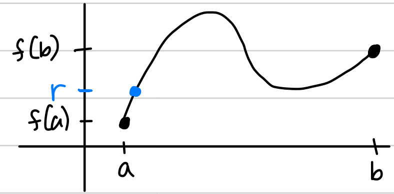
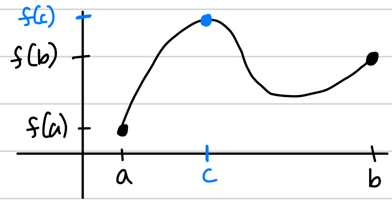
Chapter 3: Connectedness and compactness

Let $f:[a,b]\to\mathbb{R}$ be continuous.



Intermediate value theorem (IVT)

$\forall\ f(a)\le r\le f(b) \quad \exists\ c\in[a,b]$ with $f(c)=r$.



Maximum value theorem

$\exists c\in[a,b]$ s.t. $f(x)\le f(c)\quad \forall x\in[a,b]$.

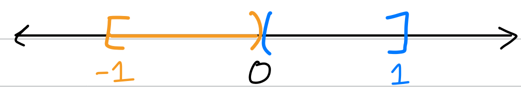
These properties are really due to the fact that $[a,b]$ is connected and compact, respectively.

Section 23: Connected spaces

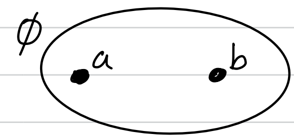
Def X a topological space. A separation of X is a pair U, V of nonempty disjoint open sets whose union is X.
X is connected if it has no separation.

Rmk Equivalently, X is connected if its only clopen subsets are $\emptyset, X$.

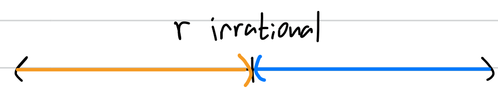
Ex $[-1,0) \cup (0,1]$ has a separation and is not connected



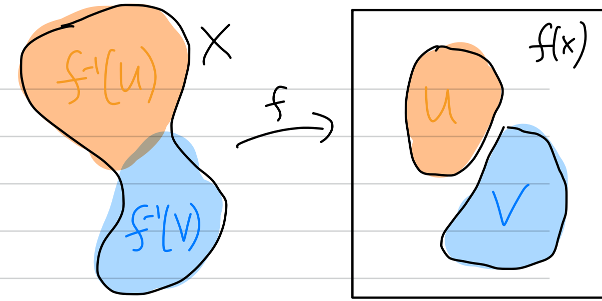
Ex $\{a,b\}$ with the indiscrete topology is connected



Ex $\mathbb{Q}$ is not connected: for r irrational, $(-\infty,r)\cap\mathbb{Q}$, $(r,\infty)\cap\mathbb{Q}$ gives a separation.



Thm If $f: X \to Y$ is continuous and X is connected, then $f(X)$ is connected.



Pf If U, V is a separation of $f(X)$, then $f^{-1}(U), f^{-1}(V)$ is a separation of X (open, nonempty, disjoint, union is X), contradicting the fact X is connected.

Cor "Connected" is a Topological property: If $X \cong Y$, then X connected $\Longleftrightarrow$ Y connected.

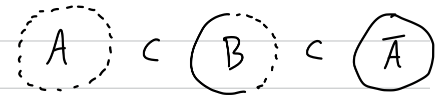
Lemma 1. X top sp $Y \subset X$. Y separated $\Leftrightarrow$ $\exists\, A, B \subset X$, $A \cap B = \emptyset$, $A \cup B = Y$ and neither of A, B contains a limit point of the other.

Pf See book.

Lemma 2 If X has separation U, V and $Y \subset X$ is connected, then $Y \subset U$ or $Y \subset V$.

Pf If both $Y \cap U$ and $Y \cap V$ were nonempty, then these open sets in Y would form a separation of Y.

