Consider the system IVP:

\[ s' = +c(1 + s^2), \quad s(0) = 0, \]
\[ c' = -s(1 + c^2), \quad c(0) = 1. \]

1. Derive the relation \( s^2 + s^2c^2 + c^2 = 1 \) (on a connected domain containing zero). Show that near zero, the initial condition \( s(0) = 0 \) is supplemented by the differential equation

\[ s' = (1 - s^4)^{1/2}. \]

2. Prove that this IVP (and hence the original system IVP) has a unique solution in the open disc \( B_r(0) \) of radius \( r = 1/\sqrt{2} \). All of what follows takes place in this disc.

3. Prove that \( s \) is odd and \( c \) is even; prove also that \( s(iz) = is(z) \) and find the corresponding transformation law expressing \( c(iz) \) in terms of \( c(z) \).

4. Prove that

\[ \int_0^{s(z)} \frac{d\sigma}{\sqrt{1 - \sigma^4}} = z. \]

5. Prove that \( s(z)^4 \in \mathbb{R} \) precisely where \( z^4 \in \mathbb{R} \).