CHAPTER VI

INTEGRAL REPRESENTATIONS OF BESSEL FUNCTIONS

6·1. Generalisations of Poisson's integral.

In this chapter we shall study various contour integrals associated with Poisson's integral (§§ 2·3, 3·3) and Bessel's integral (§ 2·2). By suitable choices of the contour of integration, large numbers of elegant formulae can be obtained which express Bessel functions as definite integrals. The contour integrals will also be applied in Chapters vii and viii to obtain approximate formulae and asymptotic expansions for \( J_\nu(z) \) when \( z \) or \( \nu \) is large.

It happens that the applications of Poisson's integral are of a more elementary character than the applications of Bessel's integral, and accordingly we shall now study integrals of Poisson's type, deferring the study of integrals of Bessel’s type to § 6·2. The investigation of generalisations of Poisson's integral which we shall now give is due in substance to Hankel*.

The simplest of the formulae of § 3·3 is § 3·3 (4), since this formula contains a single exponential under the integral sign, while the other formulae contain circular functions, which are expressible in terms of two exponentials. We shall therefore examine the circumstances in which contour integrals of the type

\[ z \int_a^b e^{i\omega t} \, T \, dt \]

are solutions of Bessel's equation; it is supposed that \( T \) is a function of \( t \) but not of \( z \), and that the end-points, \( a \) and \( b \), are complex numbers independent of \( z \).

The result of operating on the integral with Bessel’s differential operator \( \nabla_r \), defined in § 3·1, is as follows:

\[
\nabla_r \left\{ z \int_a^b e^{i\omega t} \, T \, dt \right\} = z^{r+1} \left\{ \int_a^b e^{i\omega t} \, T(1 - \nu) \, dt + (2\nu + 1) \int_a^b e^{i\omega t} \, T \, dt \, dt \right\}.
\]

** Math. Ann. 1 (1869), pp. 473—485. The discussion of the corresponding integrals for \( I_r(z) \) and \( K_r(z) \) is due to Schlöfli, Ann. di Mat. (2) 1 (1889), pp. 231—292, though Schlöfli's results are expressed in the notation explained in § 4·19. The integrals have also been examined in great detail by Gubler, Zürich Vierteljahreschrift, XXXIII. (1888), pp. 147—172, and, from the aspect of the theory of the linear differential equations which they satisfy, by Graf, Math. Ann. XLV. (1894), pp. 285—292; LVI. (1903), pp. 435—444. See also de la Vallée Poussin, Ann. de la Soc. Sci. de Bruxelles, XXXI. (1905), pp. 140—143.
by a partial integration. Accordingly we obtain a solution of Bessel's equation if \( T, a, b \) are so chosen that

\[
\frac{d}{dt} \left[ T (t^2 - 1) \right] = (2\nu + 1) T t, \quad \left[ e^{it} T (t^2 - 1) \right]^b_a = 0.
\]

The former of these equations shows that \( T \) is a constant multiple of \((t^2 - 1)^{-\frac{1}{2}}\), and the latter shows that we may choose the path of integration, either so that it is a closed circuit such that \( e^{it} (t^2 - 1)^{-\frac{1}{2}} \) returns to its initial value after \( t \) has described the circuit, or so that \( e^{it} (t^2 - 1)^{\frac{1}{2}} \) vanishes at each limit.

A contour of the first type is a figure-of-eight passing round the point \( t = 1 \) counter-clockwise and round \( t = -1 \) clockwise. And, if we suppose temporarily that the real part of \( \epsilon \) is positive, a contour of the second type is one which starts from \(+\infty\) and returns there after encircling both the points \(-1, +1\) counter-clockwise (Fig. 1 and Fig. 2). If we take \( a, b = \pm 1 \), it is necessary to suppose that \( R(\nu + \frac{1}{2}) > 0 \), and we merely obtain Poisson's integral.

To make the many-valued function \((t^2 - 1)^{-\frac{1}{2}} \) definite, we take the phases of \( t - 1 \) and \( t + 1 \) to vanish at the point \( A \) where the contours cross the real axis on the right of \( t = 1 \).