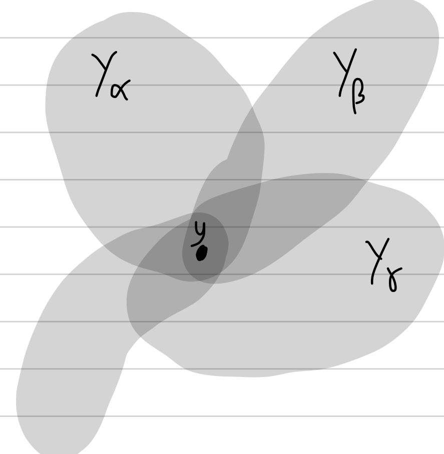
Thm If $A \subset X$ is connected and $A \subset B \subset \overline{A}$, then B is connected.



Pf Uses Lemma 1. See book.

Rmk Adding in a subset of limit points preserves connectedness.

Thm Unions of connected subspaces with a point in common are connected.

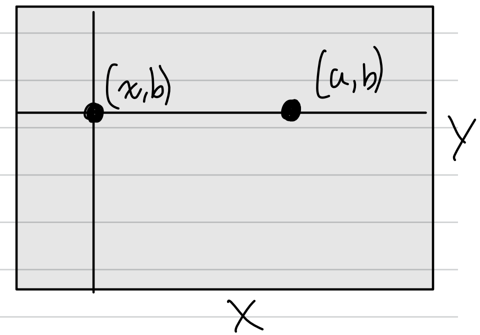


Pf Let $Y = \bigcup_\alpha Y_\alpha$ with $y \in Y_\alpha \underset{\text{connected}}{\subset} X$ $\forall \alpha$.

Assume Y has a separation U, V. Suppose $y \in U$.
Then by Lemma 2 above, $Y_\alpha \subset U$ $\forall \alpha$.
So $Y \subset U$ and V is empty, a contradiction.

Thm A finite product of connected spaces is connected

Pf $n=2$ $X \times Y$ Let $(a,b) \in X \times Y$.



Consider the "horizontal slice" $X \times b = \{(x,y) \in X \times Y \mid y = b\}$

$X \times b \cong X$ and hence is connected.

Let $x \in X$. Similarly the "vertical slice" $x \times Y \cong Y$ is connected.

For $x \in X$, let $T_x = (X \times \{b\}) \cup (\{x\} \times Y)$

T_x is a union of two connected spaces with the point (x,b) in common.

By the previous Theorem, T_x is connected.

$X \times Y = \bigcup_{x \in X} T_x$ is a union of connected spaces with the point (a,b) in common.

By the previous Theorem, $X \times Y$ is connected.

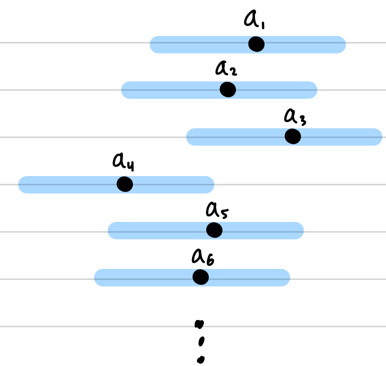
The general case of an n-fold product then follows by induction,

since $X_1 \times \cdots \times X_n \cong (X_1 \times \cdots \times X_{n-1}) \times X_n$. $\square$

Cor $\mathbb{R}^n$ is connected (assuming $\mathbb{R}$ is).

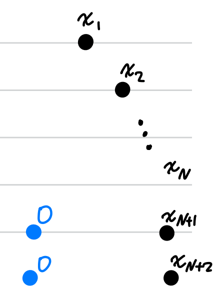
Ex $\mathbb{R}^\omega$ with box topology is not connected.

Pf Let $U = \{\text{bounded sequences}\}$ and $V = \{\text{unbounded sequences}\}$.
They're nonempty, disjoint, and $U \cup V = \mathbb{R}^\omega$. Open in box topology?
Given $a \in \mathbb{R}^\omega$, note $(a_1-1, a_1+1) \times (a_2-1, a_2+1) \times (a_3-1, a_3+1) \times \cdots$
is an open neighborhood of a that is contained in
U (resp. V) if a is bounded (resp. unbounded).
Hence U and V are open in box topology,
and $(\mathbb{R}^\omega, \text{box topology})$ is not connected.



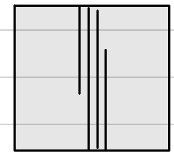
Ex $\mathbb{R}^\omega$ (with the product topology) is connected.

