Lecture 11

Collective motion.
Semi-direct products.

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In this lecture I will attempt to give a mathematical formulation to the notion of a “model” of a given mechanical system. The formulation will use the notion of the moment map.

An example of what I have in mind is to try to formulate what we mean when we say that a system of point particles with a given Hamiltonian behaves “as if it were a rigid body”. More generally, what do we mean when we say that one system behaves “as if it were” some other mechanical system.

In nuclear physics, for example, one has the “liquid drop model” of the nucleus, in which it is assumed that in an appropriate approximation, the nucleus which consists of N nucleons (considered as point particles) behaves “as if it were a liquid drop”. For this example we must explain exactly what we mean by the mechanical system consisting of a “liquid drop”. But we must explain in more generality what we mean by saying that one system...
1. The abstract definition of collective motion.
   - Some motivation for the definition.

2. Solving Hamilton’s equations for a collective Hamiltonian.

   - The Wigner - Mackey description.
   - Examples.
     - Euclidean type examples.
   - The quadrupole moment as part of a moment map.
   - The liquid drop.

The abstract definition of collective motion.

Let \((M, \omega_M, \Phi_M)\) be a Hamiltonian \(G\)-manifold.

**Definition**

A function \(\mathcal{H}\) on \(M\) is called **collective** if

\[
\mathcal{H} = F \circ \Phi_M
\]

where \(F\) is a function on \(g^*\).
Some motivation for the definition.

Example: the rigid body.

Consider $N$ point particles in $\mathbb{R}^3$. We can say that they are “almost rigidly attached” to one another if there is some potential energy $V = \sum_{ij} V_{ij}$ where $V_{ij}$ is a function of the distance between the $i$th and $j$th particles, and takes on a very sharp minimum at certain specified distances $r_{ij}$ between them.
So we can imagine that the Hamiltonian of such a system has the form

$$V + K + K' + \mathcal{H}$$

where $V$ is the potential energy described above, where $K$ is an “internal” kinetic energy involving the motion of the points relative to one another, where $K'$ is some kinetic energy of the overall center of mass, and where $\mathcal{H}$ (involving both kinetic and potential terms) is a function solely of the total angular momentum $L$ and “inertia tensor” $Q$ relative to the center of mass.
Some motivation for the definition.

**L and Q are components of a certain moment map.**

In more detail: Let \(q_1, \ldots, q_N\) denote the position vectors of the particles relative to the center of mass and \(p_1, \ldots, p_N\) the corresponding momenta. Then we can define the functions \(L\) and \(Q\) on the total phase space by

\[
L := \sum_i p_i \wedge q_i, \quad Q := \sum_i q_i \otimes q_i.
\]

So \(L\) takes values in \(\wedge^2(\mathbb{R}^3)\) and \(Q\) takes values in \(S^2(\mathbb{R}^3)\). As we will explain later in this lecture, the pair \((L, Q)\) can be thought of as the moment map corresponding to a certain nine dimensional group acting on the total phase space. (This will be explained in our discussion of semi-direct products and their moment maps.)
So the $\mathcal{H}$ in the Hamiltonian

$$V + K + K' + \mathcal{H}$$

is collective in the sense of our abstract definition. The contribution of $K'$ is simply to give an overall linear motion to the center of mass. By introducing coordinates relative to the center of mass we can ignore it (as an example of reduction relative to the $\mathbb{R}^3$ acting simultaneously on all particles). So we are reduced to looking at a Hamiltonian of the form

$$V + K + \mathcal{H}.$$ 

The motion of the system would be, approximately, the “collective motion” (or “rigid body motion”) coming from $\mathcal{H}$, with a superimposed rapid oscillation coming from $V + K$. 
To a good approximation, we might expect to be able to ignore the rapid oscillation which averages itself out over the “rigid body motion”.

The situation is even more convincing in quantum mechanics. There, one might expect that for low energies one can assume that the internal state of the system, and so the observed spectrum, should look like that of a rigid rotor. The fact that one sees unmistakable rotational levels in complicated nuclear spectra supports this description of the system.
Some motivation for the definition.

So this lecture will split into two parts. In the first part we discuss general properties of collective Hamiltonians. In particular what goes into the solution of the corresponding Hamiltonian equations. In the second part we discuss semi-direct products, the structure of their co-adjoint algebras, and examples of their moment maps. The results on semi-direct products have applications beyond collective motion.
Reminder: the Legendre transformation.

Let $U$ be an open subset of a vector space. In the case of interest to us, the vector space will be $\mathfrak{g}^*$, the dual of the Lie algebra of a Lie group $G$. Let $F$ be a smooth function defined on $U$. Then the **Legendre transformation** $\mathcal{L}_F$ is a map from $U$ to the dual of the vector space. In our case

$$\mathcal{L}_F : \mathfrak{g}^* \to (\mathfrak{g}^*)^* = \mathfrak{g}.$$

It is defined by

$$\langle \nu, \mathcal{L}_F(\mu) \rangle = \frac{d}{dt} F(\mu + t\nu)\big|_{t=0}.$$
Two ways of constructing a vector field on $M$.

Let $(M, \omega)$ be a Hamiltonian $G$-manifold with moment map $\Phi_M : M \to g^*$. Let $F$ be a smooth function defined on $g^*$. There are now two ways of constructing a vector field on $M$:

The first way is to construct the collective Hamiltonian $\mathcal{H} = F \circ \Phi_M$ and then pass to the corresponding Hamiltonian vector field $X_{\mathcal{H}}$.

Here is a (seemingly) second way:
At each $m \in M$, $\Phi_M(m) \in g^*$. So $\mathcal{L}_F (\Phi_M(m)) \in g$. Any $\xi \in g$ gives rise to a vector field $\xi_M$ on $M$, so in particular a tangent vector $\xi_M(m) = \text{ev}_M(m)(\xi)$. In particular we may take $\xi = \mathcal{L}_F (\Phi_M(m))$. So we may construct

$$[\mathcal{L}_F (\Phi_M(m))]_M(m) \in T_m M.$$ 

I claim that these two ways are the same, i.e.

$$X_{\mathcal{H}}(m) = [\mathcal{L}_F (\Phi_M(m))]_M(m) \quad (\ast)$$

where $\mathcal{H} = F \circ \Phi_m$. 

