MTG 4302-5316

Introduction to Topology 1

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Administrative

See the course syllabus and webpage.

Prereq: Mathematical experience: should have
4102/3 Intro to real analysis or 4211/2 Real Anal & Adv. Calc.

Overview of course topics.

What is topology?
What is point-set topology?
What is algebraic topology?

Who are the students in our class?
What are your reasons for enrolling?
Which graduate students may take the first year exam?

Book: "Topology" by James Munkres, 2nd Edition.
I will be reading it; please do the same.

Thanks to Peter Bubenik, whose notes were extremely helpful when preparing my notes!

Thanks to Henry Adams for making his notes available and allowing me to add to them!

Chapter 1: Set Theory and Logic

Section 1: Sets

We'll do a "naive" but accurate treatment of sets.

For the axioms: see the Wikipedia page on Zermelo-Fraenkel set theory (ZFC)

Set	A
Element	$a \in A$
Subset	$B \subset A$ (equivalently $B \subseteq A$)
Proper subset	$B \subsetneq A$

A set can be an element of another set

$A \in \mathcal{A}$ ← collection

Power set $\mathcal{P}(A)$ is the set of all subsets of A.

(Can't consider the set of all sets, instead the class of all sets.)

Need more than ZFC

(Binary) Cartesian product

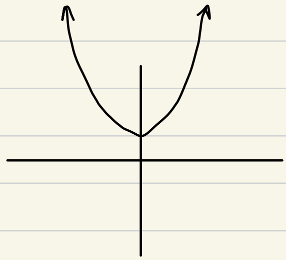
$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

Formally, $(a,b) = \{\{a\}, \{a,b\}\}$

Section 2: Functions

A <u>function</u> $f: X \to Y$ is a subset of $X \times Y$ with each $x \in X$ appearing exactly once as the first coordinate of an ordered pair in this subset.

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2+1$



$\{(x, x^2+1) \in \mathbb{R} \times \mathbb{R}\}$

Formally, $g: \mathbb{R} \to \mathbb{R}_+$, $x \mapsto x^2+1$ is a different function, where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

$f : X \longrightarrow Y$

domain   Codomain

Section 3: Relations

Def A *relation* on a set A is a subset $C \subset A \times A$.

If $(x,y) \in C$, we write xCy and say
"x is related to y" or
"x is in the relation C to y".

An *equivalence relation* $\sim \subset A \times A$ is a relation satisfying
(1) $x \sim x$ (reflexive)
(2) $x \sim y \Rightarrow y \sim x$ (symmetric)
(3) $x \sim y,\ y \sim z \Rightarrow x \sim z$ (transitive)

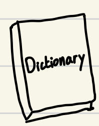
Equivalence classes form a *partition* of A
↑
disjoint nonempty sets whose union is A

An *order relation* $\leq \subset A \times A$ is a relation satisfying
(1) Either $x \leq y$ or $y \leq x$
(2) $x \leq y$ and $y \leq x \Rightarrow x = y$
(3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$

(called a *total order*)

Munkres uses $<$ as the primary symbol,
where $x < y$ when $x \leq y$ and $x \neq y$.

Ex $(\mathbb{R}, \leq)$
Ex $(\mathbb{R}^2, \leq)$ with the lexicographic order
$(a,b) \leq (c,d) \iff a < c$ or $a = c$ and $b \leq d$.



Section 4: Integers and real numbers

Def A binary operation on a set A is a function $f: A \times A \to A$.

There is a set of real numbers $\mathbb{R}$ with binary operations $+$, $\cdot$, and a linear order $\leq$ satisfying a list of axioms.

(Munkres assumes $\mathbb{R}$ exists.)

