Lecture 3
The linear symplectic category.

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The purpose of today’s lecture is to describe the category whose objects are symplectic vector spaces and whose morphisms are the linear canonical relations. This provides a vast generalization of the (linear) symplectic group as we shall see. Everything in today's lecture reflects joint work with Victor Guillemin.

References for today:

1. The language of category theory,

2. Sets and relations.

3. Categorical “points”.

4. The linear symplectic category.
   - The transpose.
   - The transpose.

5. The category \textbf{LinSym} and the symplectic group.
The definition of a category.

A category $\mathcal{C}$ consists of the following data:

(i) A family, $\text{Ob}(\mathcal{C})$, whose elements are called the **objects** of $\mathcal{C}$,

(ii) For every pair $(X, Y)$ of $\text{Ob}(\mathcal{C})$ a family, $\text{Hom}_\mathcal{C}(X, Y)$, whose elements are called the **morphisms** or **arrows** from $X$ to $Y$,

(iii) For every triple $(X, Y, Z)$ of $\text{Ob}(\mathcal{C})$ a map from $\text{Hom}_\mathcal{C}(X, Y) \times \text{Hom}_\mathcal{C}(Y, Z)$ to $\text{Hom}_\mathcal{C}(X, Z)$ called the **composition map** and denoted $(f, g) \mapsto g \circ f$.

These data are subject to the following conditions:
(iv) The composition of morphisms is associative

(v) For each $X \in Ob(C)$ there is an $id_X \in \text{Hom}_C(X, X)$ such that

$$f \circ id_X = f, \ \forall f \in \text{Hom}_C(X, Y)$$

(for any $Y$) and

$$id_X \circ f = f, \ \forall f \in \text{Hom}_C(Y, X)$$

(for any $Y$).

It follows from the definitions that $id_X$ is unique.
Functors.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a **functor** $F$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

(vi) a map $F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$

and

(vii) for each pair $(X, Y)$ of $\text{Ob}(\mathcal{C})$ a map

$$F : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{D}(F(X), F(Y))$$

subject to the rules
(viii) \[ F(id_X) = id_{F(X)} \]

and

(ix) \[ F(g \circ f) = F(g) \circ F(f). \]

This is what is usually called a **covariant functor**.
Contravariant functors.

A **contravariant functor** would have

\[ F : \text{Hom}_C(X, Y) \to \text{Hom}_D(F(Y), F(X)) \text{ in (vii) and } F(f) \circ F(g) \]

on the right hand side of (ix).
Morphisms.

Let $F$ and $G$ be two functors from $C$ to $D$. A **morphism**, $m$, from $F$ to $G$ (older name: “natural transformation”) consists of the following data:

(xi) for any $f \in \text{Hom}_C(X, Y)$

$$m(Y) \circ F(f) = G(f) \circ m(X) \quad \forall f \in \text{Hom}_C(X, Y).$$

In other words, the diagram in the figure on the next slide commutes.
The language of category theory, Sets and relations. Categorical “points”. The linear symplectic category. The category $\text{LinSym}$ and the symplectic group.

Figure:

\[ F(X) \xrightarrow{m(X)} G(X) \]
\[ F(f) \]
\[ F(Y) \xrightarrow{m(Y)} G(Y) \]
\[ G(f) \]
Involutory functors.

Consider the category $\mathcal{V}$ whose objects are finite dimensional vector spaces (over some given field $\mathbb{K}$) and whose morphisms are linear transformations. We can consider the “transpose functor” $F : \mathcal{V} \to \mathcal{V}$ which assigns to every vector space $V$ its dual space

$$V^* = \text{Hom}(V, \mathbb{K})$$

and which assigns to every linear transformation $\ell : V \to W$ its transpose $\ell^* : W^* \to V^*$. In other words,

$$F(V) = V^*, \quad F(\ell) = \ell^*.$$

This is a contravariant functor which has the property that $F^2$ is naturally equivalent to the identity functor. There does not seem to be a standard name for this type of functor. We will call it an involutory functor.
Involution functor.

A special type of involutory functor is one in which $F(X) = X$ for all objects $X$ and $F^2 = \text{id}$ (not merely naturally equivalent to the identity). We shall call such a functor a involutive functor. We will refer to a category with an involutive functor as an involutive category, or say that we have a category with an involutive structure.