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To prove \( X_H(m) = [\mathcal{L}_F (\Phi_M(m))]_M (m) \) \((*)\)
it is enough to show that both sides give the same value when we
take their symplectic scalar product (relative to \( \omega_m \)) with any
vector \( v \in T_m M \). Since \( i(X_H) \omega = dH = d(F \circ \Phi_m) \), \( \omega(X_H, v) \)
is the directional derivative

\[
v(H) = v(F \circ \Phi_M) = (d(\Phi_M)_m(v)) F = \langle d(\Phi_M)_m(v), \mathcal{L}(\Phi_M(m)) \rangle.
\]

by the chain rule and the definition of \( \mathcal{L}_F \). We recall the formula
for the differential of the moment map: Writing \( \Phi \) for \( \Phi_M \) it says
that for any \( \xi \in g \),

\[
\langle d\Phi_m(v), \xi \rangle = \omega_m(\xi_M(m), v).
\]

Applied to \( \xi = \mathcal{L}_F (\Phi_M(m)) \) we get

\[
\langle d(\Phi_M)_m(v), \mathcal{L}_F (\Phi_M(m)) \rangle = \omega_m ([\mathcal{L}_F (\Phi_M(m))]_M (m), v) .
\]

\( \square \)
We now derive some important consequences of

$$X_H(m) = [\mathcal{L}_F (\Phi_M(m))]_M (m) \quad (*) :$$

If we apply $d (\Phi_M)_m$ to both sides and use the equivariance of $\Phi_M$ we obtain

$$d (\Phi_M)_m (X_H(m)) = [\mathcal{L}_F (\Phi_M(m))]_{g^*} (\Phi_M(m)) \quad (**)$$

where the right hand side is the value at $\Phi_M(m)$ of the vector field on $g^*$ associated to the element $\mathcal{L}_F (\Phi_M(m)) \in g$ by the co-adjoint action.
\[ d (\Phi_M)_m (X_H(m)) = [\mathcal{L}_F (\Phi_M(m))]_{g^*} (\Phi_M(m)). \]  

The right hand side of (**) is the value at \( \Phi_M(m) \) of the Hamiltonian vector field associated to \( F|_O \) where \( O \) is the co-adjoint orbit through \( \Phi_M(m) \). This can either be seen directly or by applying (*) to the orbit \( O \) which is a symplectic \( G \)-space whose moment map is the injection of \( O \) into \( g^* \).

Thus if \( t \mapsto m(t) \) is the trajectory of the Hamiltonian system \( X_H \) with \( m(0) = m \), we see that \( \Phi_M(m(t)) \) lies entirely on \( O \) and is the solution curve \( \gamma \) to the Hamiltonian system corresponding to \( F \) (relative to the symplectic structure on \( O \)) and with initial condition \( \gamma(0) = \Phi_M(m) \).
Let
\[ \xi(t) := \mathcal{L}_F(\gamma(t)). \]

So the curve \( t \mapsto \xi(t) \) is a curve in the Lie algebra \( \mathfrak{g} \).

We think of \(-\xi(t)\) as a right invariant vector field on \( G \), so we have a “time dependent” vector field and we can solve this system of differential equations to obtain a curve \( t \mapsto a(t) \) with the initial condition \( a(0) = e \). That is (for the case of a linear group - and by abuse of language we write it this way for any group) we are looking for the solution to the differential equation
\[ a'(t) = -\xi(t)a(t), \quad a(0) = e. \]

Then
\[ m(t) = a(t)m \]
gives the solution to our collective Hamiltonian system on \( M \).
Solving the collective Hamiltonian equations in four easy steps.

1. Find the orbit $\mathcal{O}$ through $\Phi_M(m)$.

2. Find the solution curve to the Hamiltonian system on $\mathcal{O}$ corresponding to $F$, passing through $\Phi_M(m)$ at $t = 0$. Call this curve $\gamma$.

3. Set $\xi(t) = \mathcal{L}_F(\gamma(t))$. The curve $t \mapsto \xi(t)$ is a curve in $g$.

4. Solve the differential equations $a(t) = -\xi(t)a(t)$, $a(0) = e$ to obtain a curve in $G$. Then $m(t) = a(t)m$ is the desired solution.
Step 1.

Step 1 is purely kinematic; it depends solely on the Hamiltonian group action and has nothing to do with $E$. 

$\Phi_M(m)$
Step 2 involve solving a Hamiltonian system of dimension (usually) much smaller than the dimension of $M$. Thus, for example, as we shall see, in the liquid drop model, $\mathcal{O}$ is at most 12-dimensional while $M = \mathbb{R}^{6N}$ has dimension $6N$. 
2. Find the solution curve to the Hamiltonian system on $\mathcal{O}$ corresponding to $F$, passing through $\Phi_M(m)$ at $t = 0$. Call this curve $\gamma(t)$. 

Step 2.
Step 3.

*Step 3* is an application of the Legendre transformation.

\[ \xi(t) = \mathcal{L}_F(\gamma(t)) \]
Step 4 - Solving a differential equation on $G$.

Step 4 can pose some interesting problems even if the solution of step 2 is trivial. For instance, suppose that $F$ is a $G$-invariant. Then on each $O$, the curve $\gamma(t)$ is a constant, but the map $\mathcal{L}_F$ need not be trivial. Thus, $\xi(t)$ will be a constant element of $g$ and so $a(t)$ will be a one-parameter group. Thus the motion corresponding to $F \circ \Phi_M$ when $F$ is an invariant is given by the action of a one-parameter group, the one-parameter group depending on $m$. (For the case of a spherical top, this is the spinning motion.) We might think of the solutions for noninvariant $F$ as “generalized precessions or nutations.”
Step 4 may simplify if we are only interested in partial information about the trajectory $m(t)$. For example, suppose that $M = T^*Q$ and $m = (q, p)$, where $q \in Q$. That is, $m$ is a point in phase space whose corresponding point in configuration space is $q$. We might only be interested in the time evolution of $q$, and this may involve less than the full curve $a(t)$. We shall see that this is the case for “rigid body” or “liquid drop” Hamiltonians.