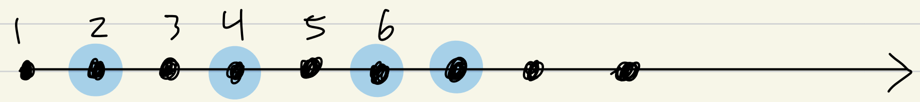
Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Positive integers $\mathbb{Z}_+$
Nonnegative integers $\mathbb{Z}_{\geq 0}$

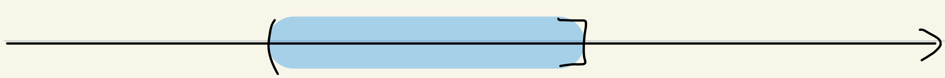
Def A set $A \subset \mathbb{R}$ is inductive if $1 \in A$, and if $x \in A \Rightarrow x+1 \in A$.

Induction principle If $A \subset \mathbb{Z}_+$ is inductive, then $A = \mathbb{Z}_+$.

Munkres uses the Inductive Principle to prove:

Well-ordering property Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.



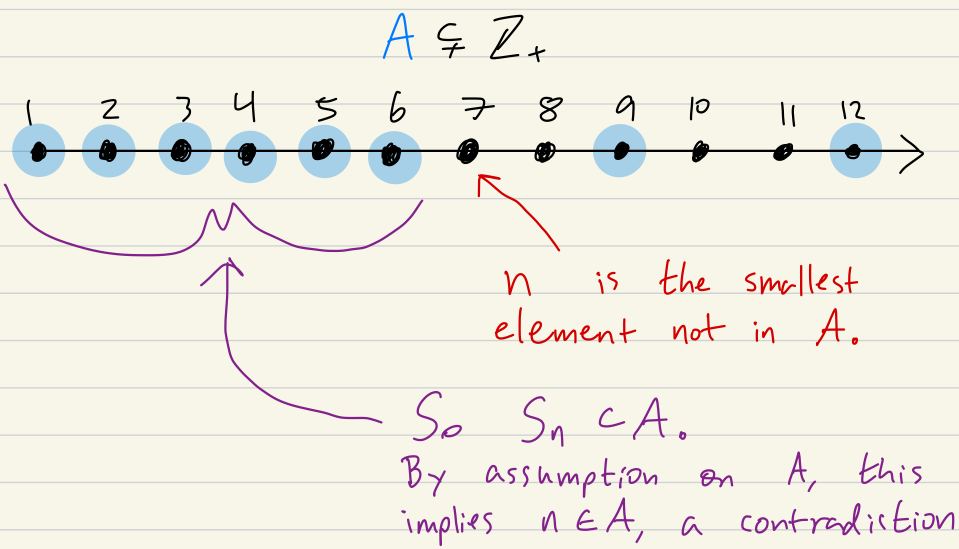


For $n \in \mathbb{Z}_+$, let S_n be the section
$S_n = \{1, 2, \dots, n-1\}$.

So $S_{n+1} = \{1, 2, \dots, n\}$, and $S_1 = \emptyset = \{\}$.

<u>Strong induction principle</u> Let $A \subset \mathbb{Z}_+$, and suppose $S_n \subset A$ implies $n \in A$ for all $n \in \mathbb{Z}_+$. Then $A = \mathbb{Z}_+$.

<u>Pf (using well-ordering)</u> If $A \subsetneq \mathbb{Z}_+$, then let n be the smallest integer in $\mathbb{Z}_+ - A \neq \emptyset$ (by well-ordering). So $S_n \subset A$, which implies $n \in A$, a contradiction. $\square$



Section 5: Cartesian products

Main point: Define the notation $\{A_\alpha\}_{\alpha\in J}$, which is not a collection of sets since we allow repeats.

<u>Def</u> An <u>indexed family of sets</u> $\{A_\alpha\}_{\alpha\in J}$ consists of
- a nonempty collection of sets $\mathcal{A}$
- an <u>indexing set</u> J
- a surjective function $f: J \to \mathcal{A}$.

(for $\alpha \in J$, let $f(\alpha) = A_\alpha$)

We may have $A_\alpha = A_\beta$ for $\alpha \neq \beta$.

<u>Def</u> $\bigcup_{\alpha\in J} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in J\}$

Note this equals $\bigcup_{A\in\mathcal{A}} A = \{x \mid x\in A \text{ for some } A \in \mathcal{A}\}$ since f is surjective.