We therefore proceed to examine the contour integrals

\[
\int_{A}^{(1+1, -1-1)} e^{it} (t^2 - 1)^{-\frac{1}{2}} dt, \quad \int_{+i}^{(-1+1, 1+1)} e^{it} (t^2 - 1)^{\frac{1}{2}} dt.
\]

* It is supposed that \( \epsilon \) has not one of the values \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \ldots \); for then the integrands are analytic at \( \pm 1 \), and both integrals vanish, by Cauchy's theorem.
It is to be observed that, when \( R(z) > 0 \), both integrals are convergent, and differentiations under the integral sign are permissible. Also, both integrals are analytic functions of \( \nu \) for all values of \( \nu \).

In order to express the first integral in terms of Bessel functions, we expand the integrand in powers of \( x \), the resulting series being uniformly convergent with respect to \( t \) on the contour. It follows that

\[
2^{1+\nu} \int_0^{1+\nu} e^{\alpha t} (t^2 - 1)^{-\nu/2} dt = \sum_{m=0}^{\infty} \frac{i^{m} z^{m+\frac{n}{2}}}{m!} \int_0^{1+\nu} t^m (t^2 - 1)^{-\nu/2} dt.
\]

Now \( t^m (t^2 - 1)^{-\nu/2} \) is an even or odd function of \( t \) according as \( m \) is even or odd; and so, taking the contour to be symmetrical with respect to the origin, we see that the alternate terms of the series on the right vanish, and we are then left with the equation

\[
2^{1+\nu} \int_0^{1+\nu} e^{\alpha t} (t^2 - 1)^{-\nu/2} dt = 2 \sum_{m=0}^{\infty} \frac{(-)^{m} z^{m+\frac{n}{2}}}{(2m)!} \int_0^{1+\nu} t^m (t^2 - 1)^{-\nu/2} dt = \sum_{m=0}^{\infty} \frac{(-)^{m} z^{m+\frac{n}{2}}}{(2m)!} \int_0^{1+\nu} u^{m+\frac{n}{2}} (u - 1)^{-\nu/2} du,
\]
on writing \( t = \sqrt{u} \); in the last integral the phases of \( u \) and \( u - 1 \) vanish when \( u \) is on the real axis on the right of \( u = 1 \).

To evaluate the integrals on the right, we assume temporarily that \( R(\nu + \frac{1}{2}) > 0 \); the contour may then be deformed into the straight line from 0 to 1 taken twice; on the first part, going from 0 to 1, we have \( u - 1 = (1 - u) e^{-\pi i} \), and on the second part, returning from 1 to 0, we have \( u - 1 = (1 - u) e^{+\pi i} \), where, in each case, the phase of \( 1 - u \) is zero.

We thus get

\[
\int_0^{1+\nu} u^{m+\frac{n}{2}} (u - 1)^{-\nu/2} du = \left[ e^{-(r-\nu)\pi i} - e^{(r-\nu)\pi i} \right] \int_0^{1+\nu} u^{m+\frac{n}{2}} (1 - u)^{-\nu/2} du = 2i \cos \nu \pi \frac{\Gamma(m+\frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(m + \nu + 1)}.
\]

Now both sides of the equation

\[
\int_0^{1+\nu} u^{m+\frac{n}{2}} (u - 1)^{-\nu/2} du = 2i \cos \nu \pi \frac{\Gamma(m+\frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{\Gamma(m + \nu + 1)}
\]
are analytic functions of \( \nu \) for all values of \( \nu \); and so, by the general theory of analytic continuation, this result, which has been proved when \( R(\nu + \frac{1}{2}) > 0 \), persists for all values of \( \nu \).