First we must study semi-direct products.
The semi-direct product of a group with a vector space.

Let $H$ be a Lie group. Suppose we are given a representation of $H$ on a (real) vector space $V$. We make $= H \times V$ into a Lie group by defining the multiplication

$$(a, v) \cdot (b, w) := (ab, aw + v).$$

$H \times V$ endowed with this multiplication is called the **semi-direct product** of $H$ with $V$ and is denoted by

$$H \circledast V.$$
The matrix mnemonic.

An easy way to remember this multiplication is to write the element \((a, v)\) as the “matrix”

\[
\begin{pmatrix}
  a & v \\
  0 & 1
\end{pmatrix}
\]

Then matrix multiplication

\[
\begin{pmatrix}
  a & v \\
  0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
  b & w \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  ab & aw + v \\
  0 & 1
\end{pmatrix}
\]

gives the semi-direct multiplication (if we ignore the bottom row).
The Lie algebra \( g \) of \( G = H \ltimes V \).

The Lie algebra in terms of the “matrix” mnemonic consists of all “matrices” of the form

\[
\begin{pmatrix}
\xi & x \\
0 & 0
\end{pmatrix}
\quad \xi \in \mathfrak{h}, \quad x \in V
\]

where \( \mathfrak{h} \) is the Lie algebra of \( H \). So the Lie algebra \( g \) of \( G = H \ltimes V \) is given, as a vector space, as \( \mathfrak{h} \times V \) and “matrix” conjugation shows that the Lie bracket on \( g \) is given by

\[
[(\xi, x), (\eta, y)] = ([\xi, \eta], \xi y - \eta x).
\]
The adjoint representation.

The inverse of the “matrix” \( \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \) is \( \begin{pmatrix} a^{-1} & -a^{-1}v \\ 0 & 1 \end{pmatrix} \) as can be checked by “matrix” multiplication, and

\[
\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \xi & x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & -a^{-1}v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\xi a^{-1} & -(a\xi a^{-1})v + ax \\ 0 & 0 \end{pmatrix}.
\]

This shows that the adjoint representation is given by

\[
\text{Ad}_{(a,v)}(\xi, x) = (\text{Ad}_a \xi, ax - (\text{Ad}_a \xi)v)
\]

Applying (\#) to \((a, v)^{-1} = (a^{-1}, -a^{-1}v)\) gives

\[
\text{Ad}_{(a,v)^{-1}}(\xi, x) = (\text{Ad}_{a^{-1}} \xi, a^{-1}x + a^{-1}\xi v).
\]
The co-adjoint representation.

Let us write an element of $\mathfrak{g}^*$ as $(\alpha, p)$ where $\alpha \in \mathfrak{h}^*$ and $p \in V^*$ so that

$$\langle (\alpha, p), (\xi, x) \rangle = \langle \alpha, \xi \rangle + \langle p, x \rangle.$$  

The group $H$ acts on $\mathfrak{h}^*$ via its co-adjoint representation - so

$$\langle a \cdot \alpha, \xi \rangle = \langle \alpha, \text{Ad}_{a^{-1}} \xi \rangle$$

and it acts on $V^*$ as $(a^{-1})^*$ i.e.

$$\langle ap, x \rangle = \langle p, a^{-1}x \rangle.$$  

Finally, if $p \in V^*$ and $v \in V$, define $p \circledcirc v \in \mathfrak{h}^*$ by

$$\langle p \circledcirc v, \xi \rangle := \langle p, \xi v \rangle.$$
We have defined \( \langle p \odot v, \xi \rangle := \langle p, \xi v \rangle \). We have also established that

\[
\text{Ad}^{(a,v)}_{(a,v)^{-1}}(\xi, x) = \left( \text{Ad}_{a^{-1}} \xi, a^{-1}x + a^{-1}\xi v \right).
\]

So

\[
\langle (\alpha, p), \text{Ad}^{(a,v)}_{(a,v)^{-1}}(\xi, x) \rangle = \langle \alpha, \text{Ad}_{a^{-1}} \xi \rangle + \langle p, a^{-1}\xi v \rangle + \langle p, a^{-1}x \rangle
\]

So we see that the co-adjoint action of \( G \) on \( \mathfrak{g}^* \) is given by

\[
(a, v) \cdot (\alpha, p) = (a\alpha + (ap) \odot v, ap).
\] (*
The co-adjoint representation, continued - the Wigner-Mackey classification of the co-adjoint orbits.

\[(a, v) \cdot (\alpha, p) = (a\alpha + (ap) \circ v, ap)\]. (*)

From (*) we see that the co-adjoint $G$ orbits are fibered over the $H$ orbits in $V^\ast$. Suppose that we pick an $H$ orbit $N \subset V^\ast$ and want to describe the various $G$ orbits in $g^\ast$ “sitting over” $N$. Pick some point $p \in N$. Let $H_p \subset E$ be the isotropy group of $p$, and let $\mathfrak{h}_p \subset \mathfrak{h}$ be the Lie algebra of $H_p$. Since $\mathfrak{h}_p$ is a subalgebra of $\mathfrak{h}$ $\mathfrak{h}$, we get a projection, $\pi_p : \mathfrak{h}^\ast \to \mathfrak{h}_p^\ast$ where $\pi_p \alpha$ is simply the restriction of $\alpha$, which is a linear function on all $\mathfrak{h}$ to the subspace $\mathfrak{h}_p$. We claim that if $\alpha_1$ and $\alpha_2$ are in $\mathfrak{h}^\ast$ then

$$\pi_p \alpha_1 = \pi_p \alpha_2 \iff \alpha_1 - \alpha_2 = p \circ v \text{ for some } v \in V.$$
The Wigner - Mackey description.

