Lecture 4

Lagrangian subspaces of a fixed symplectic vector space.
More on Hamilton’s method.

Shlomo Sternberg

July 14, 2010
In today’s lecture I want to study the space $\mathcal{L}(V)$ of Lagrangian subspaces of a fixed symplectic vector space $V$, and the action of $Sp(V)$ of $\mathcal{L}(V)$.

I also want to give some more illustrations of Hamilton’s method of what he call “characteristics” and what are now known as “generating functions”.
1. Lagrangians transversal to a finite number of Lagrangians.
   - Parametrizing Lagrangians transversal to a given Lagrangian.
   - Description in terms of a basis.

2. The action of $Sp(V)$ on $\mathcal{L}(V)$.

3. Generating functions - a simple example of Hamilton's idea.
   - Composition of symplectic transformations and addition of generating functions.
Let \( V \) be a symplectic vector space.

**Proposition**

*Given any finite collection of Lagrangian subspaces \( M_1, \ldots, M_k \) of \( V \) one can find a Lagrangian subspace \( L \) such that*

\[
L \cap M_j = \{0\}, \quad i = 1, \ldots k.
\]

In words, given a finite collection of Lagrangian subspaces, we can find a Lagrangian subspace which is transversal to all of them.
Proof.

We can always find an isotropic subspace $L$ with $L \cap M_j = \{0\}$, $i = 1, \ldots, k$, for example a line which does not belong to any of these subspaces. Suppose that $L$ is an isotropic subspace with $L \cap M_j = \{0\}$, $\forall j$ and is not properly contained in a larger isotropic subspace with this property. We claim that $L$ is Lagrangian. Indeed, if not, $L^\perp$ is a coisotropic subspace which strictly contains $L$. Let $\pi : L^\perp \rightarrow L^\perp/L$ be the quotient map. Each of the spaces $\pi(L^\perp \cap M_j)$ is an isotropic subspace of the symplectic vector space $L^\perp/L$ and so each of these spaces has positive codimension. So we can choose a line $\ell$ in $L^\perp/L$ which does not intersect any of the $\pi(L^\perp \cap M_j)$. Then $L' := \pi^{-1}(\ell)$ is an isotropic subspace of $L^\perp \subset V$ with $L' \cap M_j = \{0\}$, $\forall j$ and strictly containing $L$, a contradiction.
Let $V = (V, \omega)$ be a symplectic vector space of dimension $2d$. We let $\mathcal{L}(V)$ denote the space of all Lagrangian subspaces of $V$. It is called the **Lagrangian Grassmannian**.

If $M \in \mathcal{L}(V)$ is a fixed Lagrangian subspace, we let $\mathcal{L}(V, M)$ denote the subset of $\mathcal{L}(V)$ consisting of those Lagrangian subspaces which are transversal to $M$. 

The Lagrangian Grassmanian.
Let \( L \in \mathcal{L}(V, M) \) be one such subspace. The non-degenerate pairing between \( L \) and \( M \) identifies \( M \) with the dual space \( L^* \) of \( L \) and \( L \) with the dual space \( M^* \) of \( M \). The vector space decomposition

\[
V = M \oplus L = M \oplus M^*
\]

tells us that any \( N \in \mathcal{L}(V, M) \) projects bijectively onto \( L \) under this decomposition. In particular, this means that \( N \) is the graph of a linear map

\[
T_N : L \to M = L^*.
\]

So

\[
N = \{(T_N \xi, \xi), \quad \xi \in L = M^*\}.
\]
N = \{(T_N \xi, \xi), \, \, \xi \in L = M^*\}.