\* Modern Analysis, § 5-5. The reader will also find it possible to obtain the result, when \( R(\nu + \frac{1}{2}) < 0 \), by repeatedly using the recurrence formula

\[
\int_0^{1+\nu} u^{m+\frac{n}{2}} (u - 1)^{1+\nu} du = -\frac{m+\nu+n+1}{\nu+\frac{1}{2}} \int_0^{1+\nu} u^{m-\frac{1}{2}} (u - 1)^{\nu+n+\frac{1}{2}} du,
\]
which is obtained by integrating the formula

\[
\frac{d}{du} \left( u^{m+\frac{n}{2}} (u - 1)^{1+\nu} \right) = (m+\nu+n+1) u^{m-\frac{1}{2}} (u - 1)^{\nu+n+\frac{1}{2}} + (\nu+\frac{1}{2}) u^{m-\frac{1}{2}} (u - 1)^{\nu+n-\frac{1}{2}};
\]
the integral is then expressed in terms of an integral of the same type in which the exponent of \( u - 1 \) has a positive real part.
Hence, for all\(^*\) values of \(\nu\),
\[
z^\nu \int_A e^{izt} (t^2 - 1)^r \, dt = 2i \cos \nu \pi \, . \Gamma (\nu + \frac{1}{2}) \sum_{m=0}^\infty \frac{(-)^m z^{r+m}}{(2m)!} \Gamma (m + \frac{1}{2}) = 2^{r+1} i \Gamma (\frac{1}{2}) \Gamma (\nu + \frac{1}{2}) \cos \nu \pi \, . \quad J_r (z).
\]
Therefore, if \(\nu + \frac{1}{2}\) is not a positive integer,
\[
J_r (z) = \frac{\Gamma (\frac{1}{2} - \nu)}{2\pi i \Gamma (\frac{1}{2})} \int_A e^{izt} (t^2 - 1)^r \, dt,
\]
and this is Hankel's generalisation of Poisson's integral.

Next let us consider the second type of contour. Take the contour to lie wholly outside the circle \(|t| = 1\), and then \((t^2 - 1)^r\) is expansible in a series of descending powers of \(t\), uniformly convergent on the contour; thus we have
\[
(t^2 - 1)^r = \sum_{m=0}^\infty \frac{\Gamma (\frac{1}{2} - \nu + m)}{m! \Gamma (\frac{1}{2} - \nu)} \, z^{r-1-2m},
\]
and in the series the phase of \(t\) lies between \(-\frac{3}{4} \pi\) and \(+\frac{1}{4} \pi\).

Assuming\(\dagger\) the permissibility of integrating term-by-term, we have
\[
z^\nu \int_{\infty i} e^{izt} (t^2 - 1)^r \, dt = \sum_{m=0}^\infty \frac{2\pi i}{m! \Gamma (\frac{1}{2} - \nu)} \Gamma (2m - 2\nu + 1) \int_{\infty i} e^{izt} z^{r-2} \, dt.
\]
But
\[
\int_{\infty i} e^{izt} z^{r-2} \, dt = (-\nu - r) e^{-\nu \pi i} \Gamma (2m - 2\nu + 1) \int_{\infty i} e^{izt} \, dt,
\]
where \(\alpha\) is the phase of \(z\) (between \(\pm \frac{1}{4} \pi\)); and, by a well-known formula\(\S\), the last integral equals \(-2\pi i / \Gamma (2m - 2\nu + 1)\).

Hence
\[
z^\nu \int_{\infty i} e^{izt} (t^2 - 1)^r \, dt = \sum_{m=0}^\infty \frac{2\pi i (-)^m e^{-\nu \pi i} \Gamma (\frac{1}{2} - \nu + m)}{m! \Gamma (\frac{1}{2} - \nu) \Gamma (2m - 2\nu + 1)} = 2^{r+1} \pi i \Gamma (\frac{1}{2} - \nu) \int_{\infty} J_r (z),
\]
when we use the duplication formula\(\S\) to express \(\Gamma (2m - 2\nu + 1)\) in terms of \(\Gamma (\frac{1}{2} - \nu + m)\) and \(\Gamma (\frac{1}{2} - \nu + m + 1)\).

* If \(\nu - \frac{1}{2}\) is a negative integer, the simplest way of evaluating the integral is to calculate the residue of the integrand at \(u = 1\).

