Differential Equations
Test 2 sample

Answer FOUR questions; show all work neatly and clearly.

1. Consider a population that is governed by the logistic equation:

\[ \frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right). \]

(i) Determine the ranges of \( N \) over which the solutions are increasing/decreasing.
(ii) Determine the ranges of \( N \) over which the solutions are concave up/concave down.

2. A population has size \( N \) at time \( t \) and evolves according to the differential equation

\[ \frac{dN}{dt} = r(N - k) \left( 1 - \frac{N}{K} \right)^2 \]

where \( r > 0 \) and \( K > k > 0 \) are constants. (i) Find the equilibrium solutions.
(ii) Determine the ranges of \( N \) over which solutions are increasing/decreasing.
(iii) Sketch the graphs of representative solutions.

3. An exponentially growing population is harvested, so that

\[ \frac{dN}{dt} = rN - h \]

where \( r > 0 \) and \( h > 0 \) are constants. Solve this DE to obtain \( N \) as a function of \( t \). Show that if the initial population size \( N_0 \) is strictly less than \( h/r \) then the population becomes extinct in a finite time \( T \), which should be found.

4. A bullet of mass \( m \) is fired horizontally with initial speed \( v_0 \) in a channel filled with a medium that offers a resistive force equal to \( k \) times its speed. How far does the bullet travel before coming to rest? How long does this journey take?

5. A spherical raindrop loses mass by evaporation as it falls from rest under gravity alone. The rate at which it loses volume is equal to \( k \) times its area. Its initial radius is \( r_0 \); its initial height above ground is \( h_0 \). Decide whether the raindrop completely evaporates before it can reach the ground.