Lecture 8
Reduction.

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1. The Frobenius theorem.
   - Differential systems.
   - Foliations, submersions, and fibrations.

2. Reduction of a closed form.

3. Horizontal and basic forms of a fibration.
   - Reduction of a co-isotropic immersion.
   - Reduction and Poisson brackets.
Let $X$ be a vector field on manifold $M$. The fundamental existence theorem of ordinary differential equations tells us, at least locally, that $X$ determines a flow, i.e. a locally defined one parameter group. Through each $p \in M$ there is a curve $t \mapsto \phi_t p$, the “trajectory through the point $p$”. If we replace $X$ by $fX$ where $f$ is a non-vanishing smooth function, the new flow has the “same” trajectories, the change is in the “speed” along each trajectory. Thus the curves, up to reparametrization, are determined by a “field of lines” (if $X$ is nowhere zero) rather than a vector field.
Differential systems also known as distributions.

We can study a more general situation: a “field of $k$-dimensional subspaces”. We assume that we are given at each point a $k$-dimensional subspace of the tangent space, and that this subspace varies smoothly from point to point. Such a family will be denoted by $\mathcal{D}$. We say that $\mathcal{D}$ is a “sub-bundle of the tangent bundle” or we say that $\mathcal{D}$ is a differential system or we say that $\mathcal{D}$ is a distribution. These are all different names for the same thing.
A manifold $N$ together with an immersion $\iota : N \rightarrow M$ is called an integral manifold of $\mathcal{D}$ if at each $q \in N$

$$d\iota_q(T_q N) \subset \mathcal{D}(\iota(q)) \quad (1)$$

where $\mathcal{D}(m)$ is the subspace of $T_m(M)$ given by $\mathcal{D}$. 

\[ \]
The example of a one form.

For example, if $\alpha$ a nowhere vanishing one form on $M$, then at each $x \in M$ we get a differential system of codimension one where $\mathcal{D}_x \subset T_x M$ consists of those $v \in T_x M$ such that $\langle \alpha_x, v \rangle = 0$.

If there are functions (locally defined) $T$ and $S$ such that $\alpha = TdS$ then at each $x_0$ the space $\mathcal{D}_{x_0}$ is tangent to the hypersurface $S = S(x_0)$.

Conversely, suppose there is a (locally defined) function $S$ with $dS$ nowhere 0 such that at each $x_0$ the hyperplane in $TM_{x_0}$ is tangent to the hypersurface $S = S(x_0)$ passing through $x_0$. Then $\alpha$ must be some multiple of $dS$, so (locally) $\alpha = TdS$ for some function $T$. 
A non-integrable one form.

If $\alpha = TdS$ then $d\alpha = dT \wedge dS$ and so

$$\alpha \wedge d\alpha \equiv 0.$$ 

Consider the form $\alpha = dz + xdy$ (say on three dimensional space). Then

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

is nowhere zero. So we can not write $\alpha = TdS$ on any neighborhood. The planes of the differential system $\mathcal{D}$ defined by $\alpha$ do not “fit together” to be tangent to surfaces.
Submersions.

Let $Q$ and $B$ be manifolds. A **submersion** is a smooth map $f : Q \rightarrow B$ such that for all $q \in Q$ the differential

$$df_q : T_q Q \rightarrow T_{f(q)} B$$

is surjective. For any submersion the **fibers**

$$f^{-1}(a), \quad a \in B$$

are smooth embedded submanifolds of $Q$ of dimension $k = n - \dim B$. This follows from the implicit function theorem.
In fact $Q$ is **foliated** by such submanifolds in the following sense: Every point $q \in Q$ has a coordinate neighborhood $U$ with coordinates $x_1, \ldots, x_n$ with $q$ corresponding to $x = 0$, and $f(q)$ a coordinate neighborhood with coordinates $y_1, \ldots y_{n-k}$ with $f(q)$ corresponding to $y = 0$ such that in terms of these coordinates, the map $f$ is given by projection to the first $n-k$ coordinates. Of course, the set of tangent spaces (at all points of $Q$) to the fibers of a submersion form a differential system.
More generally, we say that a differential system $\mathcal{D}$ is a \textbf{foliation} or is \textbf{completely integrable} if it is tangent to a foliation in the preceding sense: Every point $q \in Q$ has a coordinate neighborhood $U$ with coordinates $x_1, \ldots, x_n$ with $q$ corresponding to $x = 0$ and a map $f : U \to \mathbb{R}^{n-k}$ with

