Differential Equations
Test 4 sample

Answer FOUR questions and show all work.

In each of the following questions, \( s \) and \( c \) are real-valued functions of a real variable that satisfy the initial value problem

\[ s' = c, \quad c' = s; \quad s(0) = 0, \quad c(0) = 1. \]

1. Show that \( c(t)^2 - s(t)^2 = 1 \) for each \( t \). Why does this imply that \( c(t) > 0 \) for each \( t \)?

2. If also \( f \) and \( g \) satisfy the differential equations \( f' = g, \quad g' = f \) and the initial conditions \( f(0) = 0, \quad g(0) = 1 \), show that \( g = c \) (suggestion: start by differentiating the ratio \( g/c \)) and deduce that \( f = s \).

3. Show that the function \( s \) is odd (that is, \( s(-t) = -s(t) \) always) and that the function \( c \) is even (that is, \( c(-t) = c(t) \) always).

4. Show that \( s \) and \( c \) satisfy the ‘half-angle’ formulae

\[ s(t) = 2s\left(\frac{1}{2}t\right)c\left(\frac{1}{2}t\right) \]
\[ c(t) = c\left(\frac{1}{2}t\right)^2 + s\left(\frac{1}{2}t\right)^2 \]

everywhere.

5. More generally, show that \( s \) and \( c \) satisfy the following addition formulae:

\[ s(t) = s(t-a)c(a) + c(t-a)s(a) \]
\[ c(t) = c(t-a)c(a) + s(t-a)s(a) \]

for every \( t \) and every \( a \).