Intermediate Differential Equations
Test 3 sample

Answer FOUR questions and show all work.

1. (i) Write down the Bessel equation of order \( \nu \).
(ii) State a version of the Sturm comparison theorem.

2. Let \( y \) be a solution of the Airy equation \( y'' + xy = 0 \) and let \( \omega > 0 \). Show that each interval of length \( \pi / \omega \) in the interval \( (\omega^2, \infty) \) contains a zero of \( y \).

3. Let \( y \) satisfy the Bessel equation of order \( \nu \). Show that the function \( z \) defined by \( z(x) = x^{-\nu} y(x) \) satisfies the (simpler) differential equation
\[
xyz'' + (2\nu + 1)z' + xz = 0.
\]

4. The modified Bessel functions are defined by
\[
I_\nu(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} x\right)^{2n+\nu}}{n! (n+\nu)!}
\]
for each integer \( \nu \geq 0 \). Establish one of the following two identities
(i) \( \frac{d}{dx}\{x^\nu I_\nu(x)\} = x^\nu I_{\nu-1}(x) \)
(ii) \( \frac{d}{dx}\{x^{-\nu} I_\nu(x)\} = x^{-\nu} I_{\nu+1}(x) \)
and use both to deduce the identity
(iii) \( I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x) \).

5. Seek a solution of the (cylindrical polar) Laplace equation
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0
\]
in the form \( u(r, \theta, z) = R(r)\Theta(\theta)Z(z) \) as follows. (i) Show that \( Z \) satisfies \( Z'' = \lambda Z \) for some constant \( \lambda \). Take this constant to be negative: say \( \lambda = -\omega^2 \).
(ii) Derive corresponding differential equations satisfied by \( \Theta \) and \( R \); also, express the \( R \) equation in terms of the new variable \( u = rw \).