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-k})$$

and such that at every point $q$ of $U$, the space $\mathcal{D}_q$ is the tangent space to the fiber of $f$ through $q$. 
For example, as we have seen, every one dimensional differential system is completely integrable. This was an immediate consequence of the existence theorem for ordinary differential equations. But not every one dimensional foliation is a submersion. For example, the irrational line foliation on a two dimensional torus.

Also, as we have seen, the two dimensional differential system on $\mathbb{R}^3$ defined by the null planes of $dz + xdy$ is not completely integrable. The Frobenius theorem to be stated below gives a useful necessary and sufficient condition for a differential system to be a foliation.
A more stringent condition on a submersion is for \( f : Q \to B \) to be a \textbf{fibration}. We say that \( f \) is a fibration if it is surjective and satisfies the \textbf{local triviality condition}: There exists a manifold \( F \) (called the \textbf{standard fiber}) such that about every point in \( B \) there is a neighborhood \( W \) and a diffeomorphism \( \phi : f^{-1}(W) \to W \times F \) which conjugates \( f \) into projection to the first factor.
The vector fields of a differential system.

Let $\mathcal{D}$ be a differential system on some manifold $M$. We let $\mathfrak{X}(\mathcal{D})$ denote the space of vector fields $X$ on $M$ with the property that $X_p \in \mathcal{D}_p$ at all $p$.

We can now state the Frobenius theorem:
Frobenius theorem.

**Theorem of Frobenius.** A necessary and sufficient condition for $\mathfrak{D}$ to be completely integrable is that $\mathfrak{X}(\mathfrak{D})$ be closed under Lie bracket, i.e.

$$X, Y \in \mathfrak{X}(\mathfrak{D}) \Rightarrow [X, Y] \in \mathfrak{X}(\mathfrak{D}).$$
Proof of necessity.

If $\mathcal{D}$ is completely integrable of dimension $k$, every point has a coordinate neighborhood with coordinates $x_1, \ldots, x_n$ such that

$$\frac{\partial}{\partial x_i} \in \mathcal{X}(\mathcal{D}) \quad \text{for } i = 1, \ldots, k.$$  

So the values of these vector fields span $\mathcal{D}_p$ at each point of the neighborhood and hence any element of $\mathcal{X}(\mathcal{D})$ is a sum of elements of $\mathcal{X}(\mathcal{D})$ of the form $a\frac{\partial}{\partial x_i}$ for some $1 \leq i \leq k$ and where $a$ is smooth function. But

$$\left[ a\frac{\partial}{\partial x_i}, b\frac{\partial}{\partial x_j} \right] = a\frac{\partial b}{\partial x_i} \frac{\partial}{\partial x_j} - b\frac{\partial a}{\partial x_j} \frac{\partial}{\partial x_i}$$

lies in $\mathcal{X}(\mathcal{D})$. $\Box$.  

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The proof of sufficiency is a bit harder and is by induction on $k$ where $n - k$ is the codimension of the differential system, i.e. the differential system is a system of $k$-planes.

We know the theorem to be true for $k = 1$ where the condition is vacuous and all differential systems are completely integrable.
Proof of sufficiency, 2.