\(\dagger\) To justify the term-by-term integration, observe that \(\int_{\infty i} e^{izt} \) is convergent; let its value be \(K\). Since the expansion of \((t^2 - 1)^r\) converges uniformly, it follows that, when we are given a positive number \(\epsilon\), we can find an integer \(M\) independent of \(t\), such that the remainder after \(M\) terms of the expansion does not exceed \(\epsilon / K\) in absolute value when \(M \geq M_0\). We then have at once
\[
\left| \int_{\infty i} e^{izt} (t^2 - 1)^r \, dt - \sum_{m=0}^{K-1} \frac{\Gamma (\frac{1}{2} - \nu + m)}{m! \Gamma (\frac{1}{2} - \nu)} \int_{\infty i} e^{izt} z^{r-2} \, dt \right| < \epsilon K^{-1} \int_{\infty i} e^{izt} \, dt = \epsilon,
\]
and the required result follows from the definition of the sum of an infinite series.

\(\S\) Cf. Modern Analysis, § 19-22.

613] INTEGRAL REPRESENTATIONS

Thus, when \( R(z) > 0 \) and \( \nu + \frac{1}{2} \) is not a positive integer,

\[
J_{-\nu}(z) = \frac{\Gamma \left( \frac{1}{2} - \nu \right) e^{\pi i \frac{1}{2}}}{2\pi i \Gamma \left( \frac{1}{2} \right)} \int_{\alpha i}^{(-1+1+)} e^{zt} (t^2 - 1)^{-1} dt.
\]

This equation was also obtained by Hankel.

Next consider

\[
\int_{\alpha i}^{(-1+1+)} e^{zt} (t^2 - 1)^{-1} dt,
\]

where \( \omega \) is an acute angle, positive or negative. This integral defines a function of \( z \) which is analytic when

\[-\frac{1}{2}\pi + \omega < \arg z < \frac{1}{2}\pi + \omega;\]

and, if \( z \) is subject to the further condition that \( |\arg z| < \frac{1}{2}\pi \), the contour can be deformed into the second of the two contours just considered. Hence the analytic continuation of \( J_{-\nu}(z) \) can be defined by the new integral over an extended range of values of \( \arg z \); so that we have

\[
J_{-\nu}(z) = \frac{\Gamma \left( \frac{1}{2} - \nu \right) e^{\pi i \frac{1}{2}}}{2\pi i \Gamma \left( \frac{1}{2} \right)} \int_{\alpha i \exp(-i\omega)}^{(-1+1+)} e^{zt} (t^2 - 1)^{-1} dt,
\]

where \( \arg z \) has any value between \(-\frac{1}{2}\pi + \omega\) and \(\frac{1}{2}\pi + \omega\).

By giving \( \omega \) a suitable value*, we can obtain a representation of \( J_{-\nu}(z) \) for any assigned value of \( \arg z \) between \(-\pi\) and \(\pi\).

When \( R(z) > 0 \) and \( R(\nu + \frac{1}{2}) > 0 \) we may take the contour to be that shown in Fig. 3,

![Fig. 3](image)

in which it is supposed that the radii of the circles are ultimately made indefinitely small.

By taking each straight line in the contour separately, we get

\[
J_{-\nu}(z) = \frac{\Gamma \left( \frac{1}{2} - \nu \right) e^{\pi i \frac{1}{2}}}{2\pi i \Gamma \left( \frac{1}{2} \right)} \left[ \int_{\alpha i}^{0} e^{zt} e^{-2\pi i(\nu - \frac{1}{2})} (1 - t^2)^{-1} dt + \int_{0}^{1} e^{zt} e^{-2\pi i(\nu - \frac{1}{2})} (1 - t^2)^{-1} dt + \int_{1}^{0} e^{zt} e^{-2\pi i(\nu - \frac{1}{2})} (1 - t^2)^{-1} dt + \int_{0}^{\alpha i} e^{zt} e^{2\pi i(\nu - \frac{1}{2})} (1 - t^2)^{-1} dt \right].
\]