<u>Def</u> $\bigcap_{\alpha\in J} A_\alpha = \{x \mid x\in A_\alpha \text{ for all } \alpha\in J\}$,

which equals $\bigcap_{A\in\mathcal{A}} A$ since f is surjective.

The two most important examples are when the index set is $J = \{1, 2, \dots, m\}$ or $J = \mathbb{Z}_+$.

$\underline{J = \{1, 2, \dots, m\}}$ or $\{0, 1, \dots, m-1\}$

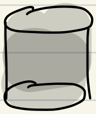
An <u>m-tuple</u> on a set X is a function $\{1, \dots, m\} \to X$, denoted $(x_1, \dots, x_m)$ with $x_i \in X$.

Let $\{A_1, \dots, A_m\}$ be a family of sets indexed by $\{1, \dots, m\}$.
Let $X = \bigcup_{i=1}^m A_i = A_1 \cup \dots \cup A_m$.
The <u>cartesian product</u> $\prod_{i=1}^m A_i = A_1 \times \dots \times A_m$ is

$$\{ \text{m-tuples } (x_1, \dots, x_m) \text{ of } X \mid x_i \in A_i \ \forall i \}.$$

<u>Ex</u> $\mathbb{R}^m$

<u>Ex</u> $S^1 \times [0, 1]$ is a cylinder



equiv class of well-ordered sets

first infinite <u>ordinal</u> $(\mathbb{Z}_+, \leq) \cong (\mathbb{Z}_{\geq 0}, \leq)$

$\underline{J = \mathbb{Z}_+}$ or $\mathbb{Z}_{\geq 0}$

An <u>infinite sequence</u> or <u>ω-tuple</u> is a function $\mathbb{Z}_+ \to X$, denoted $(x_1, x_2, \dots)$ with $x_i \in X$.

Let $\{A_1, A_2, \dots\}$ be a family of sets indexed by $\mathbb{Z}_+$.
Let $X = \bigcup_{i \in \mathbb{Z}^+} A_i$.
The <u>cartesian product</u> $\prod_{i \in \mathbb{Z}^+} A_i$ is

$$\{ \omega\text{-tuples } (x_1, x_2, \dots) \text{ of } X \mid x_i \in A_i \ \forall i \}.$$

<u>Ex</u> $\mathbb{R}^\omega$

Section 6: Finite sets

Def A set A is *finite* if there is a bijection $f: A \xrightarrow{\cong} \{1, \dots, n\}$ for $n \in \mathbb{Z}_+$, "A has cardinality n"
or if A is empty "A has cardinality 0"

Goal: Show the cardinality of a finite set is unique.

Lemma Let A be finite and $a_0 \in A$.
Then $\exists f: A \xrightarrow{\cong} \{1, \dots, n+1\} \iff \exists g: A - \{a_0\} \xrightarrow{\cong} \{1, \dots, n\}$.
See book for proof of this lemma.

Theorem Suppose $f: A \xrightarrow{\cong} \{1, \dots, n\}$ and $B \subsetneq A$.
Then $\nexists\ g: B \xrightarrow{\cong} \{1, \dots, n\}$.

(Book also proves:
And if $B \neq \emptyset$, $\exists\ h: B \xrightarrow{\cong} \{1, \dots, m\}$ for some $m < n$.)

Pf Let $C \subset \mathbb{Z}_+$ be the set of all n for which the theorem is true. We will show C is inductive.

If $n = 1$, then $B = \emptyset$, and $\nexists\ g: \emptyset \xrightarrow{\cong} \{1\}$.

If theorem is true for n, we'll show true for $n+1$.