To prove:

\[ \pi_p \alpha_1 = \pi_p \alpha_2 \iff \alpha_1 - \alpha_2 = p \circ v \quad \text{for some} \quad v \in V. \]

**Proof.**

The space \( V^* \otimes V \) is the dual to the space \( V \otimes V^* = gl(V) \) where \( \langle p \otimes v, \zeta \rangle = \langle p, \zeta v \rangle = \langle \zeta^* p, v \rangle \) for \( \zeta \in gl(V) \). So the annihilator space of \( p \otimes V \) under this pairing consists precisely of those \( \zeta \in gl(V) \) which satisfy \( \zeta^* p = 0 \). The image of \( p \otimes V \) in \( \mathfrak{h}^* \) under the map \( V^* \otimes V \) dual to the map \( \mathfrak{h} \rightarrow gl(V) = V \otimes V^* \) given by the representation of \( \mathfrak{h} \) on \( V \) is \( p \circ V \). So the annihilator space of \( p \circ V \subset \mathfrak{h}^* \) in \( \mathfrak{h} \) is the inverse image of \( gl(V)_p \) in \( \mathfrak{h} \), i.e. \( \mathfrak{h}_p \). So \( \alpha_1 - \alpha_2 \) vanishes on \( \mathfrak{h}_p \) if and only if \( \alpha_1 - \alpha_2 = p \otimes v \) for some \( v \in V \). \( \square \)
Thus, once we have picked an orbit $N$ of $H$ on $V^*$, and a point $p \in N$, the classification of the orbits above $N$ reduce to the classification of the $H_p$ orbits in $\mathfrak{h}^*_p$. So the classification of the co-adjoint orbits of $G$ acting on $\mathfrak{g}^*$ can be broken into the following steps:

- Classify the $H$ orbits on $V^*$.
- For each such orbit, $N$, pick a point $p \in N$.
- Classify the $H_p$ orbits on $\mathfrak{h}^*_p$.

This is the analogue in symplectic geometry of the famous Wigner-Mackey “little group” method of classifying the irreducible unitary representations of semi-direct products.
Over the next few slides we will consider the situation where $V$ has a non-degenerate scalar product and where $H$ is (possibly a covering group of a component of) the group $O(V)$. We will denote the scalar product of $x$ and $y \in V$ by $x \cdot y$. We do not assume that this scalar product is positive definite - just non-degenerate.

We may use the scalar product to identify $V$ with $V^*$ and also to identify $\mathfrak{h} = o(V)$ with $\wedge^2 V$. Indeed assign to $x \wedge y$ the operator sending

$$V \ni v \mapsto (y \cdot v)x - (x \cdot v)y.$$ 

I claim that under these identifications

$$p \circ v \quad \text{is identified with} \quad p \wedge v.$$
To prove: under all these identifications, $p \circ v$ is identified with $p \wedge v$.

Proof.

By definition,

$$\langle p \circ v, x \wedge y \rangle = \langle p, (x \wedge y) \cdot v \rangle$$

$$= \langle p, (y \cdot v)x - (x \cdot v)y \rangle$$

$$= (p \cdot x)(v \cdot y) - (p \cdot y)(v \cdot x)$$

$$= (p \wedge v) \cdot (x \wedge y).$$
The orbits of $SO(V)$ acting on $V$ are spheres.

Suppose $H$ is (possibly a covering group) of the connected component of the identity in $O(V)$. Then the orbits of $H$ acting on $V$ are the components of the $(\dim V - 1)$ “pseudo-spheres” $\|p^2\| = \text{const.}$ (and $p \neq 0$ in case this constant is zero) together with zero-dimensional orbit consisting of the origin, $\{0\}$. 
The Euclidean group in the plane.

Take $V = \mathbb{R}^2$ with its usual Euclidean metric and $H$ the group of rotations about the origin in the plane - so $G = E(2)$ is the group of (proper) Euclidean motions and is three dimensional.

There are two types of orbits of $H$ acting on $V$: The circles of radius $r > 0$ and the origin. In terms of an orthonormal basis $e_1, e_2$, on each orbit with $r > 0$ we can pick a (unique) point of the form $p = re_1$. Then $H_p = \{e\}$ and $p \wedge v$ as $v$ ranges over $V$ consists of all multiples of $e_1 \wedge e_2$, i.e. all of $\mathfrak{h}^*$ as is to be expected from the general theory.

Notice also that $e_2$ spans the tangent space (identified with the cotangent space) at $p$, and so these two dimensional orbits of $E(2)$ can be identified with the action of $E(2)$ on the cotangent bundles.
For the orbit consisting of \{0\} we have \(H_p = H\) and we must look at the orbits of \(H\) acting on \(\mathfrak{h}^*\). But since \(H = SO(2)\) is commutative, this action is trivial, and so each point of \(\mathfrak{h}^*\) (which we indentify with \(\mathbb{R}\)) is an orbit.

So there are two types of orbits \(E(2)\) acting on the dual of its Lie algebra:

- Two dimensional orbits, one for each \(r > 0\), given by the action of \(E(2)\) on the cotangent bundle of the circle of radius \(r\) and
- Zero dimensional orbits each given by a point of \(\mathfrak{h}^* \sim \mathbb{R}\).
The Euclidean group in three dimensions.

So $V = \mathbb{R}^3$ with its standard Euclidean metric and $H = SO(3)$, the group of rotations in three dimensions. So $G = H \circledast V = E(3)$ is the group of proper Euclidean motions of $\mathbb{R}^3$ and is six dimensional. The orbits of $H$ acting on $V$ are the spheres of radius $r > 0$ and the origin.

Let me first dispose of the case of the origin $p = 0$ where $H_p = H$ and we must determine the co-adjoint orbits of $H$ acting on $\mathfrak{h}^*$ which we may identify with $\mathbb{R}^3$ with $H$ acting as rotations. Thus there are two dimensional orbits (consisting of the spheres in $\mathfrak{h}^*$) and the zero dimensional orbit consisting of the origin. So there is a single zero dimensional orbit of $E(3)$ acting on $\mathfrak{g}^*$ consisting of the point $(0, 0)$ and a family of two dimensional orbits of the form $(\gamma, 0)$ where $||\gamma|| = s > 0$. 