For example, let $\mathcal{H}$ denote the category whose objects are Hilbert spaces and whose morphisms are bounded linear transformations. We take $F(X) = X$ on objects and $F(L) = L^\dagger$ on bounded linear transformations where $L^\dagger$ denotes the adjoint of $L$ in the Hilbert space sense.
The category \textbf{Set} is the category whose objects are ("all") sets and and whose morphisms are ("all") maps between sets. For reasons of logic, the word "all" must be suitably restricted to avoid contradiction.

We will take the extreme step of restricting our attention to the class of finite sets. Our main point is to examine a category whose objects are finite sets, but whose morphisms are much more general than maps. Some of the arguments and constructions that we use in the study of this example will be models for arguments we will use later on, in the context of the symplectic "category".
The category of finite relations.

We will consider the category whose objects are finite sets. But we enlarge the set of morphisms by defining

\[ \text{Mor}(X, Y) = \text{the collection of all subsets of } X \times Y. \]

A subset of \( X \times Y \) is called a relation. We must describe the map

\[ \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z) \]

and show that this composition law satisfies the axioms of a category.
So let
\[ \Gamma_1 \in \text{Mor}(X, Y) \quad \text{and} \quad \Gamma_2 \in \text{Mor}(Y, Z). \]

Define
\[ \Gamma_2 \circ \Gamma_1 \subset X \times Z \]
by
\[(x, z) \in \Gamma_2 \circ \Gamma_1 \iff \exists y \in Y \text{ such that } (x, y) \in \Gamma_1 \text{ and } (y, z) \in \Gamma_2. \]

Notice that if \( f : X \to Y \) and \( g : Y \to Z \) are maps, then
\[ \text{graph}(f) = \{(x, f(x)) \in \text{Mor}(X, Y) \quad \text{and} \quad \text{graph}(g) \in \text{Mor}(Y, Z) \]
with
\[ \text{graph}(g) \circ \text{graph}(f) = \text{graph}(g \circ f). \]

So we have indeed enlarged the category of finite sets and maps.
We still must check the axioms.
\[ \Delta_X \text{ is } \text{id}_X, \text{ the identity.} \]

Let \( \Delta_X \subset X \times X \) denote the diagonal:

\[ \Delta_X = \{(x, x), \ x \in X\}, \]

so

\[ \Delta_X \in \text{Mor}(X, X). \]

If \( \Gamma \in \text{Mor}(X, Y) \) then

\[ \Gamma \circ \Delta_X = \Gamma \quad \text{and} \quad \Delta_Y \circ \Gamma = \Gamma. \]

So \( \Delta_X \) satisfies the conditions for \( id_X \).
The associative law.

Suppose that $\Gamma_1 \in \text{Mor}(X, Y)$, $\Gamma_2 \in \text{Mor}(Y, Z)$ and $\Gamma_3 \in \text{Mor}(Z, W)$. Then both $\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$ and $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1$ consist of all $(x, w) \in X \times W$ such that there exist $y \in Y$ and $z \in Z$ with

$$(x, y) \in \Gamma_1, \ (y, z) \in \Gamma_2, \ and \ (z, w) \in \Gamma_3.$$ 

This proves the associative law.

Let us call this category $\text{FinRel}$. 
Let us pick a distinguished one element set and call it “pt.”. Giving a *map* from pt. to any set $X$ is the same as picking a point of $X$. So in the category **Set** of sets and *maps*, the points of $X$ are the same as the morphisms from our distinguished object pt. to $X$. In a more general category, where the objects are not necessarily sets, we can not talk about the points of an object $X$. However if we have a distinguished object pt., then we can *define* a “point” of any object $X$ to be an element of $\text{Mor}(\text{pt.}, X)$. For example, later on, when we study the symplectic “category” whose objects are symplectic manifolds, we will find that the “points” in a symplectic manifold are its Lagrangian submanifolds. This idea has been emphasized by Weinstein. As he points out, this can be considered as a manifestation of the Heisenberg uncertainty principle in symplectic geometry.
In the category \textbf{FinRel}, the category of finite sets and relations, an element of $\text{Mor}(\text{pt.}, X)$, i.e a subset of $\text{pt.} \times X$, is the same as a subset of $X$ (by projection onto the second factor). So in this category, the “points” of $X$ are the subsets of $X$. Many of the constructions we do here can be considered as warm ups to similar constructions in the symplectic “category”.
Morphisms act on “points”.