* If \( |\omega| \) be increased in a series of stages to an appropriate value (greater than \(\frac{1}{4}\pi\)), a representation of \( J_{-\nu}(z) \) valid for any preassigned value of \( \arg z \) may be obtained.
On bisecting the third path of integration and replacing $t$ in the various integrals by $ut$, $-t$, $\pm t$, $it$ respectively, we obtain a formula for $J_{-\nu}(z)$, due to Gubler, which corresponds to Poisson's integral for $J_{\nu}(z)$; the formula is

$$
J_{-\nu}(z) = \frac{2}{\Gamma(\nu + \frac{1}{2})} \left[ \sin \pi \int_0^\infty e^{-u} (1 + t^2)^{\nu-\frac{1}{2}} \, dt + \int_0^1 \cos (zt + \nu \pi) (1 - t^2)^{\nu-\frac{1}{2}} \, dt \right],
$$

and, if this be combined with Poisson's integral, it is found that

$$
Y_{\nu}(z) = \frac{2}{\Gamma(\nu + \frac{1}{2})} \left[ \int_0^1 \sin (zt) (1 - t^2)^{\nu-\frac{1}{2}} \, dt - \int_0^\infty e^{-u} (1 + t^2)^{\nu-1} \, dt \right],
$$

a formula which was also discovered by Gubler, though it had been previously stated by Weber in the case of integral values of $\nu$.

After what has gone before the reader should have no difficulty in obtaining a formula closely connected with (1), namely

$$
J_{\nu}(z) = \frac{\Gamma(\nu - \frac{1}{2})}{\pi i \Gamma(\nu)} \int_0^{(1+)} (t^2 - 1)^{\nu-\frac{1}{2}} \cos (zt) \, dt,
$$

in which it is supposed that the phase of $t^2 - 1$ vanishes when $t$ is on the real axis on the right of $t=1$.

### 6.11. Modifications of Hankel's contour integrals.

Taking $R(z) > 0$, let us modify the two contours of § 6.1 into the contours shown in Figs. 4 and 5 respectively.

![Fig. 4](image1)

![Fig. 5](image2)

By making those portions of the contours which are parallel to the real


axis move off to infinity (so that the integrals along them tend to zero), we obtain the two following formulae:

\[ J_\nu(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \cdot \left(\frac{1}{2}z\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \times \left[ \int_{1+i\infty}^{(1+i)} e^{zt} (t^2 - 1)^{-\nu - i} dt + \int_{-1+i\infty}^{(-1+i)} e^{zt} (t^2 - 1)^{\nu - i} dt \right], \]

\[ J_{-\nu}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \cdot \left(\frac{1}{2}z\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \times \left[ \int_{1+i\infty}^{(1+i)} e^{zt} (t^2 - 1)^{\nu - i} dt + \int_{-1+i\infty}^{(-1+i)} e^{zt} (t^2 - 1)^{-\nu - i} dt \right]. \]

In the first result the many-valued functions are to be interpreted by taking the phase of \( t^2 - 1 \) to be 0 at \( A \) and to be \( +\pi \) at \( B \), while in the second the phase of \( t^2 - 1 \) is 0 at \( A \) and \( i\pi \) at \( B \).

To avoid confusion it is desirable to have the phase of \( t^2 - 1 \) interpreted in the same way in both formulae; and when it is supposed that the phase of \( t^2 - 1 \) is \( +\pi \) at \( B \), the formula (1) is of course unaltered, while (2) is replaced by

\[ J_{-\nu}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \cdot \left(\frac{1}{2}z\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \times \left[ e^{\pi i} \int_{1+i\infty}^{(1+i)} e^{zt} (t^2 - 1)^{-\nu - i} dt + e^{-\pi i} \int_{-1+i\infty}^{(-1+i)} e^{zt} (t^2 - 1)^{\nu - i} dt \right]. \]

In the last of these integrals, the direction of the contour has been reversed and the alteration in the convention determining the phase of \( t^2 - 1 \) has necessitated the insertion of the factor \( e^{-\pi i(\nu - i)i}. \)