Let $x \in M$ and let $X_1, \ldots, X_k$ be vector fields defined in a neighborhood of $x$ and whose values at each point $p$ in the neighborhood span $\mathcal{D}_p$. Let $S$ be a submanifold of dimension $n - 1$ passing through $x$ and such that $X_1$ is nowhere tangent to $S$. Solve the ordinary differential equation corresponding to $X_1$ with initial conditions for the trajectories to lie in $S$ at time 0. If $y_2, \ldots, y_n$ are coordinates on $S$, and if $y_1(p)$ denotes the time at which a trajectory starting at $S$ reaches $p$, then the variables $y_1, \ldots, y_n$ are coordinates about $x$ in terms of which

$$X_1 = \frac{\partial}{\partial y_1}.$$
Proof of sufficiency, 3.

Let

\[ f_i := X_i y_1 \]

and set

\[ Y_1 = X_1, \quad Y_i := X_i - f_i X_1, \quad i = 2, \ldots, k. \]

Then the vector fields \( Y_1, \ldots, Y_k \) are still linearly independent at all points in a neighborhood about \( x \) and span \( \mathcal{D} \) in this neighborhood, and, in addition,

\[ Y_i y_1 = 0, \quad i = 2, \ldots, k. \]
The equation

\[ Y_i y_1 = 0, \quad i = 2, \ldots, k \]

implies that the \( Y_i \) are all tangent to the initial hypersurface \( S \) given by \( y_1 = 0 \). So there are vector fields \( Z_2, \ldots, Z_k \) on \( S \) such that

\[ d\iota_q(Z_{iq}) = Y_{iq} \]

at all \( q \in S \), where \( \iota : S \to M \) denotes the injection of \( S \) into \( M \).
Proof of sufficiency, 5.

The vector fields $Z_2, \ldots, Z_k$ define a differential system of dimension $k - 1$ on $S$ which satisfies Frobenius’s criterion that $[Z_i, Z_j]$ lie in the subspace spanned by the $Z_i$ at all points. Indeed, if this were not true at some point $q$ of $S$ then the the same would hold for $[Y_i, Y_j]$ since none of the $Y_i$ have any component in $\frac{\partial}{\partial y_1}$ direction.
Proof of sufficiency, 6, using the induction hypothesis.

So by induction we conclude that we can find coordinates \( w_2, \ldots, w_n \) on \( S \) near \( x \) so that the differential system spanned by \( Z_2, \ldots, Z_k \) is tangent to the foliation given by projection onto the last \( n - k \) coordinates. Then \( (x_1, \ldots, x_n) := (y_1, w_2, \ldots, w_n) \) is a system of coordinates about \( x \) in \( M \) and

\[
\frac{\partial x_i}{\partial y_1} = 0, \ i = 2, \ldots n
\]

so

\[
Y_1 = \frac{\partial}{\partial y_1} = \frac{\partial}{\partial x_1}.
\]
Proof of sufficiency, 7.

In particular,

\[ Y_1 x_s = 0 \quad \text{for} \quad s = k + 1, \ldots, n. \]

Therefore there are functions \( c_{ij} \) such that

\[
\frac{\partial}{\partial x_1} (Y_i x_s) = Y_1 (Y_i x_s) \\
= Y_1 (Y_i x_s) - Y_i (Y_1 x_s) \\
= [Y_1, Y_i] x_s \\
= c_{i1} Y_1 x_s + \sum_{j=2}^{k} c_{ij} Y_j x_s \quad \text{since} \quad [Y_1, Y_i] \in \mathfrak{X}(\mathcal{D}) \\
= \sum_{j=2}^{k} c_{ij} Y_j x_s.
\]
Proof of sufficiency, 8.

So the $Y_i x_s$ satisfy the system of (ordinary) homogeneous linear differential equations

$$\frac{\partial}{\partial x_1} (Y_i x_s) = \sum_{j=2}^{k} c_{ij} Y_j x_s$$

with the initial conditions at $x_1 = 0$ given by

$$Y_i x_s = Z_i x_s = 0.$$
The uniqueness theorem for differential equations then implies that

\[ Y_i \equiv 0, \quad i \leq k, \quad s > k. \]

So the vector fields \( Y_i \) are function linear combinations of

\[ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \]

and since the \( Y_1, \ldots, Y_k \) are linearly independent everywhere, this shows that \( \mathcal{D} \) is completely integrable. \( \square \)
Our main application of Frobenius.