Suppose we have a category with a distinguished object $pt$. A morphism $\Gamma \in \text{Mor}(X, Y)$ yields a map from “points” of $X$ to “points” of $Y$. Namely, a “point” of $X$ is an element $p \in \text{Mor}(pt., X)$ so if $f \in \text{Mor}(X, Y)$ we can form

$$f \circ p \in \text{Mor}(pt., Y)$$

which is a “point” of $Y$. So $f$ maps “points” of $X$ to “points” of $Y$.

We will sometimes use the more suggestive language $f(p)$ instead of $f \circ p$. 
Back to the category $\text{FinRel}$. 

Consider three objects $X, Y, Z$. Inside 

$$X \times X \times Y \times Y \times Z \times Z$$

we have the subset 

$$\Delta_X \times \Delta_Y \times \Delta_Z.$$ 

Let us move the first $X$ factor past the others until it lies to immediate left of the right $Z$ factor, so consider the subset 

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$ 

By introducing parentheses around the first four and last two factors we can write 

$$\tilde{\Delta}_{X,Y,Z} \subset (X \times Y \times Y \times Z) \times (X \times Z).$$
\[ \tilde{\Delta}_{X,Y,Z} \subset (X \times Y \times Y \times Z) \times (X \times Z). \]

In other words,

\[ \tilde{\Delta}_{X,Y,Z} \in \text{Mor}(X \times Y \times Y \times Z, X \times Z). \]

Let \( \Gamma_1 \in \text{Mor}(X, Y) \) and \( \Gamma_2 \in \text{Mor}(Y, Z) \). Then

\[ \Gamma_1 \times \Gamma_2 \subset X \times Y \times Y \times Z \]

is a “point” of \( X \times Y \times Y \times Z \). We identify this “point” with an element of \( \text{Mor}(\text{pt.}, X \times Y \times Y \times Z) \) so that we can form

\[ \tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2). \]
\[ \tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2) \] consists of all \((x, z)\) such that

\[ \exists (x_1, y_1, y_2, z_1, x, z) \text{ with } \]

\[
\begin{align*}
(x_1, y_1) & \in \Gamma_1, \\
(y_2, z_1) & \in \Gamma_2, \\
x_1 &= x, \\
y_1 &= y_2, \\
z_1 &= z.
\end{align*}
\]

Thus

\[ \tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1. \quad (2) \]
Similarly, given four sets $X, Y, Z, W$ we can form

$$\tilde{\Delta}_{X, Y, Z, W} \subset (X \times Y \times Y \times Z \times Z \times W) \times (X \times W)$$

$$\tilde{\Delta}_{X, Y, Z, W} = \{(x, y, y, z, z, w, x, w)\}$$

so

$$\tilde{\Delta}_{X, Y, Z, W} \in \text{Mor}(X \times Y \times Y \times Z \times Z \times W, X \times W).$$

If $\Gamma_1 \in \text{Mor}(X, Y), \Gamma_2 \in \text{Mor}(Y, Z), \text{ and } \Gamma_3 \in \text{Mor}(Z, W)$ then

$$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = (\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \tilde{\Delta}_{X, Y, Z, W}(\Gamma_1 \times \Gamma_2 \times \Gamma_3).$$
From this point of view the associative law is a reflection of the fact that

$$(\Gamma_1 \times \Gamma_2) \times \Gamma_3 = \Gamma_1 \times (\Gamma_2 \times \Gamma_3) = \Gamma_1 \times \Gamma_2 \times \Gamma_3.$$
The transpose on \textbf{FinRel}.

If $\Gamma \in \text{Mor}(X, Y)$ define $\Gamma^\dagger \in \text{Mor}(Y, X)$ by

$$
\Gamma^\dagger := \{(y, x) | (x, y) \in \Gamma\}.
$$

We have defined a map

$$
\dagger : \text{Mor}(X, Y) \to \text{Mor}(Y, X)
$$

for all objects $X$ and $Y$ which clearly satisfies

$$
\dagger^2 = \text{id}
$$

and

$$
(\Gamma_2 \circ \Gamma_1)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger.
$$

So $\dagger$ is a contravariant functor and satisfies our conditions for an involution. This makes our category \textbf{FinRel} of finite sets and relations into an involutive category.
The linear symplectic category.

