouin precursors, are the dominant singular vectors associated with transmission through media such as foliage or water using basic techniques of operator theory and linear algebra.

In addition, we showed in Part 1 that there are an infinite number of “precursors”, in the sense that there are an infinite number of orthogonal functions which are not exponentially attenuated. These “precursors” are the singular functions of the compact operator associated with the transmission of a short pulse through a dispersive and absorptive media. These singular functions are shown to asymptotically converge to the Legendre polynomials. The result of this asymptotic singular value decomposition is that no causal function will decay exponentially through the standard physical models which we consider.

We will now restate this result as the major theorem of Part 1.

THEOREM 1. *[The Singular Value Decomposition for Absorbing Media] Let L_z be the compact operator associated with transmitting a short pulse through a uniform dispersive and absorbing medium of length z. Then for each distance z the operator L_z will have a singular value decomposition*

$$L_z(f) = \sum_{k=0}^{\infty} \sigma_k^z \langle f, \psi_k^z \rangle \phi_k^z.$$

If the absorption coefficient $\alpha(w) = O(w^\beta)$ in a neighborhood about the origin, then $\sigma_k^z = O(z^{-(2k+1)/2\beta})$. Moreover, $\|\psi_k^z(t) - P_k(t)\|_2 \to 0$, where the $P_k(t)$ are the Legendre polynomials on the interval $[0, l]$.

In this paper we will numerically verify the contents of Theorem 1.

1.1. Overview

In Section 2 we will review the historical origins of the approach which we will take. Section 3 will give our first major theorem, showing that the operator norm of a function transmitted through an absorbing media is not exponentially absorbed. Section 4 demonstrates that a wide class of functions will exhibit precursor behavior. Section 5 shows only one subspace, or singular vector, will decay at the optimal rate in a medium. This dominant singular vector correspond to the Brillouin precursor. In addition we show that the subdominant singular vectors decay at a rate which is slower than the expected exponential attenuation, and is much closer to the decay rate of the Brillouin precursor. Section 6 discusses the differences between Brillouin's work, which focused on the amplitude of the precursor, and this work, which focuses on the energy of the precursor.

2. Background material and notation

2.1. Brillouin and Sommerfeld's Analysis

Brillouin and Sommerfeld worked together in 1913, concerned about the concepts of group velocity and causality. In this situation causality means that no transmitted signal exceeds the speed of light. It had been observed that when group velocity was used to determine the speed of a pulse, some pulses traveled at a speed which exceeded the speed of light. Sommerfeld was interested in understanding this phenomenon and suggested the problem to Brillouin.

Sommerfeld showed that the first precursor, called the forerunner in these papers, traveled at the vacuum speed of light but was absolutely causal, in that it did not exceed the speed of light.[1] The problem with causality was the definition of group velocity.

Brillouin observed that there was another function whose group velocity traveled at a speed which was above that of the expected speed of light in the medium. This Brillouin precursor followed the Sommerfeld precursor, both in understanding and time. This precursor, or forerunner, came after the Sommerfeld precursor. This precursor was also not exponentially attenuated.

Recent interest in these pulses has centered around this non-exponential attenuation property. This property diverges from all of the easy standards of mathematics and physics and therefore needs understanding. Brillouin's paper states in its conclusion that the second forerunner or precursor, which is now referred to as the Brillouin precursor,[2] is attenuated at a rate of

$$\frac{1}{\sqrt{z}} \exp\left(-\frac{2}{3}\rho\frac{\delta'}{c}z\right).$$

At first examination one wonders why this is surprising, given that this is exponential attenuation. Further examination of the paper reveals that δ' is a moving space-time coordinate, which is 0 at exactly the point of the maximum of the Brillouin precursor. Thus there is one space-time coordinate where there is no exponential attenuation.

We will now present a numerical approach which follows the results of Part 1. We will show that this is not an anomaly, but in fact no causal function will be attenuated exponentially in this broad class of models.

2.2. Slepian and Pollack's approach

We begin in the spirit of Slepian and Pollack.[6] They studied the question "How much energy can a finite-time signal put through a finite-frequency window?" from the viewpoint of communications.

Convolution operators which describe the evolution of a pulse $r(t, z)$ through a homogeneous linear medium have a very simple form. Given an initial plane wave signal which is incident on a homogeneous medium, $s(t)$, the pulse at time t, and distance z is appropriately modeled by

$$r(t, z) = \int s(\tau)A_z(t-\tau)d\tau = L_z(s(t)). \tag{1}$$

Unless otherwise noted, all integrals are over the real line.