Let $\Omega$ be a closed form of any degree. Consider the set of vector fields which satisfy

$$i(X)\Omega = 0.$$  

This may or may not define a differential system, since the dimension of the space spanned by such $X_p$ may vary with $p$. Suppose that this dimension is constant.

**Proposition**

*The differential system defined by $i(X)\Omega = 0$ is completely integrable, i.e. is a foliation.*
Proof.

If \( i(X)\Omega = 0 \) then Weil’s formula implies that

\[
D_X \Omega = di(X)\Omega + i(X)d\Omega = 0.
\]

Hence if \( i(Y)\Omega = 0 \) then

\[
i([X, Y])\Omega = D_X (i(Y)\Omega) = 0.
\]

We call this foliation, that is the foliation spanned by the vector fields satisfying \( i(X)\Omega = 0 \) the **null foliation** of \( \Omega \).
There is a very pretty generalization of this Proposition due to Cartan: Let $\mathcal{I}$ be an ideal in the ring of differential forms on a manifold $M$ which is homogeneous: if $\sigma \in \mathcal{I}$ then all the homogeneous components of $\sigma$ belong to $\mathcal{I}$. Consider the set of vector fields $X$ on $M$ which satisfy

$$i(X)\mathcal{I} \subset \mathcal{I}.$$ 

In other words, the set of all vector fields $X$ with the property that

$$\sigma \in \mathcal{I} \Rightarrow i(X)\sigma \in \mathcal{I}.$$ 

Once again, this may or may not define a differential system, since the dimension of the space spanned by the $X_p$ may vary from one $p$ to another.
Suppose that this dimension is constant. So we get a differential system which is called the **characteristic system** of the ideal $\mathcal{I}$.

**Theorem**

**[Cartan.]** *If $d\mathcal{I} \subset \mathcal{I}$, its characteristic system is completely integrable.*

The condition $d\mathcal{I} \subset \mathcal{I}$ means that if $\sigma \in \mathcal{I}$ then $d\sigma \in \mathcal{I}$.
Proof.

If $i(X)\mathcal{I} \subset \mathcal{I}$ and $\sigma \in \mathcal{I}$ then

$$D_X \sigma = i(X)d\sigma + di(X)\sigma \in \mathcal{I}.$$ 

So if $i(Y)\sigma \in \mathcal{I}$ then $D_X(i(Y)\sigma) \in \mathcal{I}$. But

$$D_X(i(Y)\sigma) = i([X, Y])\sigma + i(Y)D_X\sigma$$

so if $i(X)\mathcal{I} \subset \mathcal{I}$ and $i(Y)\mathcal{I} \subset \mathcal{I}$ then

$$i([X, Y])\mathcal{I} \subset \mathcal{I}.$$
Horizontal and basic forms of a submersion.

Let $\pi : Q \rightarrow B$ be a submersion. If $\tau$ is a differential form on $B$ then $\sigma = \pi^* \tau$ is a differential form on $Q$ with the following two properties:

- It is **horizontal** in the sense that if $X$ is a vector field which is everywhere tangent to the fiber (we say that $X$ is **vertical**), then
  \[ i(X)\sigma = 0 \]
  and

- It is vertically invariant in the sense that $D_X \sigma = 0$ for every vertical vector field.
Suppose that $\pi : Q \to B$ is a fibration with fiber $F$, and suppose that $\sigma$ is horizontal. Let us introduce local product coordinates $(x_1, \ldots, x_f, y_1, \ldots, y_b)$ where $(x_1, \ldots, x_f)$ are local coordinates on the fiber $F$ and $(y_1, \ldots, y_b)$ are local coordinates on the base $B$. If $\sigma$ is horizontal then $\sigma$ must be a linear combination with function coefficients of products of the $dy$’s. These functions $a$ might depend on all the variables. But if $\sigma$ also vertically invariant, they must satisfy

$$\frac{\partial a}{\partial x_i} \equiv 0. \quad i = 1, \ldots, f.$$ 

In other words they are locally constant in the fiber direction. If $F$ is connected, this implies that $\sigma = \pi^* \tau$ for some form $\tau$ on $B$. We say that $\sigma$ is **basic** in the sense that it come from the base. We have proved:
**Proposition**

Let $Q \rightarrow B$ be a fibration with connected fibers. Then a differential form on $Q$ is basic if and only if it is horizontal and is vertically invariant.
Back to reduction of a closed form.