We now come to the main point of today’s lecture:

Let $V_1$ and $V_2$ be symplectic vector spaces with symplectic forms $\omega_1$ and $\omega_2$. We will let $V_1^-$ denote the vector space $V_1$ equipped with the symplectic form $-\omega_1$. So $V_1^- \oplus V_2$ denotes the vector space $V_1 \oplus V_2$ equipped with the symplectic form $-\omega_1 \oplus \omega_2$.

A Lagrangian subspace $\Gamma$ of $V_1^- \oplus V_2$ is called a linear canonical relation. The main purpose of today’s lecture is to show that if we take the collection of symplectic vector spaces as objects, and the linear canonical relations as morphisms we get a category. Here composition is in the sense of composition of relations as in the category $\text{FinRel}$. 
In more detail: Let $V_3$ be a third symplectic vector space, let

$$\Gamma_1 \text{ be a Lagrangian subspace of } V_1^- \oplus V_2$$

and let

$$\Gamma_2 \text{ be a Lagrangian subspace of } V_2^- \oplus V_3.$$ 

Recall that as a set the composition

$$\Gamma_2 \circ \Gamma_1 \subset V_1 \times V_3$$

is defined by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \iff \exists y \in V_2 \text{ such that } (x, y) \in \Gamma_1 \text{ and } (y, z) \in \Gamma_2.$$ 

We must show that this is a Lagrangian subspace of $V_1^- \oplus V_3$. 
\[(x, z) \in \Gamma_2 \circ \Gamma_1 \iff \exists y \in V_2 \text{ such that } (x, y) \in \Gamma_1 \text{ and } (y, z) \in \Gamma_2.\]

We must show that this is a Lagrangian subspace of $V_1^- \oplus V_3$. It will be important for us to break up the definition of $\Gamma_2 \circ \Gamma_1$ into two steps:
The space $\Gamma_2 \ast \Gamma_1$.

Define $\Gamma_2 \ast \Gamma_1 \subset \Gamma_1 \times \Gamma_2$ to consist of all pairs $((x, y), (y', z))$ such that $y = y'$. We will restate this definition in two convenient ways. Let

$$\pi : \Gamma_1 \to V_2, \quad \pi(v_1, v_2) = v_2$$

and

$$\rho : \Gamma_2 \to V_2, \quad \rho(v_2, v_3) = v_2.$$ 

Let

$$\tau : \Gamma_1 \times \Gamma_2 \to V_2$$

be defined by

$$\tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

Then $\Gamma_2 \ast \Gamma_1$ is determined by the exact sequence

$$0 \to \Gamma_2 \ast \Gamma_1 \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \to \text{Coker } \tau \to 0.$$
Fiber products also known as exact squares.

Let $f : A \to C$ and $g : B \to C$ be maps, say between sets. Then we express the fact that $F \subset A \times B$ consists of those pairs $(a, b)$ such that $f(a) = g(b)$ by saying that

$$
\begin{array}{c}
F & \longrightarrow & A \\
\downarrow & & \downarrow f \\
B & \longrightarrow & C \\
& \underset{g}{\longrightarrow} &
\end{array}
$$

is an **exact square** or a **fiber product diagram**.
Thus another way of expressing the definition of $\Gamma_2 \star \Gamma_1$ is to say that

$$
\begin{array}{ccc}
\Gamma_2 \star \Gamma_1 & \longrightarrow & \Gamma_1 \\
\downarrow & & \downarrow \pi \\
\Gamma_2 & \longrightarrow & V_2 \\
\rho & \end{array}
$$

is an exact square.
The transpose.

If $\Gamma \subset V_1^- \oplus V_2$ is a linear canonical relation, we define its transpose $\Gamma^\dagger$ just as in \textbf{FinRel}:

$$\Gamma^\dagger := \{(y, x) | (x, y) \in \Gamma\}. \quad (9)$$

Here $x \in V_1$ and $y \in V_2$ so $\Gamma^\dagger$ as defined is a linear Lagrangian subspace of $V_2 \oplus V_1^-$. But replacing the symplectic form by its negative does not change the set of Lagrangian subspaces, so $\Gamma^\dagger$ is also a Lagrangian subspace of $V_2^- \oplus V_1$, i.e. a linear canonical relation between $V_2$ and $V_1$. It is also obvious that just as in \textbf{FinRel} we have

$$\left(\Gamma^\dagger\right)^\dagger = \Gamma.$$
The projection $\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$.