Giving a map from a vector space to its dual is the same as giving a bilinear form on the original vector space. In other words, $N \in \mathcal{L}(V, M)$ determines, and is determined by, the bilinear form $\beta_N$ on $L = M^*$ where

$$\beta_N(\xi, \xi') = \frac{1}{2} \langle T_N \xi', \xi \rangle = \frac{1}{2} \omega(\xi, T_N \xi').$$

This is true for any $d$-dimensional subspace transversal to $M$. What is the condition on $\beta_N$ for $N$ to be Lagrangian?
Well, if \( w = (T_N\xi, \xi) \) and \( w' = (T_N\xi', \xi') \) are two elements of \( N \) then

\[
\omega(w, w') = \omega(T_N\xi, \xi') - \omega(T_N\xi', \xi)
\]

since \( L \) and \( M \) are Lagrangian. So the condition for \( N \) to be Lagrangian is that \( \beta_N \) be symmetric. We have proved:
Proposition

If $M \in \mathcal{L}(V)$ and we choose $L \in \mathcal{L}(V, M)$ then we get an identification of $\mathcal{L}(V, M)$ with $S^2(L)$, the space of symmetric bilinear forms on $L$.

So every choice of a pair of transverse Lagrangian subspaces $L$ and $M$ gives a coordinate chart on $\mathcal{L}(V)$ which is identified with $S^2(L)$. In particular, $\mathcal{L}(V)$ is a smooth manifold and we get another proof of the fact that

$$\dim \mathcal{L}(V) = \frac{d(d + 1)}{2}$$

where $d = \frac{1}{2} \dim V$. 

Shlomo Sternberg
Lecture 4
Description in terms of a basis.

Suppose that we choose a basis $e_1, \ldots, e_d$ of $L$ and so get a dual basis $f_1, \ldots, f_d$ of $M$. If $N \in \mathcal{L}(V, M)$ then we get a basis $g_1, \ldots, g_d$ of $N$ where

$$g_i = e_i + \sum_j S_{ij} f_j$$

where

$$S_{ij} = 2\beta_N(e_i, e_j).$$

We record the following fact: Let $N$ and $N'$ be two elements of $\mathcal{L}(V, M)$. The symplectic form $\omega$ induces a (possibly singular) bilinear form on $N \times N'$. In terms of the bases given above for $N$ and $N'$ we have

$$\omega(g_i, g_j') = S'_{ij} - S_{ij}. \quad (1)$$
$Sp(V)$ acts transitively on transverse pairs of elements of $\mathcal{L}(V)$.

We already know that $Sp(V)$ acts transitively on $\mathcal{L}(V)$.

Suppose that $L_1$ and $L_2$ are elements of $\mathcal{L}(V)$. An obvious invariant is the dimension of their intersection. Suppose that they are transverse, i.e. that $L_1 \cap L_2 = \{0\}$. We have seen that a basis $e_1, \ldots, e_d$ of $L_1$ determines a (dual) basis $f_1, \ldots, f_d$ of $L_2$ and together $e_1, \ldots, e_d, f_1, \ldots, f_d$ form a symplectic basis of $V$. Since $Sp(V)$ acts transitively on the set of symplectic bases, we see that it acts transitively on the space of pairs of transverse Lagrangian subspaces.
Sp(V) does not act transitively on transverse triples of elements of \( \mathcal{L}(V) \).

We can see this already in the plane: Every line through the origin is a Lagrangian subspace. If we fix two lines, the set of lines transverse to both is divided into two components corresponding to the two pairs of opposite cones complementary to the first two lines. See the figure on the next slide:
Lagrangians transversal to a finite number of Lagrangians.

The action of $Sp(V)$ on $\mathcal{L}(V)$.

Generating functions - a simple example of Hamilton's idea.
We can see this more analytically as follows: By an application of $SL(2, \mathbb{R}) = Sp(\mathbb{R}^2)$ we can arrange that $L_1$ is the $x$-axis and $L_2$ is the $y$-axis. The subgroup of $SL(2, \mathbb{R})$ which preserves both axes consists of the diagonal matrices (with determinant one), i.e. of all matrices of the form

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}.
$$

If $\lambda > 0$ such a matrix preserves all quadrants, while if $\lambda < 0$ such a matrix interchanges the first and third and the second and fourth quadrants.

