Sampling real algebraic varieties for topological data analysis

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Sampling real varieties

$$y^2 - x^2(x + 1) = 0$$
Topological data analysis
Persistent homology overview

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Figure: A $(\delta, \epsilon)$ sample of an algebraic curve.
Theorem (Cohen-Steiner et al. 2007, Chazal and Lieutier 2005)

Let $X \subseteq \mathbb{R}^N$ be compact with $\text{reach}(X) > 2(\epsilon + \delta)$. $\beta_p$ is the number of points in above and to the left of $(\epsilon, 2\epsilon + \delta)$ in the Čech diagram for a $(\delta, \epsilon)$ sample of $X$. 
Algorithm overview

Input

- A system of polynomial equations \( f : \mathbb{C}^N \rightarrow \mathbb{C}^{N-d}, \)
  \( f = (f_1, \ldots, f_{N-d}) \)
- A density goal \( \epsilon > 0 \)
- A rectangular region \( R = [a_1, b_1] \times \cdots \times [a_N, b_N] \) to search
- A homotopy continuation error bound \( 0 \leq \delta < \epsilon. \)

Output

A \((\delta, \epsilon)\)-sample of \( V_{\mathbb{R}}(f) \cap R \) that has as few points in the sample as possible.
A subset of related work

Sampling algebraic sets from equations


Sampling from a distribution

- Sampling from the uniform distribution on real algebraic manifolds: Breiding and Marigliano, 2018.

Computing homology for semialgebraic sets

Tools from numerical algebraic geometry

Minimum distance problem

Figure: The minimum distance problem for a curve and point in $\mathbb{R}^2$. 
Tools from numerical algebraic geometry

Homotopy continuation

\[ H(z(t), t) = 0 \]
\[ H = (1 - t)f(z) + tg(z) \]

\( f(z) \)
\( \infty \)
\( C^N \)
nonsingular endpoint
singular endpoint
endgame boundary
finite start points

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Theorem (J. Hauenstein, 2012)

For \( y \in \mathbb{R}^N \) the minimum distance problem can be solved with homotopy continuation. The homotopy is as follows.

\[
H(x, \lambda_0, \ldots, \lambda_{N-d}, t) = \begin{pmatrix}
    f(x) - t \gamma \\
    \lambda_0(x - y) + \sum_{i=1}^{N-d} \lambda_i \nabla f_i(x) \\
    \sum_{i=0}^{N-d} \alpha_i \lambda_i - 1
\end{pmatrix}
\]
Sampling Algorithm

- Pick a point and find the critical points of the minimal distance equations with the variety.
- Record sample points, plus exclusion zone around these new sample points and the test point.
Sampling Algorithm

- Pick a point and find the critical points of the minimal distance equations with the variety.
- Record sample points, plus exclusion zone around these new sample points and the test point.
- Pick another test point do the same. Repeat until sample and exclusion balls cover the space.
Sampling algorithm

Termination

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Example: quartic surfaces in $\mathbb{R}^3$

$$V_1 = V_{\mathbb{R}}(4x^4 + 7y^4 + 3z^4 - 3 - 8x^3 + 2x^2y - 4x^2 - 8xy^2 - 5xy + 8x - 6y^3 + 8y^2 + 4y)$$

$$V_2 = V_{\mathbb{R}}(144x^4 + 144y^4 - 225(x^2 + y^2)z^2 + 350x^2y^2 + 81z^4 + x^3 + 7x^2y + 3x^2 + 3xy^2 - 4x - 5y^3 + 5y^2 + 5y)$$
Example
Deformable pentagonal linkage

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https://arxiv.org/abs/1802.07716

Software
https://github.com/P-Edwards/tdasampling