Consider the map

$$\alpha : (x, y, y, z) \mapsto (x, z).$$  \hspace{1cm} (10)

By definition

$$\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1.$$
Let $V_1$ and $V_2$ be symplectic vector spaces and let $\Gamma \subset V_1^- \times V_2$ be a linear canonical relation. Let

$$\pi : \Gamma \to V_2$$

be the projection onto the second factor.

Define

- $\text{Ker } \Gamma \subset V_1$ by $\text{Ker } \Gamma := \{v \in V_1 | (v, 0) \in \Gamma\}$.
- $\text{Im } \Gamma \subset V_2$ by
  $$\text{Im } \Gamma := \pi(\Gamma) = \{v_2 \in V_2 | \exists v_1 \in V_1 \text{ with } (v_1, v_2) \in \Gamma\}.$$
Now $\Gamma^\dagger \subset V_2^- \oplus V_1$ and hence both $\ker \Gamma^\dagger$ and $\text{Im} \; \Gamma$ are linear subspaces of the symplectic vector space $V_2$. We claim that

$$(\ker \Gamma^\dagger)^\perp = \text{Im} \; \Gamma.$$  \hspace{2cm} (11)

Here $\perp$ means perpendicular relative to the symplectic structure on $V_2$.

**Proof.**

Let $\omega_1$ and $\omega_2$ be the symplectic bilinear forms on $V_1$ and $V_2$ so that $\tilde{\omega} = -\omega_1 \oplus \omega_2$ is the symplectic form on $V_1^- \oplus V_2$. So $\nu \in V_2$ is in $\text{Ker} \; \Gamma^\dagger$ if and only if $(0, \nu) \in \Gamma$. Since $\Gamma$ is Lagrangian, $(0, \nu) \in \Gamma \Leftrightarrow (0, \nu) \in \Gamma^\perp$ and

$$(0, \nu) \in \Gamma^\perp \Leftrightarrow 0 = -\omega_1(0, \nu_1) + \omega_2(\nu, \nu_2) = \omega_2(\nu, \nu_2) \; \forall (\nu_1, \nu_2) \in \Gamma.$$

But this is precisely the condition that $\nu \in (\text{Im} \; \Gamma)^\perp$. \hfill \square
The kernel of $\alpha$ consists of those $(0, \nu, \nu, 0) \in \Gamma_2 \star \Gamma_1$. We may thus identify

$$\ker \alpha = \ker \Gamma_1^\dagger \cap \ker \Gamma_2$$

(12)

as a subspace of $V_2$. 

\vspace{20pt}
If we go back to the definition of the map $\tau$, we see that the image of $\tau$ is given by

$$\text{Im}\tau = \text{Im}\Gamma_1 + \text{Im}\Gamma_2^\dag,$$  \hspace{1cm} (13)

a subspace of $V_2$. If we compare (12) with (13) we see that

$$\ker\alpha = (\text{Im}\tau)^\perp$$ \hspace{1cm} (14)

as subspaces of $V_2$ where $\perp$ denotes orthocomplement relative to the symplectic form $\omega_2$ of $V_2$. 
Proof that $\Gamma_2 \circ \Gamma_1$ is Lagrangian.

Since $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$ and $\Gamma_2 \star \Gamma_1 = \ker \tau$ it follows that $\Gamma_2 \circ \Gamma_1$ is a linear subspace of $V_1^- \oplus V_3$.

It is equally easy to see that $\Gamma_2 \circ \Gamma_1$ is an isotropic subspace of $V_1^- \oplus V_3$. Indeed, if $(x, z)$ and $(x', z')$ are elements of $\Gamma_2 \circ \Gamma_1$, then there are elements $y$ and $y'$ of $V_2$ such that

$$(x, y) \in \Gamma_1, \ (y, z) \in \Gamma_2, \ (x', y') \in \Gamma_1, \ (y', z') \in \Gamma_2.$$

Then

$$\omega_3(z, z') - \omega_1(x, x') = \omega_3(z, z') - \omega_2(y, y') + \omega_2(y, y') - \omega_1(x, x') = 0.$$ 

So we must show that $\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$. 
We must show that \( \dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3 \). It follows from (14) that

\[
\dim \ker \alpha = \dim V_2 - \dim \Im \tau
\]

and from the fact that \( \Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1) \) that

\[
\dim \Gamma_2 \circ \Gamma_1 = \dim \Gamma_2 \star \Gamma_1 - \dim \ker \alpha = \\
= \dim \Gamma_2 \star \Gamma_1 - \dim V_2 + \dim \Im \tau.
\]
dim \Gamma_2 \circ \Gamma_1 = \dim \Gamma_2 \star \Gamma_1 - \dim V_2 + \dim \text{Im } \tau.

