Modern Analysis
Midterm practice

Answer FOUR questions; show all work.

1. For each $\lambda \in \Lambda$ let $A_\lambda \subseteq \mathbb{R}$ be nonempty; assume that the union $A = \bigcup_{\lambda \in \Lambda} A_\lambda \subseteq \mathbb{R}$ is bounded above. True or false (proof or counterexample):

$$\sup A = \sup \{\sup A_\lambda : \lambda \in \Lambda\}?$$

2. A rectangular parallelepiped (‘box’) has variable side-lengths $a, b, c$ but fixed surface area; maximize its volume.

3. Let $X$ be a metric space. True or false? (proof or counterexample):
   (i) if $A \subseteq B \subseteq X$ then $A \subseteq \overline{B}$;
   (ii) if $A_n \subseteq X$ for each positive integer $n$ then $\bigcup_{n > 0} A_n = \bigcup_{n > 0} \overline{A_n}$.

4. Let $d$ be a metric on $X$ and assume that there exists $\delta > 0$ such that $d(a, b) \geq \delta$ whenever $a, b \in X$. Determine precisely which subsets of $X$ are:
   (i) open; (ii) compact; (iii) connected.

5. For $1 \leq n \leq N$ let $A_n$ be a compact subset of the metric space $X$. Prove:
   (i) $A_1 \cup \cdots \cup A_N$ is compact; (ii) $A_1 \cap \cdots \cap A_N$ is compact. Can either of these statements be improved (extended, generalized)?