Pf Let $\widetilde{\mathbb{R}}^n = \{x \in \mathbb{R}^\omega \mid x_i = 0 \text{ for } i > n\}$. Connected since $\widetilde{\mathbb{R}}^n \cong \mathbb{R}^n$.
Let $\mathbb{R}^\infty = \bigcup_{n \geq 1} \widetilde{\mathbb{R}}^n$. Connected since union of connected sets containing $(0,0,0,\ldots)$.
We claim $\mathbb{R}^\infty \subset \mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$, giving that $\mathbb{R}^\omega$ is connected. Exercise: HW4
[Indeed, for $x \in \mathbb{R}^\omega$ and $x \in \prod_i U_i$ a basic open set,
$\exists N$ with $U_i = \mathbb{R}$ $\forall i > N$, meaning $(x_1, x_2, \ldots, x_N, 0, 0, \ldots) \in \mathbb{R}^\infty \cap \prod_i U_i$.]



Section 24: Connected subspaces of $\mathbb{R}$

Def A simply ordered set L with more than one element is a <u>linear continuum</u> if
(1) L has the least upper bound property
(2) If $x<y$, then $\exists z$ with $x<z<y$.

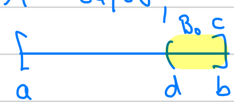


Ex $\mathbb{R}$, $[0,1]\times[0,1]$ with dictionary order.

Thm If L is a linear continuum with the order topology then L is connected and so is every interval and ray in L.

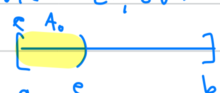
Proof Let $Y\subset L$ be <u>convex</u> ($a,b\in Y$ with $a<b \implies [a,b]\subset Y$).
We will show Y is connected from which the result follows.
Suppose A, B separate Y. Choose $a\in A$, $b\in B$. WLOG assume $a<b$.
Then $A_0 = A\cap[a,b]$, $B_0 = B\cap[a,b]$ separate $[a,b]$. Let $c=\sup A_0$.

Cases: $c\in B_0$

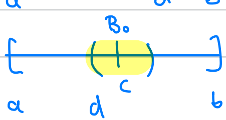


$\Rightarrow\Leftarrow$

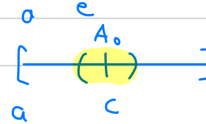
$c\in A_0$



$\Rightarrow\Leftarrow$



$\Rightarrow\Leftarrow$



$\Rightarrow\Leftarrow$ $\square$

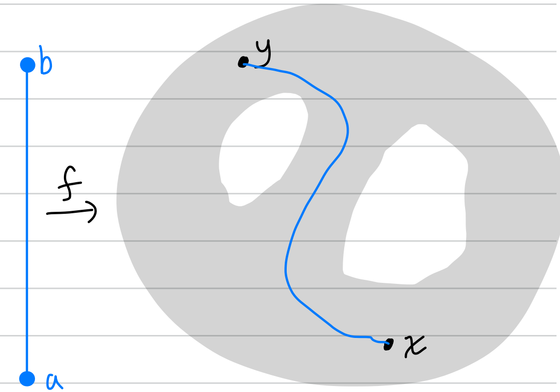
Cor $\mathbb{R}$ is connected, as are intervals and rays in $\mathbb{R}$.

Thm (Intermediate Value Theorem) Let $f: X \to Y$ be continuous, X connected, Y totally ordered with order topology. Let $a, b \in X$, $r \in Y$ with $f(a) < r < f(b)$. Then $\exists\ x \in X$ s.t. $f(x) = r$.

Proof Let $A = f(X) \cap (-\infty, r)$, $B = f(X) \cap (r, \infty)$.
Then $A \cap B = \emptyset$, A, B nonempty since $f(a) \in A$, $f(b) \in B$.
$A, B \underset{\text{open}}{\subset} f(X)$.