In any event, such a matrix carries a line passing through the first and third quadrant into another such line and the group of such matrices acts transitively on the set of all such lines. Similarly for lines passing through the second and fourth quadrant.
The situation depicted in the figure above has a 2d-dimensional analogue: Let $M_1$ and $M_2$ be Lagrangian subspaces of a symplectic vector space $V$. For the moment we will assume that they are transverse to each other, i.e., $M_1 \cap M_2 = \{0\}$. Let

$$\mathcal{L}(V, M_1, M_2) = \mathcal{L}(V, M_1) \cap \mathcal{L}(V, M_2)$$

be the set of Lagrangian subspaces, $L$ of $V$ which are transverse both to $M_1$ and to $M_2$. Since $M_1$ and $M_2$ are transverse, $V = M_1 \oplus M_2$, so $L$ is the graph of a bijective mapping: $T_L : M_1 \rightarrow M_2$, and, as we saw in the preceding section, this mapping defines a bilinear form, $\beta_L \in S^2(M_1)$ by the recipe

$$\beta_L(v, w) = \frac{1}{2} \omega(v, T_L w).$$
\[ \beta_L(v, w) = \frac{1}{2} \omega(v, T_L w). \]

Moreover since \( T_L \) is bijective this bilinear form is non-degenerate. Thus, denoting by \( S^2(M_1)_{\text{non-deg}} \) the set of non-degenerate symmetric bilinear forms on \( M_1 \), the bijective map

\[ \mathcal{L}(V, M_1) \rightarrow S^2(M_1) \]

gives, by restriction, a bijective map

\[ \mathcal{L}(V, M_1, M_2) \rightarrow S^2(M_1)_{\text{non-deg}}. \] (2)
We have a bijective map \( \mathcal{L}(V, M_1, M_2) \rightarrow S^2(M_1)_{\text{non-deg}} \) and so the connected components of \( S^2(M_1)_{\text{non-sing}} \) are characterized by the signature invariant

\[
\beta \in S^2(M_1)_{\text{non-sing}} \rightarrow \text{sgn} \beta,
\]

so, via the identification (2) the same is true of \( \mathcal{L}(V, M_1, M_2) \): its connected components are characterized by the invariant \( L \rightarrow \text{sgn} \beta_L \). For instance in the two-dimensional case depicted in the figure above, \( \text{sgn} \beta_L \) is equal to 1 on one of the two components of \( \mathcal{L}(V, M_1, M_2) \) and \(-1\) on the other. Let

\[
\sigma(M_1, M_2, L) =: \text{sgn} \beta_L
\]

(3)

This is by definition a *symplectic invariant* of the triple, \( M_1, M_2, L \), so this shows that just as in two dimensions the group \( Sp(V) \) does *not* act transitively on triples of mutually transversal Lagrangian subspaces.
Explicit computation of $\text{sgn} \beta_L$.

We now describe how to compute this invariant explicitly in some special cases. Let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be a system of Darboux coordinates on $V$ such that $M_1$ and $M_2$ are the spaces, $\xi = 0$ and $x = 0$. Then $L$ is the graph of a bijective linear map $\xi = Bx$ with $B^\dagger = B$ and hence

$$\sigma(M_1, M_2, L) = \text{sgn}(B).$$  \hfill (4)

Next we consider a slightly more complicated scenario:
Let $M_2$ be, as above, the space, $x = 0$, but let $M_1$ be a Lagrangian subspace of $V$ which is transverse to $\xi = 0$ and $x = 0$, i.e., a space of the form $x = A\xi$ where $A^\dagger = A$ and $A$ is non-singular. In this case the symplectomorphism

$$(x, \xi) \rightarrow x, \xi - A^{-1}x$$

maps $M_1$ onto $\xi = 0$ and maps the space $L : \xi = Bx$

onto the space $L_1 : \xi = (B - A^{-1})x$.

and hence by the previous computation

$$\sigma(M_1, M_2, L) = \text{sgn}(B - A^{-1})$$.
\[ \sigma(M_1, M_2, L) = \text{sgn}(B - A^{-1}). \]