Since \Gamma_2 \star \Gamma_1 is the kernel of the map \tau : \Gamma_1 \times \Gamma_2 \to V_2 it follows that
\dim \Gamma_2 \star \Gamma_1 = \dim \Gamma_1 \times \Gamma_2 - \dim \text{Im } \tau =
\frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_2 + \frac{1}{2} \dim V_2 + \frac{1}{2} \dim V_3 - \dim \text{Im } \tau.

Putting these two equations together we see that
\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3

as desired. We have thus proved

**Theorem**

The composite \Gamma_2 \circ \Gamma_1 of two linear canonical relations is a linear canonical relation.
The associative law.

The associative law can be proved exactly as for FinRel: given four symplectic vector spaces $X, Y, Z, W$ we can form

$$\tilde{\Delta}_{X,Y,Z,W} \subset [(X^{-} \times Y) \times (Y^{-} \times Z) \times (Z^{-} \times W)]^{-} \times (X^{-} \times W)$$

$$\tilde{\Delta}_{X,Y,Z,W} = \{(x, y, y, z, z, w, x, w)\}.$$

It is immediate to check that $\tilde{\Delta}_{X,Y,Z,W}$ is a Lagrangian subspace, so

$$\tilde{\Delta}_{X,Y,Z,W} \in \text{Mor}((X^{-} \times Y) \times (Y^{-} \times Z) \times (Z^{-} \times W), X^{-} \times W).$$
If $\Gamma_1 \in \text{Mor}(X, Y)$, $\Gamma_2 \in \text{Mor}(Y, Z)$, and $\Gamma_3 \in \text{Mor}(Z, W)$ then

$$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = (\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \tilde{\Delta}_{X,Y,Z,W}(\Gamma_1 \times \Gamma_2 \times \Gamma_3),$$

as before. From this point of view the associative law is again a reflection of the fact that

$$(\Gamma_1 \times \Gamma_2) \times \Gamma_3 = \Gamma_1 \times (\Gamma_2 \times \Gamma_3) = \Gamma_1 \times \Gamma_2 \times \Gamma_3.$$

The diagonal $\Delta_V$ gives the identity morphism and so we have verified that

**Theorem**

LinSym is a category whose objects are symplectic vector spaces and whose morphisms are linear canonical relations.
The category **LinSym** is a vast generalization of the symplectic group because of the following observation: Let $X$ and $Y$ be symplectic vector spaces. Suppose that the Lagrangian subspace $\Gamma \subset X^- \oplus Y$ projects bijectively onto $X$ under the projection of $X \oplus Y$ onto the first factor. This means that $\Gamma$ is the graph of a linear transformation $T$ from $X$ to $Y$:

$$\Gamma = \{(x, Tx)\}.$$ 

$T$ must be injective. Indeed, if $Tx = 0$ the fact that $\Gamma$ is isotropic implies that $x \perp X$ so $x = 0$. 
Also \( T \) is surjective since if \( y \perp \text{im}(T) \), then \( (0, y) \perp \Gamma \). This implies that \( (0, y) \in \Gamma \) since \( \Gamma \) is maximal isotropic. By the bijectivity of the projection of \( \Gamma \) onto \( X \), this implies that \( y = 0 \). In other words \( T \) is a bijection. The fact that \( \Gamma \) is isotropic then says that

\[
\omega_Y(Tx_1, Tx_2) = \omega_X(x_1, x_2),
\]

i.e. \( T \) is a symplectic isomorphism. If \( \Gamma_1 = \text{graph } T \) and \( \Gamma_2 = \text{graph } S \) then

\[
\Gamma_2 \circ \Gamma_1 = \text{graph } S \circ T
\]

so composition of Lagrangian relations reduces to composition of symplectic isomorphisms in the case of graphs. In particular, if we take \( Y = X \) we see that \( \text{Symp}(X) \) is a subgroup of \( \text{Morph } (X, X) \) in our category.