Assume $\nexists\ x \in X$ s.t. $f(x) = r$. Then A, B separate $f(X)$.
But X connected $\Rightarrow$ $f(X)$ connected. $\square$

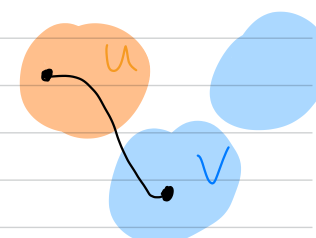
Images of closed intervals in $\mathbb{R}$, which are connected, give a sufficient condition for showing a space is connected:



<u>Def</u> A space X is <u>path connected</u> if every $x, y \in X$ can be joined by a path, i.e. a continuous map $f : [a,b] \to X$ with $f(a) = x$ and $f(b) = y$.

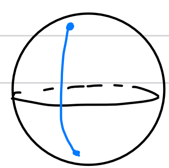
<u>Lemma</u> A path connected space X is connected.

<u>Pf</u> Suppose U, V separate X.
Let $f : [a,b] \longrightarrow X$ be continuous with $f(a) \in U$, $f(b) \in V$.
But the continuous image of the connected set $[a,b]$ is connected, meaning $f([a,b])$ must be contained in U or in V. $\Rightarrow\Leftarrow$ $\square$



<u>Ex</u> Ball $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is path connected
Sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is path connected for $n \geq 1$


B^2

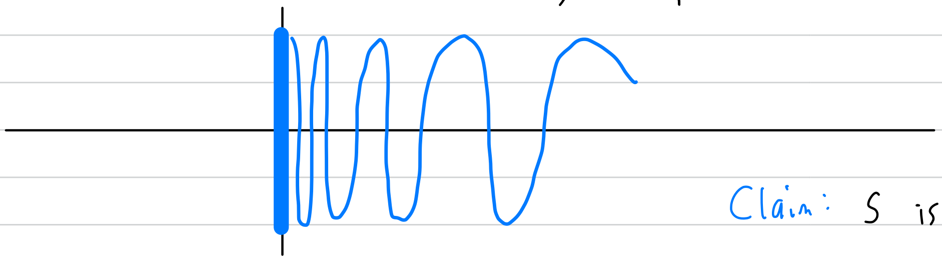

S^2

Ex Topologist's sine curve is $\overline{S} = \{(x, \sin(1/x)) \mid 0 < x \le 1\} \cup \{(0,y) \mid -1 \le y \le 1\}$.

Here $S = \{(x, \sin(1/x)) \mid 0 < x \le 1\}$ is path-connected $\Rightarrow$ S connected.

$\Downarrow$

$\overline{S}$ Connected.



Claim: $\overline{S}$ is not path connected.

Proof Assume $\exists$ path $f: [0,1] \to \overline{S}$ with $f(0) = (0,0)$, $f(1) \in S$.

$f^{-1}\{0 \times [-1,1]\}$ $\underset{\text{closed}}{\subseteq}$ $[0,1]$ has a maximum b. ($\exists$ least upper bound and closed sets contain their limit points)

For convenience, replace $[b,1]$ with $[0,1]$. Let $f(t) = (x(t), y(t))$.

Then $x(0) = 0$ and $x(t) > 0$ for $t > 0$.

For $n \in \mathbb{Z}_+$, choose u_n with $0 < u_n < x(\frac{1}{n})$ s.t. $\sin(\frac{1}{u_n}) = (-1)^n$.

Use IVT to find t_n with $0 < t_n < \frac{1}{n}$ s.t. $x(t_n) = u_n$

Then $t_n \to 0$, but $y(t_n) = (-1)^n$ does not converge.

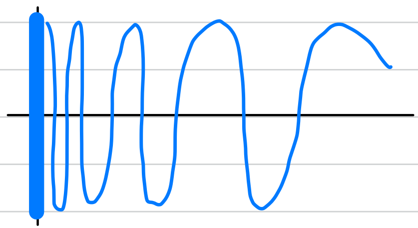
But f is continuous. $\Rightarrow\Leftarrow$. $\square$

Section 25: Components and local connectedness

Def X topological space.
Declaring $x \sim y$ when $\exists$ connected subspace containing x, y gives equivalence relation; equivalence classes called <u>components</u>.

Declaring $x \sim y$ when $\exists$ a path in X from x to y gives equivalence relation; equivalence classes called <u>path components</u>.