Let $f: A \xrightarrow{\cong} \{1, \dots, n+1\}$, let $B \subsetneq A$.
If $B = \emptyset$, same as before.
If $B \neq \emptyset$, choose $a_0 \in B$ and $a_1 \in A - B$.
Apply lemma to get $A - \{a_0\} \xrightarrow{\cong} \{1, \dots, n\}$.
Note $B - \{a_0\} \subsetneq A - \{a_0\}$ (consider a_1).
Since the theorem is true for n, $\nexists\ g: B - \{a_0\} \xrightarrow{\cong} \{1, \dots, n\}$.
By lemma, $\nexists$ bijection $B \xrightarrow{\cong} \{1, \dots, n+1\}$. $\square$

Corollary 1 If A is finite, there is no bijection of A with a proper subset of itself.

Pf

$$\begin{tikzcd} A \arrow[r, "\cong", "f"'] \arrow[d, "\cong"', "g"] & B \arrow[dl, "\cong", "g \circ f^{-1}"'] \\ \{1, \dots, n\} & \end{tikzcd}$$

$g \circ f^{-1}$ would contradict Theorem.

Corollary 2 The cardinality of a finite set A is unique.

Pf For $m < n$, suppose we had bijections

$$\begin{tikzcd} A \arrow[r, "\cong", "f"'] \arrow[d, "\cong"', "g"] & \{1, \dots, n\} \\ \{1, \dots, m\} \arrow[ur, "\cong", "g \circ f^{-1}"'] & \end{tikzcd}$$

$g \circ f^{-1}$ would contradict Corollary 1.

Corollary 3 $\mathbb{Z}_+$ is not finite.

$$\begin{aligned} f : \mathbb{Z}_+ &\longrightarrow \mathbb{Z}_+ - \{1\} \\ n &\longmapsto n+1 \end{aligned}$$

is a bijection of $\mathbb{Z}_+$ with a proper subset.

From now on, we'll freely use basic facts about finite sets, such as:

Corollary 4 Set $A \neq \emptyset$ is finite

$\Leftrightarrow$ $\exists$ surjection $\{1, \dots, n\} \twoheadrightarrow A$ for some $n \in \mathbb{Z}_+$.

$\Leftrightarrow$ $\exists$ injection $A \hookrightarrow \{1, \dots, k\}$ for some $k \in \mathbb{Z}_+$.

See book for a proof of this.

Section 7: Countable and uncountable sets

Def A set A is
- infinite if it is not finite
- countably infinite if $\exists$ bijection $A \xrightarrow{\cong} \mathbb{Z}_+$
- countable if it is finite or countably infinite
- uncountable if it is not countable.

Ex $\mathbb{Z}$ is countably infinite.

$$\begin{array}{cccccccccccc}
 & \mathbb{Z}_+ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 2i & 2i+1 \\
\cong & \downarrow f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
 & \mathbb{Z} & 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \cdots & i & -i
\end{array}$$

Lemma If $C \subset \mathbb{Z}_+$ is infinite, then C is countably infinite.

Pf Define $f : \mathbb{Z}_+ \xrightarrow{\cong} C$.
Let $f(1) =$ smallest element of C.
If $f(1), \ldots, f(n-1)$ have been defined,
then let $f(n) =$ smallest element of $C - \{f(1), \ldots, f(n-1)\}$.

This is called a recursive definition. Must do things "in order": certainly can't define $f(n) =$ smallest element of $C - \{f(1), \ldots, f(n)\}$.

f injective: For $m < n$, note $C - \{f(1), \ldots, f(n-1)\}$ contains $f(n)$ but not $f(m)$. So $f(m) \neq f(n)$.

f surjective: Let $c \in C$.

Note $f(\mathbb{Z}_+) \not\subseteq \{1, \dots, c-1\}$ since $\mathbb{Z}_+$ infinite and f injective.

Hence $\exists\, n \in \mathbb{Z}_+$ with $f(n) \geq c$

Let $m \in \mathbb{Z}_+$ be the smallest integer with $f(m) \geq c$.

So $\forall i < m$, $f(i) < c \implies c \notin \{f(1), \dots, f(m-1)\}$

$\implies f(m) \leq c$ by def$^{\underline{n}}$ of f.