Convolution operators L of the type (1) have been heavily studied and are well understood. The Fourier transform diagonalizes the operator and the spectrum of the operator is the continuous Fourier transform of A_z, for any fixed distance z. A monochromatic signal $s(t)$ transmitted at a frequency w_k, will be exponentially absorbed according the real part of the Fourier transform $\hat{A}_z(w_k)$. Dispersion is described by the complex portion of $\hat{A}_z(w_k)$. Appropriate physics generally dictates that the absorption and dispersion are heavily tied to each other. If the signal is monochromatic, or consisting of just one frequency, the real part of $\hat{A}_z(w_k)$ will give its absorption and the complex part of $\hat{A}_z(w_k)$ will give its space-time-displacement or dispersion from the normal signal velocity. When the signal is not monochromatic, then the resulting signal $r(t, z) \equiv r_z(t)$ has a Fourier transform which is the product of $\hat{A}$, and $\hat{s}$, i.e. $\hat{r}_z(w) = \hat{A}_z(w)\hat{s}(w)$.

In Brillouin's words a "physicist is interested in the results (of standard single frequency analysis), but he immediately asks some indiscreet questions about waves in a dispersive medium, where the velocity of propagation is not a constant, but strongly depends upon the frequency. The well known differential equation (y' = -ky) is no longer satisfied and must be replaced by a more complicated systems of equations, which include the model, the physical mechanism,.."[3] etc. This distinction between a simple narrow-band formulation where the dispersion and absorption are constant, and a wide-band understanding is the key to understanding this phenomenon.

Slepian and Pollack utilized the finite length of signals to alter an operator of the type (1). This alteration creates a compact operator, with a corresponding discrete set of singular values and singular vectors as opposed to the continuous spectrum of L_z. The setting of the compact operator allows one to shift the concentration of study from amplitude of signals to the energy of signals. Following this development we will consider pulses of finite length l, which by assumption will be non-zero only on the interval $[0, l]$. The corresponding new operators L_z describe a finite pulse on $[0, l]$ evolving through a distance z of a medium. Formally we have

$$r(t, z) = \int_{-\infty}^{\infty} s(\tau)A_z(\tau - t)d\tau \equiv \int_{\infty}^{\infty} s(\tau)\mathcal{X}_l(\tau)A_z(t-\tau)d\tau = L_z(s(t)), \tag{2}$$

where $\mathcal{X}_l(\tau) = 1$ for $\tau \in [0, l]$, and is 0 otherwise. Thus our old kernel was $A_z(t)$ and our new kernel is $K_z(t, \tau) = \mathcal{X}_l(\tau)A_z(t-\tau)$. Note that if $A_z(t)$ is square integrable, then $K_z(t, \tau)$ will be square integrable in both variables. A basic result of functional analysis states that when a kernel of the type $K_z(t, \tau)$ is square integrable in both variables, the corresponding operator L_z is a compact operator. This is stated clearly in

THEOREM 2 (THE HILBERT-SCHMIDT THEOREM:[9]). *Let an operator L be defined by*

$$L(f)(t) = \int f(\tau)G(t, \tau)d\tau \tag{3}$$

and let $\|G(t, \tau)\|_2 < \infty$. Then L is a compact operator, and it follows that there exist singular vectors and singular values $\{u_k\}$, $\{v_k\}$, and $\{\sigma_k\}$ such that

$$L(f)(t) = \int f(\tau)G(t, \tau)d\tau = \sum_{k=0}^{\infty} \sigma_k \langle f, u_k \rangle v_k. \tag{4}$$

The values σ_k are called the singular values and the vectors $\{u_k\}$ and $\{v_k\}$ are correspondingly called the left and right singular vectors. In addition, we have $\sigma_k v_k = L(u_k)$, or that v_k is the image of u_k, with magnitude σ_k. Moreover, the energy of the singular values is exactly that of the kernel, or

$$\int\int |G(t,\tau)|^2 dt d\tau = \sum_{k=0}^{\infty} \sigma_k^2. \tag{5}$$

At this time, let us adopt some notation. We are dealing with a class of compact operators which deal with signals on $[0,l]$, and are indexed by the propagation distance z, so we refer to the kernels of these operators as K_z. Similarly we will refer to the corresponding singular values σ_k^z, where k is the index, and likewise the left singular vectors $u_k^z(t) \in L^2[0,l]$, and right singular vectors $v_k^z(t) \in L^2(R)$. Thus k runs from 0 to ∞, and the necessarily positive singular values decrease by convention. Thus the dominant left and right singular vectors are always u_0^z, and v_0^z. We will also refer to the transmission operator, without regard to the finite pulse length as L_z.

3. Signal or Pulse Processing

We will now explain how one can utilize these individual signals to create a joint resolution which is far from that expected of the individual signals. This allows one to equalize the signal to noise ratio along the spectrum. This will allow one to increase the resolution of the system beyond that which is expected from the dielectrics of the medium.

We begin with a basic theorem of Fourier Analysis.

