Differential Equations
Test 3 sample

Answer FOUR questions; show all work neatly and clearly.

1. Solve the differential equation $x'' + 4x' + 8x = 0$ with initial conditions $x(0) = 2$ and $x'(0) = -2$.

2. Find the general solution to the differential equation $x'' + 2x' - 8x = 0$. Suppose that $x(0) = 1$; for what value (or values) of $x'(0)$ does the solution $x(t)$ remain bounded for all positive $t$?

3. For each of the following differential equations, write down the appropriate form of a particular solution; do not calculate the ‘undetermined coefficients’.
   (i) $x'' - x' - 2x = t^2 + e^{-t}$;
   (ii) $x'' + 4x = 2e^{2t} + \cos 2t$;
   (iii) $x'' + 2x' + x = e^t + e^{-t}$.

4. A crate of density $\rho$, height $H$ and cross-sectional area $A$ bobs up and down at the surface of a lake having (greater) density $\rho_0$. If the base of the bobbing crate is at depth $h$ (less than $H$) below the surface of the lake, show that

   \[
   \frac{d^2h}{dt^2} = \left(1 - \frac{\rho_0 h}{\rho H}\right)g.
   \]

   Derive the differential equation satisfied by the new variable $x = h - \frac{\rho}{\rho_0}H$ and deduce that the bobbing motion has period $2\pi \sqrt{\frac{H}{\rho_0 g}}$.

5. A mass $m$ is attached to a fixed point by a spring of stiffness $k$, the whole system resting on a frictionless horizontal table. The mass is also acted upon by a force that varies as $F \sin(\omega t)$ so that

   \[
   mx'' + kx = F \sin(\omega t)
   \]

   where $x$ is the displacement from equilibrium. Find $x$ as a function of time $t$ given that $x(0) = 0$ and $x'(0) = 0$. It may be assumed that $\omega^2$ and $\omega_0^2 = k/m$ are different.