Notice however that the matrix

\[ \begin{bmatrix} A & I \\ I & B \end{bmatrix} \]

can be written as the product

\[ \begin{bmatrix} I & 0 \\ A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B - A^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ A^{-1} & I \end{bmatrix}^\dagger \]

so

\[ \text{sgn} A + \text{sgn}(B - A^{-1}) = \text{sgn} \begin{bmatrix} A & I \\ I & B \end{bmatrix}. \]

Hence

\[ \sigma(M_1, M_2, L) = \text{sgn} \begin{bmatrix} A & I \\ I & B \end{bmatrix} - \text{sgn} A. \]
\[ \sigma(M_1, M_2, L) = \text{sgn} \begin{bmatrix} A & I \\ I & B \end{bmatrix} - \text{sgn} A. \]

In particular if \( L_1 \) and \( L_2 \) are Lagrangian subspaces of \( V \) which are transverse to \( M_1 \) and \( M_2 \) the difference,

\[ \sigma(M_1, M_2, L_1) - \sigma(M_1, M_2, L_2) \]

is equal to

\[ \text{sgn} \begin{bmatrix} A & I \\ I & B_1 \end{bmatrix} - \text{sgn} \begin{bmatrix} A & I \\ I & B_0 \end{bmatrix}. \]
In other words the quantity
\[ \sigma(M_1, M_2, L_1, L_2) = \sigma(M_1, M_2, L_1) - \sigma(M_1, M_2, L_2) \]
is a symplectic invariant of \( M_1, M_2, L_1, L_2 \) which satisfies
\[ \sigma(M_1, M_2, L_2, L_2) = \text{sgn} \left[ \begin{array}{cc} A & I \\ I & B_1 \end{array} \right] - \text{sgn} \left[ \begin{array}{cc} A & I \\ I & B_2 \end{array} \right]. \] (8)

In the derivation of this identity we’ve assumed that \( M_1 \) and \( M_2 \) are transverse, however, the right hand side is well-defined provided the matrices
\[ \left[ \begin{array}{cc} A & I \\ I & B_i \end{array} \right] \quad i = 1, 2 \]
are non-singular, i.e., provided that \( L_1 \) and \( L_2 \) are transverse to \( M_1 \). Hence to summarize, we’ve proved:
Theorem

Given Lagrangian subspaces $M_1, M_2, L_1, L_2$ of $V$ such that the $L_i$’s are transverse to the $M_i$’s the formula

$$
\sigma(M_1, M_2, L_2, L_2) = \text{sgn} \begin{bmatrix} A & I \\ I & B_1 \end{bmatrix} - \text{sgn} \begin{bmatrix} A & I \\ I & B_2 \end{bmatrix}
$$

defines a symplectic invariant $\sigma(M_1, M_2, L_1, L_2)$ of $M_1, M_2, L_1, L_2$.

If $M_1$ and $M_2$ are transverse

$$
\sigma(M_1, M_2, L_1, L_2) = \sigma(M_1, M_2, L_1) - \sigma(M_1, M_2, L_2).
$$

(9)
Let us go back to the situation described above where we have a vector space $V = M \oplus M^* (= T^* M)$ and we have a Lagrangian subspace $N \subset V$ which is transversal to $M$. This determines a linear map $T_N : M^* \rightarrow M$ and a symmetric bilinear form $\beta_N$ on $M^*$. Suppose that we choose a basis of $M$ and so identify $M$ with $\mathbb{R}^n$ and so $M^*$ with $\mathbb{R}^{n*}$. Then $T = T_N$ becomes a symmetric matrix and if we define

$$\gamma_N(\xi) := \beta_N(\xi, \xi) = \frac{1}{2} T \xi \cdot \xi$$

then

$$T \xi = T_N \xi = \frac{\partial \gamma_N}{\partial \xi}.$$
Consider the function $\phi = \phi_N$ on $M \oplus M^*$ given by
\[
\phi(x, \xi) = x \cdot \xi - \gamma_N(\xi), \quad x \in M, \; \xi \in M^*.
\] (10)