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The four dimensional co-adjoint orbits of $E(3)$. 

Let us now consider the case where the orbit of $SO(3)$ on $\mathbb{R}^3$ is a sphere of radius $k > 0$. In terms of an orthonormal basis $e_1, e_2, e_3$ of $\mathbb{R}^3$, we can choose $p$ on this sphere to be the point $p = ke_3$, so that $H_p$ consists of rotations about the $e_3$ axis.

To see how the general theory works in this case, let $v = ae_1 + be_2 + ce_3$ so that $p \wedge v = kae_3 \wedge e_1 + kbe_3 \wedge e_2$.

Writing the most general element of $\mathfrak{h}^* \sim \wedge^2 (\mathbb{R})^3$ as $\gamma = xe_1 \wedge e_3 + ye_2 \wedge e_3 + se_1 \wedge e_2$, we see (since $k > 0$) that we can uniquely choose $a$ and $b$ so that

$$\gamma + p \wedge v = se_1 \wedge e_2 \in \mathfrak{h}_p^*.$$
In other words, on every co-adjoint orbit of $E(3)$ “lying over” an orbit of $H = SO(3)$ on $V$ consisting of a sphere of radius $k$, there is a unique point of the form

$$\mu = (se_1 \wedge e_2, ke_3),$$

and all of these co-adjoint orbits are four dimensional. Indeed, it is clear that the subgroup of $E(3)$ which fixes $\mu$ consists of those Euclidean motions which carry the line through $e_3$ into itself (built up out of rotations about the $e_3$ actions and translations in the $e_3$ direction). So as homogenous $E(3)$ spaces (i.e. as the space $E(3)/E(3)_\mu$) all of these orbits look like the space of all straight lines in $\mathbb{R}^3$ which is four dimensional. But the symplectic structures are different, depending on $k$ and $s$. 
For $\mu = (se_1 \wedge e_2, ke_3)$, we have $se_1 \wedge e_2 \wedge ke_3 = s e_1 \wedge e_2 \wedge e_3$. For a general element $(\beta, p) \in \mathfrak{g}^*$, 

$$\beta \wedge p \in \wedge^3(\mathbb{R}^3)$$

is invariant under the action of $E(3)$. Indeed, for $v \in V$ and $a \in SO(V)$ we have 

$$(a\beta + ap \wedge v) \wedge ap = a\beta \wedge ap = \beta \wedge p$$

since $\det a = 1$. So
The functions $k$ defined by $k(\beta, p) := \|p\|$ and $s$ defined for those $(\beta, p)$ with $k \neq 0$ by

$$\beta \wedge p = kse_1 \wedge e_2 \wedge e_3$$

are invariant under the action of $E(3)$ and the four dimensional co-adjoint orbits are described by the different possible values of the functions $k$ and $s$.

As we shall see over the next few slides, the function $s$ can be regarded as a sort of “intrinsic angular momentum” or “spin”. In fact, as we shall see, the image under the moment map of the phase space of a “classical free particle” includes the orbits with $s = 0$. 

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Examples.

The moment map for $E(3)$ acting on six-dimensional phase space.

Consider $E(3)$ acting on $\mathbb{R}^3$ as Euclidean motions, so \[\begin{pmatrix} a & w \\ 0 & 1 \end{pmatrix}\] acts on $q \in \mathbb{R}^3$ by the matrix multiplication \[\begin{pmatrix} a & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} aq + w \\ 1 \end{pmatrix} .\]

Thus \[\exp -t \begin{pmatrix} \xi & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} q - t(\xi q + v) + \cdots \\ 1 \end{pmatrix} \]
so the fundamental vector field corresponding to $(\xi, v)$ is \[q \mapsto -\xi q - v.\]
The fundamental vector field corresponding to \((\xi, v)\) is

\[ q \mapsto -\xi q - v, \]

so by our general for moment maps on exact symplectic manifolds

\[ \langle \Phi(q, p), (\xi, v) \rangle = -p \cdot \xi q - p \cdot v, \]

that is

\[ \Phi(q, p) = -(p \land q, p). \]

So if \(p \neq 0\) the image of \((q, p)\) under \(\Phi\) is the four dimensional orbit with \(k = \|p\|\) and \(s = 0\). (The image of a point \((q, 0)\) is the zero dimensional co-adjoint orbit consisting of the origin.)
The inverse image of the moment map.

\[ \Phi(q, p) = -(p \wedge q, p). \]

If \( p \neq 0 \), two points \((q, p)\) and \((q', p')\) have the same image under \( \Phi \) if and only if \( p = p' \) and \( p \wedge q = p \wedge q' \), and this last condition holds if and only if \( q - q' \) is some multiple of \( p \). So the inverse image of a four dimensional orbit with \( s = 0 \) is a five dimensional submanifold of \( T^*\mathbb{R}^3 \) where \( \Phi^{-1}(\Phi(q, p)) = \{q + tp, p\} \).

This is a straight line which we may think of as the trajectory of a particle with momentum \( p \). In this case, these straight lines give the null foliation of the co-istropic manifold given by the inverse image of the co-adjoint orbit.
If we want to think of $p \wedge q$ as an “angular momentum”, we find that this has no intrinsic meaning, because we can change our origin of coordinates (i.e. apply a translation by $-q$) to eliminate it. So we can say that $s = 0$ because a standard classical particle has no “intrinsic angular momentum”.

The existence of co-adjoint orbits with $s \neq 0$ suggests that there should be particles with an “intrinsic angular momentum” or “spin”. In fact, an experiment done by Beck in the 1930’s at the suggestion of Einstein showed the the photon does have just such an intrinsic angular momentum measured mechanically.
Here is another important point. If we think of $\|q\|$ as having units of length, then $k = \|p\|$ should have units of inverse length, since $p$ lies in the dual space to the space of $q$.

But in classical mechanics, $p$ or $k$ is measured in terms of an independent unit known as “momentum”. What is the relation between the two?

Let me try to give an answer in the form of historical science fiction:
Examples.