Let us go back to the situation of a closed form $\Omega$ with the property that the vector fields satisfying $i(X)\Omega = 0$ form a differential system, which we proved is completely integrable and hence give a foliation which we called the null-foliation of $\Omega$.

By definition, $i(X)\Omega = 0$ for vector fields tangent to this null foliation, and by Weil’s formula $D_X\Omega = i(X)d\Omega + di(X)\Omega = 0$ for vector fields tangent to this null foliation since $d\Omega = 0$.

Suppose that this null foliation is a fibration $\pi : Q \to B$ with connected fibers. Then by the preceding proposition, we know that $\Omega = \pi^*\omega$ for a uniquely determined form $\omega$ on $B$.

Since $\pi^*$ is an injection and $0 = d\Omega = d\pi^*\omega = \pi^*d\omega$ we conclude that $d\omega = 0$. 

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Finally, I claim that $\omega$ is non-degenerate in the following sense: For any $b \in B$ and $v \in T_bB$,

$$i(v)\omega_b = 0 \Rightarrow v = 0.$$  

Indeed, let us introduce local product coordinates

$$(x_1, \ldots, x_f, y_1, \ldots, y_b)$$

around a point $q = (f, b)$ as above. If $i(v)\omega_b = 0$, then the vector $w \in T_qQ \cong T_fF \oplus T_bB$ given by $w = (0, v)$ satisfies

$$i(w)\Omega_q = 0.$$  

So $w \in D_q$. But all vectors of $D_q$ are vertical, i.e of the form $(z, 0)$. So $v = 0$. 

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We have proved:

**Proposition**

*Let $\Omega$ be a closed form of constant rank on a manifold $Q$. Suppose that its null foliation is a fibration $\pi : Q \to B$ with connected fibers. Then*

$$\Omega = \pi^* \omega$$

*where $\omega$ is a non-degenerate closed form on $B$.***
Reduction of a co-isotropic immersion.

I want to apply the preceding Proposition to the following situation: \((M, \omega)\) is a symplectic manifold and

\[ \iota : Q \to M \]

is a co-isotropic immersion. This means that \(\iota\) is an immersion such that \(d\iota_q(T_qQ) \subset T_{\iota(q)}M\) is a co-isotropic subspace relative to the symplectic form \(\omega_{\iota(q)}\) on \(T_{\iota(q)}(M)\) for each \(q \in Q\).

Recall that to say that \(d\iota_q(T_qQ) \subset T_{\iota(q)}M\) is co-isotropic means that

\[ (d\iota_q(T_qQ))^\perp \subset d\iota_q(T_qQ) \]

where \(^\perp\) is orthogonal complement relative to \(\omega_{\iota(q)}\).
For each $q \in Q$, consider

$$(d\iota_q)^{-1} \left((d\iota_q(T_qQ))^\perp\right) \subset T_qQ.$$ 

Since $\dim((d\iota_q(T_qQ))^\perp) = \dim M - \dim Q$ is constant and since $\iota$ is an immersion, we see that $(d\iota_q)^{-1} \left((d\iota_q(T_qQ))^\perp\right)$ has constant dimension and hence $q \mapsto (d\iota_q)^{-1} \left((d\iota_q(T_qQ))^\perp\right)$ defines a differential system on $Q$. By definition, this is differential system associated to the closed two form

$$\Omega := \iota^*\varpi.$$ 

We know that this differential system is completely integrable. By abuse of language, we will refer to the associated null-foliation as the null foliation of $Q$ (where $\iota$ is understood). We conclude from the Proposition:
Theorem