Hence $f(m) = c$, as desired. $\square$

Thm For $B \neq \emptyset$, the following are equivalent:

(1) B is countable

(2) $\exists$ surjection $f : \mathbb{Z}_+ \twoheadrightarrow B$

(3) $\exists$ injection $g : B \hookrightarrow \mathbb{Z}_+$.

Pf $(1) \Rightarrow (2)$ B countably infinite By definition, $\exists\, f : \mathbb{Z}_+ \xrightarrow{\cong} B$ ✓

B finite $\{1, \dots, n\} \xrightarrow{\cong} B$ ✓

$$\begin{tikzcd} \{1,\dots,n\} \arrow[r, "\cong"] & B \\ \mathbb{Z}_+ \arrow[u] \arrow[ur] & \end{tikzcd} \qquad m \mapsto \min(m,n)$$

$(2) \Rightarrow (3)$ Given $f : \mathbb{Z}_+ \twoheadrightarrow B$, define $g : B \to \mathbb{Z}_+$ by

$g(b) =$ smallest element of $f^{-1}(b)$. (nonempty since f surjective)

Note g is injective since $b \neq b' \Rightarrow f^{-1}(b) \cap f^{-1}(b') = \emptyset \Rightarrow g(b) \neq g(b')$.

$(3) \Rightarrow (1)$

$$\begin{tikzcd} B \arrow[rr, hook, "g"] \arrow[dr, "\cong"'] & & \mathbb{Z}_+ \\ & \mathrm{image}(B) \arrow[ur, hook] & \end{tikzcd}$$

If image(B) finite ✓

If image(B) infinite — apply last lemma ✓

$\mathrm{image}(B) := \{n \in \mathbb{Z}_+ : n = g(b) \text{ for some } b \in B\}$ $\square$

Corollary $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.

Pf Define $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$

$$(n, m) \longmapsto 2^n \cdot 3^m$$

Note f is injective by the uniqueness of prime factorizations. $\square$

Thm A finite product of countable sets is countable

Pf Proceed by induction. $\square$

Thm $\{0,1\}^\omega$ is uncountable (So the countable product of countable sets need not be countable.)

Pf Recall an element of $\{0,1\}^\omega$ is an infinite tuple $(x_1, x_2, x_3, \ldots)$ with $x_i \in \{0,1\}$.

We show any $g: \mathbb{Z}_+ \longrightarrow \{0,1\}^\omega$ is not surjective.

$$\begin{aligned}
g(1) &= (x_{11}, & x_{12}, && x_{13}, && x_{14}, && \ldots) \\
g(2) &= (x_{21}, & x_{22}, && x_{23}, && x_{24}, && \ldots) \\
g(3) &= (x_{31}, & x_{32}, && x_{33}, && x_{34}, && \ldots) \\
g(4) &= (x_{41}, & x_{42}, && x_{43}, && x_{44}, && \ldots)
\end{aligned}$$

Define $y = (y_1, y_2, y_3, \ldots) \in \{0,1\}^\omega$ by $y_i = \begin{cases} 0 & \text{if } x_{ii} = 1 \\ 1 & \text{if } x_{ii} = 0. \end{cases}$

Note y is not in the image of g. $\square$

Fact $\mathbb{R}$ is uncountable. (Munkres: "decimal expansion proof unsatisfying." Later proof using order properties.)

For A a set, recall the <u>power set</u> $\mathcal{P}(A)$ is the set of all subsets of A.

<u>Thm</u> $\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

This follows from the following stronger theorem:

<u>Thm</u> For A a set,
$\nexists$ a surjection $g: A \twoheadrightarrow \mathcal{P}(A)$
and $\nexists$ an injection $f: \mathcal{P}(A) \hookrightarrow A$

<u>Pf</u> Let $g: A \to \mathcal{P}(A)$.
Let $B = \{a \in A \mid a \in A - g(a)\}$.
If we had $B = g(a_0)$ for some $a_0 \in A$, we'd have
$a_0 \in B \iff a_0 \in A - g(a_0) \iff a_0 \in A - B$.
This is a contradiction.
Hence g is not surjective.