Suppose that mechanics had developed before the invention of clocks, so we could observe the trajectories of particles, their collisions and deflections, but not their velocities. For instance, we might be able to observe tracks in a bubble chamber or on a photographic plate. In the case of light, we might have conducted these experiments before there was an accurate measurement of the velocity of light.

As we have seen, each free classical particle, that is each of the five dimensional orbits of $E(3)$ acting on the six dimensional phase space is parametrized by its value of $k = \|p\|$. In the absence of clocks, we can not measure velocity, so we can not distinguish between a “light particle moving fast” or a “heavy particle moving slowly”. 
The "classical meaning" of Planck's constant.

Without some way of relating momentum to length, we would introduce "independent units" of momentum, perhaps by combining particles in various ways and performing collision experiments. But we would know that the "natural units" should be inverse length. A single experiment, the photoelectric effect, involving an interaction between light and one of our "particles" would give a conversion factor and allow us to write $\|p\| = h/\lambda$. Thus, from our group theoretical point of view, Planck's constant $h$ is a conversion formula from the "independent" units of momentum to the the "natural" units of inverse length. Of course the story did not develop this way, the "conversion factor" was first found between "energy" and "inverse time".
The co-adjoint orbits of the Poincaré group.

Here $V = \mathbb{R}^{1,3}$ and $H$ is the connected component of the identity in $O(1,3)$ or its (double) cover (which is $Sl(2,\mathbb{C})$). We know from our general discussion that we can identify $g^*$, the dual of the Lie algebra of $H \otimes V$, with

$$\bigwedge^2 V \oplus V.$$

We also know that if we write an element of this ten dimensional space as $(\Gamma, P)$ then

$$\| P \|^2 := P \cdot P$$

and

$$\| \Gamma \wedge P \|^2$$

are invariants of the co-adjoint action.
Our first task is to describe the orbits of $H$ acting on $V^* \cong V$. Since the function $\|P\|^2$ is invariant, each orbit must lie on a level set of this function, which can take on positive, zero, or negative values. If we choose a space time splitting in terms of which $P = (E, p)$ we have $P \cdot P = E^2 - c^2 p^2$, where $c$ is the speed of light and $p^2$ is the squared length of the three vector $p$. So there are six types of orbits of $H$ acting on $V^*$:
Examples.

1. Orbits with $\|P\|^2 > 0$ and $E > 0$,
2. Orbits with $\|P\|^2 > 0$ and $E < 0$,
3. The orbits with $\|P\|^2 = 0$ and $E > 0$,
4. The orbit with $\|P\|^2 = 0$ and $E < 0$.
5. Orbits with $\|P\|^2 < 0$, and
6. The single point orbit \{0\}.

For orbits of type 1 or 2, we may write $\|P\|^2 = m^2 c^4$ where $m$ is called the **rest mass**. Then orbits of type 3 or 4 correspond to particles of rest mass zero, while orbits of type 5 correspond to the so-called “tachyons” which are not believed to exist, nor are particles of type 6.
The positive mass co-adjoint orbits.

Let us pass to units in which $c = 1$ and examine the co-adjoint orbits which “live over” the $V$ orbits of type 1. On such a $V$ orbit there will be a unique point of the form $(m, 0)$. The subgroup $H_P$ of $H$ fixing this point will be $SO(3)$ (or its double cover). The co-adjoint orbits of $H_P$ are the spheres of various radii and the origin. By our general method we can subtract off $P \wedge Q$ from $\Gamma$ if necessary so as to arrange that we have $\Gamma \in \wedge^2 \mathbb{R}^3$ so that

$$\| \Gamma \wedge P \|^2 = -m^2 s^2$$

where $s$ is the radius of the sphere describing the co-adjoint orbits of $SO(3)$ (or the origin if $s = 0$).
Mass and spin.

So co-adjoint orbits corresponding to orbits of type 1 of $H$ acting on $V$ are:

- Eight dimensional orbits with $m > 0$ and $s > 0$. These are particles with positive mass and intrinsic spin $s > 0$.
- Six dimensional orbits with $m > 0$ and $s = 0$. These are particles with positive mass and no intrinsic spin.

The classification of co-adjoint orbits “lying over” orbits of type 2 is similar.
Examples.

For orbits “lying over” $H$ orbits of type 3 or 4, it turns out that (by appropriate choice of $P$ the isotropy group $H_P$ is the group $E(2)$ of Euclidean motions in the plane (or its double cover). So we can apply the results above to get the co-adjoint orbits of $H_P$. It turns our that the orbits given by the circles of positive radius do not correspond to known particles. The remaining co-adjoint orbits of $H_P$ are determined by points of $\mathfrak{h}_P^* \sim \mathbb{R}$ and the value $s \in \mathbb{R}$ is (unfortunately) also known as spin.

I will not go into this further here.
Consider the case $G = H \ltimes V$ where $H$ is a subgroup of $Gl(n)$ and $V = S^2(\mathbb{R}^n^*)$ and $a \in H$ acts on $S \in V$ sending it to $(a^{-1})^\dagger S a^{-1}$. The group multiplication in $G$ is thus

$$(a, S)(b, T) = (ab, S + (a^{-1})^\dagger T a^{-1}).$$

I will write a “matrix version” of this multiplication in a different fashion than before. I will write the element $(a, S)$ as the $2n \times 2n$

$$
\begin{pmatrix}
a & 0 \\
S a & (a^{-1})^\dagger
\end{pmatrix}.
$$
The quadrupole moment as part of a moment map.

Then

\[
\begin{pmatrix}
  a & 0 \\
  Sa & (a^{-1})^\dagger
\end{pmatrix}
\cdot
\begin{pmatrix}
  b & 0 \\
  Tb & (b^{-1})^\dagger
\end{pmatrix}
= \begin{pmatrix}
  ab & 0 \\
  Sab + (a^{-1})^\dagger Tb & ((ab)^{-1})^\dagger
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  ab & 0 \\
  Sab + (a^{-1})^\dagger Ta^{-1}ab & ((ab)^{-1})^\dagger
\end{pmatrix}
\]

as required.
The quadrupole moment as part of a moment map.