Suppose that the null-foliation of $Q$ is a fibration

$$\pi : Q \rightarrow B$$

with connected fibers. Then $B$ is a symplectic manifold with symplectic form $\omega$ uniquely determined by

$$\pi^* \omega = \iota^* \varpi.$$ 

The content of the theorem is summarized in the diagram on the next slide:
Diagrammatically we have

\[ \pi^* \omega = \iota^* \overline{\omega}. \]
I next want to study the relation between Poisson brackets on $M$ and on $B$: Let $f$ be a smooth function on $M$ and $X_f$ its corresponding Hamiltonian vector field so $i(X_f)\omega = df$. So if $q \in Q$ and $v \in T_qQ$ we have

$$\langle df_{\iota(q)}, d\iota_q(v) \rangle = \omega_q(X_f(\iota(q)), d\iota_q(v)).$$

Suppose we choose $v \in \mathcal{D}_q$ where $\mathcal{D}$ is the null foliation of $Q$. Since the orthocomplement of $d\iota_Q(\mathcal{D}_q)$ relative to $\omega_q$ is $d\iota_q(T_qQ)$ we see that:
Proposition

\[ d(\iota^* f)_q \text{ vanishes on } D_q \text{ if and only if } X_f(q) \in d\iota_q(T_q Q). \]

Applied to all points of \( Q \) we conclude that

Proposition

The function \( \iota^* f \) on \( Q \) is constant along the null foliation of \( Q \) if and only if \( X_f \) is tangent to the image of \( Q \) in \( M \).
If the null foliation of $Q$ is a fibration with connected fibers, then a function is constant along the null foliation of $Q$ if and only if its is basic, i.e. is of the form $\pi^* F$ where $F$ is a function on the base, $B$. So if $f$ is a function on $M$ such that $X_f$ is tangent to the image of $Q$, we have

$$\iota^* f = \pi^* F$$

for a unique smooth function $F$ on $B$. 
Suppose that $i^*f = \pi^*F$. Let $v \in T_qQ$ and let

$$m = i(q) \in M \quad \text{and} \quad b = \pi(q) \in B$$

so that $d\pi_q(v) \in T_b(B)$. Then

$$\langle df(m), d\iota_q(v) \rangle = \langle dF(b), d\pi_q(v) \rangle = \omega_b(Y_F, d\pi_q(v))$$

where $Y_F$ is the Hamiltonian vector field on $B$ corresponding to the function $F$ and the symplectic form $\omega$. Now

$$\langle df(m), d\iota_q(v) \rangle = \varpi_q(X_f(q), d\iota_q(v))$$

$$= \omega_b (d\pi_q ((d\iota_q)^{-1}X_f(q)), d\pi_q(v)).$$
We have shown that

\[ \langle df(m), d\iota_q(v) \rangle = \omega_b(d\pi_q(X_f(q)), d\pi_q(v)) = \omega_b(Y_F, d\pi_q(v)) \]

where we have written \( X_f(q) \) instead of \( (d\iota_q)^{-1}(X_f(q)) \). Since vectors of the form \( d\pi_q \nu \) span \( T_bB \) we conclude that

\[ d\pi_q(X_f(q)) = Y_F(b). \]

So if \( f_1 \) and \( f_2 \) are two functions on \( M \) whose Hamiltonian vector fields are tangent to \( Q \) then

\[ \{f_1, f_2\}_M(\iota(q)) = \{F_1, F_2\}_B(\pi(q)). \]

To summarize:
Theorem

Let \( \iota : Q \to M \) be a coisotropic immersion whose null foliation is a fibration \( \pi : Q \to B \). If \( f_1 \) and \( f_2 \) are two functions on \( M \) such that \( \iota^* f_i = \pi^* F_i \), \( i = 1, 2 \) then

\[
\iota^* \{ f_1, f_2 \}_M = \pi^* \{ F_1, F_2 \}_B.
\]