(If $B \neq \emptyset$ and $\exists f: B \hookrightarrow A$, then let $b \in B$ and define $g: A \twoheadrightarrow B$ by letting
$$g(a) = \begin{cases} f^{-1}(a) & \text{for } a \in \mathrm{im}(f), \\ b & \text{for } a \notin \mathrm{im}(f) \end{cases}$$
) $\square$

__Ex__ The set $\mathbb{Q}_+$ of positive rationals is countable.

$$\begin{tikzcd}
 & (n, m) \arrow[r, mapsto] & m/n \\
 & \mathbb{Z}_+ \times \mathbb{Z}_+ \arrow[r, two heads] & \mathbb{Q}_+ \\
\mathbb{Z}_+ \arrow[ur, two heads] \arrow[urr, two heads] & &
\end{tikzcd}$$

__Thm__ A countable union of countable sets is countable.

__Pf__ Let $\{A_n\}_{n \in J}$ be an indexed family of countable sets with J countable.

Get surjections $\mathbb{Z}_+ \xrightarrow{f_n} A_n \quad \forall n$

$\mathbb{Z}_+ \xrightarrow{g} J$

Define surjection

$$\begin{tikzcd}
 & \mathbb{Z}_+ \times \mathbb{Z}_+ \arrow[r, two heads] & \bigcup_{n \in J} A_n \\
 & (k, m) \arrow[r, mapsto] & f_{g(k)}(m) \\
\mathbb{Z}_+ \arrow[uur] & &
\end{tikzcd}$$

Def Sets A, B have the <u>same cardinality</u> if $\exists$ bijection $f: A \xrightarrow{\cong} B$.

(Section 7, Ex 6 + wiki)

Cantor - Schröder - Bernstein Thm

If $\exists$ injections $f: A \hookrightarrow B$ and $g: B \hookrightarrow A$, then A and B have the same cardinality.

Pf Assume WLOG A and B are disjoint.

For $a \in A$ consider

$$\cdots \longmapsto b_{-3} \overset{g}{\longmapsto} a_{-2} \overset{f}{\longmapsto} b_{-1} \overset{g}{\longmapsto} \overset{a}{\overset{\shortparallel}{a_0}} \overset{f}{\longmapsto} b_1 \overset{g}{\longmapsto} a_2 \overset{f}{\longmapsto} b_3 \overset{g}{\longmapsto} a_4 \overset{f}{\longmapsto} \cdots$$

This sequence is uniquely determined!

Similarly for $b \in B$.

Three possibilities: The sequence

(1) Stops at some $b_{-k} \in B$ (<u>B-stopper</u>)

(2) Stops at some $a_{-k} \in A$ (<u>A-stopper</u>)

(3) Is bi-infinite or cyclic.

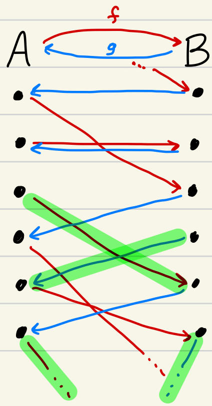
Since f, g injective, these sequences partition $A \amalg B$.

Define $h: A \xrightarrow{\cong} B$ via

$h(a) = f(a)$ if a is in A-stopper seq.

$h(a) = g^{-1}(a)$ if a is in B-stopper seq.

(either works) if a is in bi-infinite/cyclic seq.



$\square$

Sections 9-11

Perhaps the most commonly used axiomatic system for mathematics is Zermelo-Fränkel set theory

- ZFC with Axiom of Choice
- ZF without Axiom of Choice

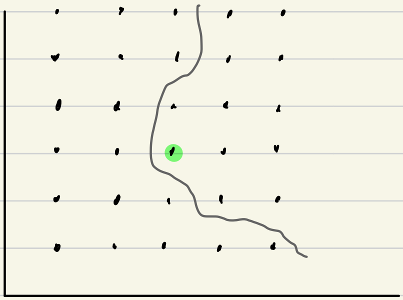
Thm In ZF, the following are equivalent:

- Axiom of Choice (§9)
- Well-ordering theorem (§10)
- Hausdorff maximal principle (§11)
- Zorn's lemma (§11)

Section 10: Well-ordered sets

Def A well-order on a set A is an order relation (total order) s.t. every nonempty subset of A has a smallest element.