Relation to the symplectic group.

Recall that the condition for a $2n \times 2n$ matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be symplectic is that

\[
A^\dagger C = C^\dagger A, \quad B^\dagger D = D^\dagger B, \quad \text{and} \quad A^\dagger D - C^\dagger B = I.
\]

So we see that the matrices \( \begin{pmatrix} a & 0 \\ Sa & (a^{-1})^\dagger \end{pmatrix} \) are symplectic. In other words, we have defined an injective homomorphism of $G$ into $Sp(2n)$.

The Lie algebra $\mathfrak{g}$ of $G$ is then injected into $sp(2n)$ by

\[
\nu((\xi, S)) = \begin{pmatrix} \xi & 0 \\ S & -\xi^\dagger \end{pmatrix}.
\]
The quadrupole moment as part of a moment map.

We can use our formula for the moment map of a symplectic representation to compute the moment map of $G$ acting on $\mathbb{R}^{2n}$:

$$\langle \Phi, (\xi, S) \rangle = \frac{1}{2} \omega \left( \begin{pmatrix} q \\ p \end{pmatrix}, \iota((\xi, S)) \begin{pmatrix} q \\ p \end{pmatrix} \right)$$

$$= \frac{1}{2} \omega \left( \begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ S & -\xi^\dagger \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right)$$

$$= \frac{1}{2} q \cdot Sq - \frac{1}{2} q \cdot \xi^\dagger p - \frac{1}{2} p \cdot \xi q.$$
We have shown that

\[ \langle \Phi, (\xi, S) \rangle = \frac{1}{2} q \cdot S q - \frac{1}{2} q \cdot \xi^\dagger p - \frac{1}{2} p \cdot \xi q. \]

If we think of \( \mathbb{R}^{n^*} \otimes \mathbb{R}^n \) as the dual space of \( gl(n) = \mathbb{R}^n \otimes \mathbb{R}^{n^*} \) we can combine the last two terms to get \( -\langle p \otimes q, \xi \rangle \).

The first term is \( \frac{1}{2} \langle q \otimes q, S \rangle \) where we are thinking of the symmetric tensor \( q \otimes q \) as lying in \( S^2(\mathbb{R}^n) \), the dual space of \( V = S^2(\mathbb{R}^{n^*}) \). So

\[ \Phi \left( \begin{pmatrix} q \\ p \end{pmatrix} \right) = (-p \otimes q, \frac{1}{2} q \otimes q). \]
If we had replaced the embedding $G \to Sp(2n)$ given above by

$$(a, S) \mapsto \begin{pmatrix} a & 0 \\ mSa & (a^{-1})^\dagger \end{pmatrix}$$

where $m$ is a real parameter (eventually thought of as a mass) then the entire discussion above goes through with the change that now

$$\Phi \left( \begin{pmatrix} q \\ p \end{pmatrix} \right) = (-p \otimes q, \frac{1}{2}mq \otimes q).$$
Suppose we consider $N$ copies of $\mathbb{R}^{2n}$ with $Sp(2n)$ acting diagonally, and $G$ embedded in the $i$-th copy with parameter $m_i$. The moment map for the induced action is just the sum of the individual moment maps and so sends a point of $\mathbb{R}^{2Nn}$ into

$$- \sum_{i=1}^{N} p_i \otimes q_i + \frac{1}{2} \sum_{i=1}^{N} m_i q_i \otimes q_i.$$ 

If $n = 3$ the term $I := \sum m_i q_i \otimes q_i$ is known as the “moment of inertia” or the “quadrupole moment” or the “inertia tensor” of the $N$ particles. If we restrict to the subgroup $SO(3)$ of $Gl(3)$ then $\sum p_i \otimes q_i$ restricts to $\sum p_i \wedge q_i$, the “total angular momentum.

This is the claim we made earlier - that the total angular momentum and the quadrupole moment could be regarded as components of a moment map.
The liquid drop model.

In the liquid drop model of the nucleus, one imagines that the system behaves as an incompressible fluid, but as in the case of the rigid body, it is only the quadrupole moments that matter and that the group $H$ is now the group of all linear volume preserving transformations, i.e., $H = SL(3; \mathbb{R})$ the group of all matrices $a$ with $\det a = 1$. We want to consider the group $H \ltimes S^2(\mathbb{R}^3^*)$ which is $8+6=14$ dimensional.
The liquid drop.

**Orbits of** $H$ **acting on** $V$.

$a \in H = Sl(3, \mathbb{R})$ sends $S \in V = S^2(\mathbb{R}^3^*)$ into $(a^{-1})^*Sa^{-1}$ when we think of $S$ as a symmetric matrix, so $\det S$ and the signature of $S$ are preserved. So we expect that a typical orbit of $H$ acting on $V$ would be five dimensional. Indeed, if $S$ is positive definite, we can, by an appropriate $a$, bring $S$ to the form $cl$ where $c = (\det S)^{\frac{1}{3}}$.

The isotropy group for $cl$ is $SO(3)$ and the corresponding orbit is $8-3=5$ dimensional. Such an orbit $N$ is the configuration space of a liquid drop whose total volume is $\det Q$. The corresponding orbit, $T^*N$, in $g^*$ is 10 dimensional.
Notice that $z = cl$ is the unique point in $N$ which is left fixed by $SO(3)$; it corresponds to a spherical globule of liquid. The tangent space to $N$ at $z$ is five dimensional and (since $\det S = \text{const on } N$) can be identified the space of traceless symmetric tensors, and hence is irreducible under $SO(3)$, and hence so is $T^*N_z$. The unique point in $T^*N$ that is left fixed by $SO(3)$ is the point $(z, 0)$. 
Suppose that $\mathcal{H}$ is a function on $T^*N$ that is invariant under $SO(3)$. Then the only point where it could have an absolute minimum would be $(z, 0)$. If $\mathcal{H}$ is the sum of kinetic and potential terms, and we take the quadratic approximation $\mathcal{H}_0$ to $\mathcal{H}$ at $(z, 0)$, we see that

$$\mathcal{H}_0 = AP^2 + BQ^2$$

where $P^2$ and $Q^2$ are the (unique up to constant) quadratic forms on $TN_z$ and $T^*N_z$, invariant under $SO(3)$ and $A$ and $B$ are constants. The corresponding collective function is the Bohr-Mottleson Hamiltonian.
Collective and invariant Hamiltonians Poisson commute.