On comparing equations (1) and (3) with § 3-61 equations (1) and (2), we see that

\[ H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \cdot \left(\frac{1}{2}z\right)^\nu}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{1+i\infty}^{(1+i)} e^{zt} (t^2 - 1)^{-\nu - i} dt, \]

\[ H_{-\nu}^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \cdot \left(\frac{1}{2}z\right)^\nu}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{-1+i\infty}^{(-1+i)} e^{zt} (t^2 - 1)^{\nu - i} dt, \]

unless \( \nu \) is an integer, in which case equations (1) and (3) are not independent.

We can, however, obtain (4) and (5) in the case when \( \nu \) has an integral value \( (n) \), from a consideration of the fact that all the functions involved are continuous functions of \( \nu \) near \( \nu = n \). Thus

\[ H_n^{(1)}(z) = \lim_{\nu \to n} H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - n\right) \cdot \left(\frac{1}{2}z\right)^n}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{1+i\infty}^{(1+i)} e^{zt} (t^2 - 1)^{n - i} dt, \]

and similarly for \( H_n^{(0)}(z) \).
As in the corresponding analysis of § 6.1, the ranges of validity of (4) and (5) may be extended by swinging round the contours and using the theory of analytic continuation.

Thus, if $-\frac{1}{2}\pi < \omega < \frac{3}{2}\pi$, we have

$$
H_{(n)}^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right)}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{\infty e^{-i\omega}}^{(1+)} e^{iz(t-1)^{\nu-1}} dt,
$$

while, if $-\frac{3}{2}\pi < \omega < -\frac{1}{2}\pi$, we have

$$
H_{(n)}^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right)}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{\infty e^{i\omega}}^{(-1)} e^{iz(t-1)^{\nu-1}} dt,
$$

provided that, in both (6) and (7), the phase of $z$ lies between $-\frac{3}{2}\pi + \omega$ and $\frac{1}{2}\pi + \omega$.

Representations are thus obtained of $H_{(n)}^{(1)}(z)$ when arg $z$ has any value between $-\pi$ and $2\pi$, and of $H_{(n)}^{(1)}(z)$ when arg $z$ has any value between $-2\pi$ and $\pi$.

If $\omega$ be increased beyond the limits stated, it is necessary to make the contours coil round the singular points of the integrand, and numerical errors are liable to occur in the interpretation of the integrals unless great care is taken. Weber, however, has adopted this procedure, Math. Ann. XXXVII. (1890), pp. 411–412, to determine the formulae of § 3.52 connecting $H_{(n)}^{(1)}(-x)$, $H_{(n)}^{(1)}(-z)$ with $H_{(n)}^{(1)}(z)$, $H_{(n)}^{(1)}(z)$.

Note. The formula $\Re Y_{(n)}(z) = H_{(n)}^{(1)}(z) - H_{(n)}^{(1)}(z)$ makes it possible to express $Y_{(n)}(z)$ in terms of loop integrals, and in this manner Hankel obtained the series of § 3.52 for $Y_{(n)}(z)$; this investigation will not be reproduced in view of the greater simplicity of Hankel's other method which has been described in § 3.52.

6.12. Integral representations of functions of the third kind.

In the formula § 6.11 (6) suppose that the phase of $z$ has any given value between $-\pi$ and $2\pi$, and define $\beta$ by the equation

$$
\arg z = \omega + \beta,
$$

so that $-\frac{1}{2}\pi < \beta < \frac{3}{2}\pi$.