Then the equation
\[
\frac{\partial \phi}{\partial \xi} = 0
\] (11)

is equivalent to $x = T_N \xi$. Of course, we have
\[
\xi = \frac{\partial \phi}{\partial x}
\]

and at points where (11) holds, we have
\[
\frac{\partial \phi}{\partial x} = d\phi,
\]

the total derivative of $\phi$ in the obvious notation.
Proposition

Let $M$ be a vector space and $V = M \oplus M^*$ with its standard symplectic structure. Let $N$ be a Lagrangian subspace of $M \oplus M^*$ which is transversal to $M$. Then

$$N = \{(x, d\phi(x, \xi))\}$$

where $\phi$ is the function on $M \times M^*$ given by

$$\phi(x, \xi) = x \cdot \xi - \gamma_N(\xi), \quad x \in M, \; \xi \in M^* \quad (10)$$

and where $(x, \xi)$ satisfies

$$\frac{\partial \phi}{\partial \xi} = 0. \quad (11)$$
Subspaces transversal to $M^*$.  

Suppose that $L$ is a Lagrangian subspace of $M \oplus M^*$ which is transversal to $M^*$ and hence projects isomorphically onto $M$ and so is the graph of a linear map from $M \to M^*$ which corresponds therefore to a quadratic function $\phi$ on $M$ so that  

$$L = \{(x, d\phi(x))\}.$$
We can consider a common generalization of these examples:
Suppose that $S$ is some auxiliary space and that $\phi$ is a function on $M \times S$ with the property that the Lagrangian subspace $L$ is given as

$$L = \{(x, d\phi(x, s))\}$$

where $(x, s)$ is constrained to satisfy

$$\frac{\partial \phi}{\partial s} = 0. \quad (*)$$

We then say that $\phi$ is a **generating function** for $L$. In our first example $S = M^*$ and the constraint $(*)$ was (11). In our second example $S$ was a point and the constraint $(*)$ was empty.
We now recall some notation from Lecture 3. Let $V = (V, \omega)$ be a symplectic vector space. Recall that we let $V^{-} = (V, -\omega)$. In other words, $V$ is the same vector space as $V$ but with the symplectic form $-\omega$.

We saw that if $T \in Sp(V)$, then its graph

$\Gamma := \text{graph } T = \{(v, Tv), \ v \in V\}$ is a Lagrangian subspace of $V^{-} \oplus V$. 
Suppose that $V = X \oplus X^*$ where $X$ is a vector space and where $V$ is given the usual symplectic form:

$$\omega\left(\left(\begin{array}{c} x \\ \xi \end{array}\right), \left(\begin{array}{c} x' \\ \xi' \end{array}\right)\right) = \langle \xi', x \rangle - \langle \xi, x' \rangle.$$

The map $\varsigma: V \to V$

$$\varsigma\left(\left(\begin{array}{c} x \\ \xi \end{array}\right)\right) = \left(\begin{array}{c} x \\ -\xi \end{array}\right)$$

is a symplectic isomorphism of $V$ with $V^-$. So $\varsigma \oplus \text{id}$ gives a symplectic isomorphism of $V^- \oplus V$ with $V \oplus V$.

A generating function for $(\iota \oplus \text{id})(\Gamma)$ will also (by abuse of language) be called a generating function for $\Gamma$ or for $T$. 
We consider the simplest case - the case of Gaussian optics that we studied in Lecture 1: We take $X = \mathbb{R}$. Then

$$V \oplus V = \mathbb{R} \oplus \mathbb{R}^* \oplus \mathbb{R} \oplus \mathbb{R}^* = T^*(\mathbb{R} \oplus \mathbb{R}).$$