Suppose that $f \in C^\infty(M)$ is a $G$-invariant function on a symplectic manifold $M$ with a Hamiltonian $G$-action. We write this as $f \in C^\infty(M)^G$. If $T_m M \ni v \in g_M(m)$ then $vf = 0$. But we know that if $\mathcal{H}$ is a collective Hamiltonian, and if $X_{\mathcal{H}}$ is the corresponding vector field, then $X_{\mathcal{H}}(m) \in g_M(m)$. Hence $X_{\mathcal{H}}f = 0$, i.e.,

$$\{\mathcal{H}, f\} = 0$$

if $\mathcal{H}$ is collective and $f$ is invariant.
Do they mutually centralize one another?

At any \( m \in M \), the subspace \( g_M(m)^\perp \subset T^*_m M \) is spanned by those covectors which are of the form \( df(m) \) where \( \xi_M f(m) = 0 \) for all \( \xi \in g \). Since \( (g_M(m)^\perp)^\perp = g_M(m) \), this suggests that the collective Hamiltonians and the invariant Hamiltonians mutually centralize one another, i.e. that the collective Hamiltonians are all the Hamiltonians which Poisson commute with all the invariant Hamiltonians and vice versa. This is not quite true without some additional assumptions which I will discuss.

The rest of this lecture is taken from the paper “The centralizer of invariant functions and the division properties of moment maps” by Y. Karshon and E. Lerman, *Ill. J. Math.* 41 #3 (1997), 462-487.
The centralizer of the invariant functions.

**Theorem**

*Suppose that $G$ is compact. The centralizer of the $G$ invariant functions consists of the set of smooth functions which are locally constant on every level set of the moment map $\Phi$.***

The Hamiltonian flow of an invariant function preserves the level sets of $\Phi$, so the Poisson bracket of an invariant function and a function which is locally constant on level sets of $\Phi$ is zero. So the centralizer of the invariant functions contains the functions which are locally constant on level sets of $\Phi$. We must show that there is nothing else in the centralizer.
Let $h$ be a smooth function in this centralizer, and let $\gamma$ be a smooth curve contained in a level set of $\Phi$. Since any two points in a connected component of a level set of $\Phi$ can be joined by a piecewise smooth curve, it is enough to show that the derivative of $t \mapsto h(\gamma(t))$ vanishes for all $t$. This derivative equals $\omega(X_h, \dot{\gamma})$. For any $\xi \in \mathfrak{g}$ we have

$$0 = d\Phi^\xi(\dot{\gamma}) = \omega(\xi_M, \dot{\gamma}),$$

so $\dot{\gamma}(t)$ lies in the symplectic perpendicular to the $G$ orbits for all $t$. So to prove the theorem it is enough to prove that

$$X_h$$

is tangent to the $G$ – orbits.
Proof.

Let \( t \mapsto \sigma(t) \) be an integral curve of \( X_h \). Let \( f \in C^\infty(M)^G \).

\[ 0 = \{ h, f \} = X_h f \] by assumption. But

\[ \frac{d}{dt}(f(\sigma(t))) = (X_h f)\sigma(t)). \]

So \( f \) is constant along \( \sigma \).

As \( G \) is compact, the \( G \)-orbits are compact, hence two distinct \( G \)-orbits are contained in two disjoint open sets, and so the \( G \)-invariant functions separate orbits. Hence \( \sigma \) lies on a fixed \( G \)-orbit and so \( X_h \) is tangent to the \( G \)-orbits.
By the Jacobi identity, the centralizer of any subset of $C^\infty(M)$ is a Poisson subalgebra of $C^\infty(M)$. So a corollary of our theorem is

**Corollary**

*The set of smooth functions which are locally constant on $G$-orbits is a Poisson subalgebra of $C^\infty(M)$.*
If \( f \in C^\infty(M) \) Poisson commutes with all the \( \Phi^\xi, \xi \in \mathfrak{g} \) then \( X^\xi_M f = 0 \) for all \( \xi \in \mathfrak{g} \) which implies that \( f \) is constant under the action of \( G^0 \), the connected component of the identity in \( G \). Of course, it is enough to check this for \( \xi \) ranging over a basis of \( \mathfrak{g} \). So we have proved:

**Proposition**

If \( f \in C^\infty(M) \) Poisson commutes with all the \( \Phi^\xi \) as \( \xi \) ranges over a basis of \( \mathfrak{g} \) then \( f \in C^\infty(M)^{G^0} \).
Three types of “collectives”.

Let $\mathcal{C}^{\infty}(M)^{\Phi}$ denote the set of smooth functions which are constant on level sets of the moment map, let $\mathcal{C}^{\infty}(M)^{\Phi}_{loc}$ denote the set of smooth functions which are locally constant on level sets of the moment map, and let $\Phi^{*}(\mathcal{C}^{\infty}(g^{*}))$ denote the set of collective functions, that is, the pull-back of smooth functions on $g^{*}$ via the moment map. So
\[ \Phi^*(C^\infty(g^*)) \subset C^\infty(M)^\Phi \subset C^\infty(M)^\Phi_{loc}. \quad (\ast) \]

Our Theorem says that (if \( G \) is compact) the centralizer of \( C^\infty(M)^G \) is \( C^\infty(M)^\Phi_{loc} \), and the Proposition says that the centralizer of the collective functions is \( C^\infty(M)^G_0 \). So we can conclude that

**Corollary**

\( C^\infty(M)^\Phi_{loc} \) and \( C^\infty(M)^G_0 \) are mutual centralizers.

If \( G \) is connected, the issue to be studied is: when are the three spaces in \((\ast)\) equal? For this I refer to the paper by Karshon and Lerman.