Ex $(\mathbb{Z}_+, \leq)$



Ex $(\mathbb{Z}_+ \times \mathbb{Z}_+, \leq \text{ lexicographic})$

Non-Ex $(\mathbb{Z}, \leq)$

Non-Ex $(\mathbb{R}_{\geq 0}, \leq)$ think $(0,1)$

Non-Ex $(\mathbb{Z}_+)^\omega := \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \times \dots$, lexicographic order.
Indeed, consider the set of all sequences with a single entry 2 and all other entries 1: $(1,1,1,1,2,1,1,1,1,\dots)$

Well-ordering theorem Every set has a well-ordering.

Proved by Zermelo in 1904.
Startled mathematical community.
Nobody has constructed specific well-ordering on $(\mathbb{Z}_+)^\omega$.
Proof uses Axiom of Choice.

Section 9: Axiom of Choice

Axiom of Choice Given a collection $\mathcal{A}$ of disjoint nonempty sets, $\exists$ a set C consisting of exactly one element from each set in $\mathcal{A}$.

(I.e., $C \subset \bigcup_{A \in \mathcal{A}} A$, and $|C \cap A| = 1$ for each $A \in \mathcal{A}$.)

Def A choice function on a collection $\mathcal{B}$ of nonempty sets is a function $f : \mathcal{B} \longrightarrow \bigcup_{B \in \mathcal{B}} B$

such that $f(B) \in B$, for all $B \in \mathcal{B}$.

need not be disjoint

Consequence of Axiom of Choice For any collection of nonempty sets, there exists a choice function.

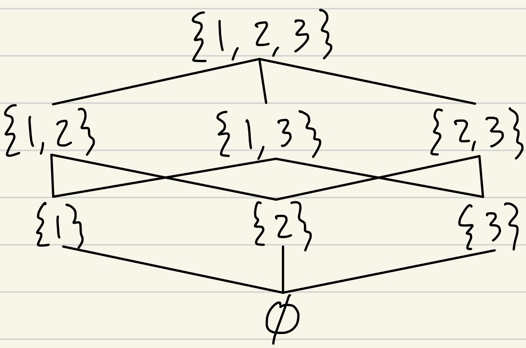
Section 11: Hausdorff maximal principle and Zorn's lemma

Def A <u>partial order</u> $\leq$ on a set S (<u>poset</u>) satisfies

- $a \leq a$
- $a \leq b,\ b \leq a \implies a = b$
- $a \leq b,\ b \leq c \implies a \leq c$

Some pairs of elements may not be comparable ($a \not\leq b$ and $b \not\leq a$ is okay).

Ex Subsets of $\{1,2,3\}$ under inclusion.



Def A <u>chain</u> is a totally ordered subset of a poset.

Ex $\emptyset \subset \{2\} \subset \{1,2,3\}$ is a chain. It is contained in a maximal chain $\emptyset \subset \{2\} \subset \{2,3\} \subset \{1,2,3\}$, for example.

<u>Maximal principle</u> In a poset, every chain is contained in a maximal chain.

Zorn's lemma Let A be a poset. If every chain in A has an upper bound in A, then A has a maximal element.

u s.t. $c \le u$ $\forall c$ in chain

m s.t. $m \le a \Rightarrow m = a$ $\forall a \in A$

Maximal principle implies Zorn's lemma

Let A be a poset in which every chain has an upper bound.
By the Maximum principle, let $B \subset A$ be a maximal chain.
let $u \in A$ be an upper bound for B.
To see u is
maximal in A, note that if $u < v$, then the chain
$B \cup \{v\}$ would contradict the maximality of B.