Then we shall write

$$
t - 1 = e^{-iz} z^{-1} (-u),
$$

so that the phase of $-u$ increases from $-\pi + \beta$ to $\pi + \beta$ as $t$ describes the contour; and it follows immediately that

$$
H_{(n)}^{(1)}(z) = \frac{i\Gamma\left(\frac{1}{2} - \nu\right)}{\pi \sqrt{2\pi}} \int_{\exp i\beta}^{(-1+)} e^{-u} (-u)^{\nu-1}(1 + \frac{i\nu}{2z}) e^{i\nu} du,
$$

where the phase of $1 + \frac{1}{z}i/\nu$ has its principal value. Again, if $\beta$ be a given acute angle (positive or negative), this formula affords a representation of $H_{(n)}^{(1)}(z)$ valid over the sector of the $z$-plane in which

$$
-\frac{1}{2}\pi + \beta < \arg z < \frac{3}{2}\pi + \beta.
$$
Similarly, from § 6.11 (7),

\[ H_n^{(0)}(z) = \frac{i\Gamma(\nu + \frac{1}{2})}{\pi \sqrt{2\pi z}} e^{-u} \left( 1 - \frac{i\nu}{2z} \right)^{-\frac{1}{2}} du, \]

where \( \beta \) is any acute angle (positive or negative) and

\[-\frac{1}{2} \pi + \beta < \arg z < \frac{1}{2} \pi + \beta.\]

Since, by § 3.61 (7), \( H_{-\nu}^{(0)}(z) = e^{\pi i} H_{\nu}^{(0)}(z) \), it follows that we lose nothing by restricting \( \nu \) so that \( R(\nu + \frac{1}{2}) > 0 \); and it is then permissible to deform the contours into the line joining the origin to \( \infty \) \( e^{i\beta} \), taken twice; for the integrals taken round a small circle (with centre at the origin) tend to zero with the radius of the circle.\(^\dagger\)

On deforming the contour of (1) in the specified manner, we find that

\[ H_{\nu}^{(0)}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} e^{i(\nu - \pi - i\nu)} \Gamma(\nu + \frac{1}{2}) \int_{0}^{\infty} e^{-u} u^{\nu - \frac{1}{2}} \left( 1 + \frac{i\nu}{2z} \right)^{-\frac{1}{2}} du, \]

where \( \beta \) may be any acute angle (positive or negative) and

\[ R(\nu + \frac{1}{2}) > 0, \quad -\frac{1}{2} \pi + \beta < \arg z < \frac{1}{2} \pi + \beta.\]

In like manner, from (2),

\[ H_{\nu}^{(1)}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} e^{i(\nu - \pi - i\nu)} \Gamma(\nu + \frac{1}{2}) \int_{0}^{\infty} e^{-u} u^{\nu - \frac{1}{2}} \left( 1 - \frac{i\nu}{2z} \right)^{-\frac{1}{2}} du, \]

where \( \beta \) may be any acute angle (positive or negative) and

\[ R(\nu + \frac{1}{2}) > 0, \quad -\frac{1}{2} \pi + \beta < \arg z < \frac{1}{2} \pi + \beta.\]

The results (3) and (4) have not yet been proved when \( 2\nu \) is an odd positive integer. But in view of the continuity near \( \nu = n + \frac{1}{2} \) of the functions involved (where \( n = 0, 1, 2, \ldots \)) it follows, as in the somewhat similar work of § 6.11, that (3) and (4) are true when \( n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \) The results may also be obtained for such values of \( \nu \) by expanding the integrands in terms of descending powers of \( z \), and integrating term-by-term; the formulae so obtained are easily reconciled with the equations of § 3.4.

The general formulae (3) and (4) are of fundamental importance in the discussion of asymptotic expansions of \( J_{\nu}(z) \) for large values of \( |z| \). These applications of the formulae will be dealt with in Chapter VII.

A useful modification of the formulae is due to Schafheitlin.\(^\S\) If we take \( \arg z = \beta \) (so that \( \arg z \) is restricted to be an acute angle), and then write \( u = 2z \cot \theta \), it follows that

\[ H_{\nu}^{(0)}(z) = \frac{2^{\nu + 1} \pi}{\pi \Gamma(\nu + \frac{1}{2})} \int_{0}^{\pi} e^{i(\nu - \pi - i\nu)} \frac{\sin \nu \pi \theta}{\sin 2\nu \pi \theta} e^{-2z \cot \theta} d\theta, \]

\[ H_{\nu}^{(1)}(z) = \frac{2^{\nu + 1} \pi}{\pi \Gamma(\nu + \frac{1}{2})} \int_{0}^{\pi} e^{i(\nu - \pi - i\nu)} \frac{\sin \nu \pi \theta}{\sin 2\nu \pi \theta} e^{-2z \cot \theta} d\theta, \]