Let $(x, y)$ be coordinates on $\mathbb{R} \oplus \mathbb{R}$ and consider a generating function (of our second type) of the form

$$\phi(x, y) = \frac{1}{2}(ax^2 + 2bxy + cy^2),$$

where

$$b \neq 0.$$

Taking into account the transformation $\varsigma$, the corresponding Lagrangian subspace of $V^- \oplus V$ is given by the equations

$$\xi = -(ax + by), \quad \eta = bx + cy.$$
The Lagrangian subspace of $V^- \oplus V$ is given by the equations

$$\xi = -(ax + by), \quad \eta = bx + cy.$$ 

Solving these equations for $y, \eta$ in terms of $x, \xi$ gives

$$y = -\frac{1}{b}(ax + \xi), \quad \eta = \left(b - \frac{ca}{b}\right)x - \frac{c}{b}\xi.$$ 

In other words, the matrix (of) $T$ is given by

$$
\begin{pmatrix}
-\frac{a}{b} & -\frac{1}{b} \\
\frac{b - ca}{b} & -\frac{c}{b}
\end{pmatrix}.
$$

(Notice that by inspection the determinant of this matrix is 1, which is that condition that $T$ be symplectic.)

Notice also that the upper right hand corner of this matrix is not zero.
Conversely, starting with a matrix

\[ T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

of determinant one, with \( \beta \neq 0 \) we can solve the equation

\[
\begin{pmatrix}
-\frac{a}{b} & -\frac{1}{b} \\
-b - \frac{ca}{b} & -\frac{c}{b}
\end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

for \( a, b, c \) in terms of \( \alpha, \beta, \gamma, \delta \). So the most general two by two matrix of determinant one with the upper right hand corner \( \neq 0 \) is represented by a generating function of the above form.
Suppose we have two functions

\[ \phi_1(x, y) = \frac{1}{2}[ax^2 + 2bxy + cy^2], \quad \phi_2(y, z) = \frac{1}{2}[Ay^2 + 2Byz + Cz^2], \]

with \( b \neq 0 \) and \( B \neq 0 \), and consider their sum:

\[ \phi(x, z, y) = \phi_1(x, y) + \phi_2(y, z). \]

Here we are considering \( y \) as an “auxiliary variable” in the sense of our general definition, i.e. as parametrizing our space \( S \).
So we want to impose the constraint

$$\frac{\partial \phi}{\partial y} = 0,$$  \hspace{2cm} (12)

and on this constrained set let

$$\xi = -\frac{\partial \phi}{\partial x}, \quad \zeta = \frac{\partial \phi}{\partial z},$$  \hspace{2cm} (13)

and use these equations to express \((z \zeta)\) in terms of \((x \xi)\).
Our constraint equation (12) gives

\[(A + c)y + bx + Bz = 0.\] (14)

There are now two alternatives:

1. If \(A + c \neq 0\) we can solve (14) for \(y\) in terms of \(x\) and \(z\). This then gives a generating function of the above type (i.e. quadratic in \(x\) and \(z\)). It is easy to check that the matrix obtained from this generating function is indeed the product of the corresponding matrices. This is an illustration of Hamilton’s principle that the composition of two symplectic transformations is given by the sum of their generating functions. Notice also that because

\[\frac{\partial^2 \phi}{\partial y^2} = A + c \neq 0,\]

the effect of (12) was to allow us to eliminate \(y\).

By the way, this shows the important principle that the same Lagrangian subspace can be given by two quite different generating functions.
2.

If $A + c = 0$, then (14) imposes no condition on $y$ but does give $bx + Bz = 0$, i.e.

$$z = -\frac{b}{B}x$$

which means precisely that the upper right hand corner of the corresponding matrix vanishes. Since $y$ is now a “free variable”, and $b \neq 0$ we can solve the first of equations (13) for $y$ in terms of $x$ and $\xi$ giving

$$y = -\frac{1}{b}(\xi + ax)$$

and substitute this into the second of the equations (13) to solve for $\zeta$ in terms of $x$ and $\xi$. 

Shlomo Sternberg 
Lecture 4
We see that the corresponding matrix is

\[
\left(\begin{array}{ccc}
-\frac{b}{B} & 0 \\
-aB & -\frac{Cb}{B} & -\frac{B}{b}
\end{array}\right).
\]

Again, this is indeed the product of the corresponding matrices.