Theorem 3. *Let $\{o_k(t)\}_k$ be any orthonormal basis for $L^2[a,b]$, and let $\{\hat{o}_k(w)\}_k$ be the respective Fourier transforms in $L^2(R)$. The quantity*

$$\sum_k |\hat{o}_k(w)|^2 \tag{6}$$

will be independent of w. Moreover we have a partition of unity, in the sense that

$$\frac{2\pi}{b-a} \sum_k |\hat{o}_k(w)|^2 = 1.$$

Thus the Legendre polynomials which we have shown to be the singular values of the medium saturate the bandwidth completely. We are thus utilizing the entire bandwidth, and can separate it and manipulate it as we please.

We will also be utilizing the expansion above (4). We will be transmitting but not all of the singular functions $\{u_k\}$ with our system. Thus let us look at a truncated expansion of the singular operator. This is represented by

$$L_N(f)(t) = \sum_{k=0}^{N} \sigma_k \langle f, u_k \rangle v_k. \tag{7}$$

Equation (7) recognizes that we will not be able to transmit an infinite number of the singular functions, but rather the select few, i.e. the ones which carry the most energy.

The question now is "What are we trying to accomplish?" That is simple. We want a well defined pulse response function through the medium in from of us. Thus we would like to have

$$L_N(f(t)) = pr(t).$$

That is not generally possible since we can only use a finite number of functions. Rather we will have to find the best possible approximation to the point response function, which is given by transmitting the pseudo inverse of $pr(t)$, which is given by

$$L_N^*(pr(t)) = U_N \Sigma^{-1} V_N^t (pr(t)).$$

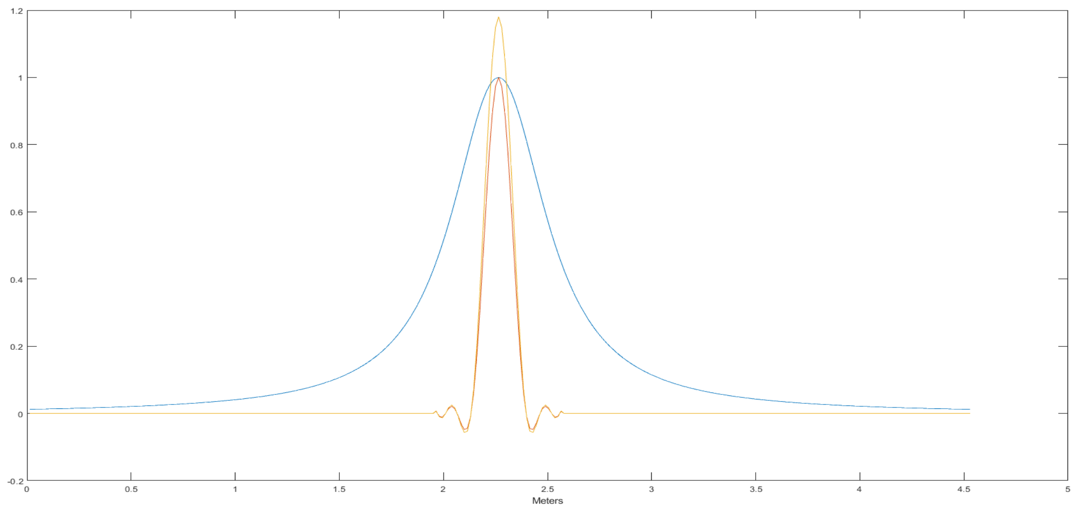

Figure 1: We illustrate the use of multiple precursors to sharpen the pulse response function of the system above. The blue curve is the point response function of a point target through an absorbing media (water in this case). The two other curves are the idealized point response function, and the point response function which is achieved by the use of an appropriate pseudoinverse.

We can rewrite this in operator notation as

$$L_N^*(pr(t)) = \sum_{k=0}^{N} \sigma_k^{-1} \langle v_k(t), pr(t) \rangle u_k(t). \tag{8}$$

The obvious problem with (8) is that σ_k^{-1} might be very large, since by necessity $\lim_{k\to\infty} \sigma_k = 0$. We will avoid this problem by taking multiple samples of the image of $u_k(t)$ or $\sigma_k v_k(t)$. This will allow us to equalize the signal to noise ratio and get a stable inverse.

One example of a pseudo inverse is demonstrated above. There are many possible versions of pseudo inverses, which will allow the sharpening of the point spread function. The final choice of which one should be used depends on the specific application.

In Figure 1 we illustrate how one can increase the resolution of the system substantially. The blue curve is the impulse response function of the system, which is equivalent to the first Brillouin precursor. The other curves represent the idolized inverse, or sharpened point spread function, and an approximate point spread function. The ability to exceed the temporal resolution of the simple Brillouin precursor is the basis and result of this research.

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