* To obtain this formula, write

\[ t + 1 = e^{2z \cot \theta} \sin \nu \pi \theta, \quad t - 1 = 2z (1 - \frac{i\nu}{2z}). \]

† There seems to be no simple direct proof that

\[ \Gamma(\frac{1}{2} - \nu) e^{\pi i} (-u)^{\nu - \frac{1}{2}} \left( 1 + \frac{i\nu}{2z} \right)^{-\frac{1}{2}} du \]

is an even function of \( \nu \).

\( \dagger \) Cf. Modern Analysis, § 12-22.

\( \S \) Journal für Math. cxxiv. (1894), pp. 31—44.
and hence that

\[ J_\nu(x) = \frac{2^{\nu+1}x^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\infty \cos^{\nu-1}\theta \sin(x \nu \theta + \frac{1}{2} \nu \theta) e^{-x \cos \theta} d\theta, \]

\[ Y_\nu(x) = -\frac{2^{\nu+1}x^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\infty \cos^{\nu-1}\theta \cos(x \nu \theta + \frac{1}{2} \nu \theta) e^{-x \cos \theta} d\theta. \]

These formulae, which are of course valid only when \( R(\nu + \frac{1}{2}) > 0 \), were applied by Schafheitlin to obtain properties of the zeros of Bessel functions (§§ 15.32—15.35). They were obtained by him from the consideration that the expressions on the right are solutions of Bessel's equation which behave in the appropriate manner near the origin.

The integral \( \int_0^\infty e^{-x \xi - u^{\nu-1}(1 + u^{2\mu-1})} du \), which is reducible to integrals of the types occurring in (3) and (4) when \( \mu = \nu \), has been studied in some detail by Nielsen, Math. Ann. LIX. (1904), pp. 89—102.

The integrals of this section are also discussed from the aspect of the theory of asymptotic solutions of differential equations by Brajtsow, Warschau Polyt. Inst. Nach. 1902, nos. 1, 2 [Jahrbuch über die Fortschritte der Math. 1903, pp. 575—577].

6.13. The generalised Mehler-Sonine integrals.

Some elegant definite integrals may be obtained to represent Bessel functions of a positive variable of a suitably restricted order. To construct them, observe that, when \( x \) is positive (= \( \omega \)) and the real part of \( \nu \) is less than \( \frac{1}{2} \), it is permissible to take \( \omega = \frac{1}{2} \pi \) in § 6.11 (6) and to take \( \omega = -\frac{1}{2} \pi \) in § 6.11 (7), so that the contours are those shown in Fig. 6. When, in addition, the real part of \( \nu \) is greater than \( -\frac{1}{2} \), it is permissible to deform the contours (after the manner of § 6.12) so that the first contour consists of the real axis from \( +1 \) to \( +\infty \) taken twice while the second contour consists of the real axis from \( -1 \) to \( -\infty \) taken twice.

\begin{figure}[h]
\centering
\includegraphics{fig6.png}
\caption{Fig. 6.}
\end{figure}

We thus obtain the formulae

\[ H_0^{(1)}(x) = \frac{\Gamma(\frac{1}{2} - \nu) \cdot \frac{1}{2} x^\nu}{\pi i \Gamma(\frac{1}{2})} \int_1^\infty e^{ist} (t^2 - 1)^{-\nu-i} dt, \]

\[ H_0^{(1)}(x) = -\frac{\Gamma(\frac{1}{2} - \nu) \cdot \frac{1}{2} x^\nu}{\pi i \Gamma(\frac{1}{2})} \int_1^\infty e^{-ist} (t^2 - 1)^{-\nu-i} dt, \]

the second being derived from § 6.11 (7) by replacing \( t \) by \( -t \).