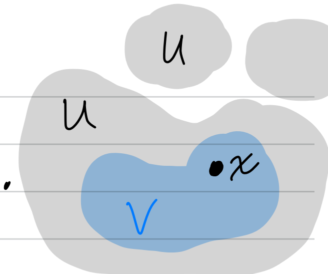
Ex



One component
Two path components

Each path component is connected, hence contained in a single component.

When do components and path components coincide?
X being "locally path connected" suffices.

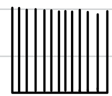


Def A topological space X is <u>locally (path) connected</u> if $\forall x \in X$ and nbhds $U \ni x$, $\exists$ a (path) connected open nbhd $x \in V \subset U$.

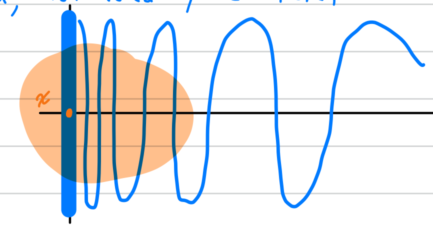
Locally path connected $\Longrightarrow$ locally connected, since path connected $\Longrightarrow$ connected.

<u>Ex</u> Path connected, not locally path connected

$(\mathbb{Q} \cap [0,1]) \times [0,1] \cup [0,1] \times \{0\}$



<u>Ex</u> Connected, not locally connected



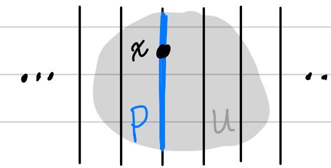
topologist's sine curve

<u>Ex</u>	Connected	Not connected
Locally connected	Intervals in $\mathbb{R}$	$[-1,0) \cup (0,1]$
Not locally connected	Above two examples	$\mathbb{Q}$

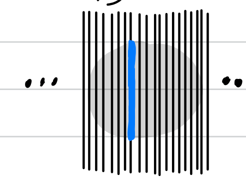
Thm A space X is locally (path) connected $\iff$ $\forall$ open $U \subset X$, each (path) component P of U is open.

Pf Let's prove the path connected version.
($\Rightarrow$) Open $U \subset X$ has path component P.
For $x \in P$, let open nbhd $x \in V \subset U$ be path connected; hence $V \subset P$. So P is open.
($\Leftarrow$) Given open nbhd $U \ni x$, let P be the open path component of U containing x. Note $x \in P \subset U$.
So X is locally path connected.

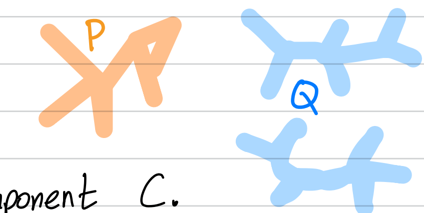
Ex $\mathbb{Z} \times [0,1]$



Non-Ex $\mathbb{Q} \times [0,1]$



Thm If X is locally path connected, then the components and path components coincide.

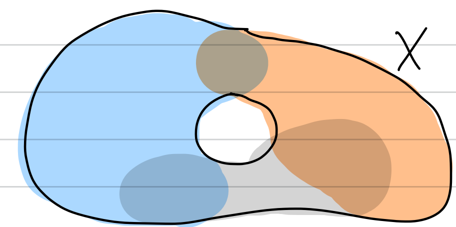


Pf Let P be a path component contained in a component C.
If $P \subsetneq C$, then let Q be the union of all other path components in C.
X locally connected, so prior theorem (connected version with $U = X$) says C open.
Prior theorem (path connected version with $U = C$) then says P and Q are open.
Then $C = P \cup Q$ gives a separation of C. So it must be that $P = C$.

Section 26: Compact spaces

Analogy: (Sets, functions) $\rightsquigarrow$ (Topological spaces, continuous functions)

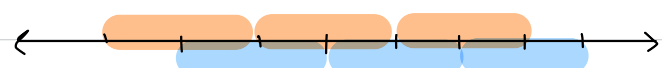
$\cup$ Finite sets $\qquad$ $\cup$ Compact spaces



A cover $\mathcal{U}$ of a topological space X is a collection of subsets whose union is X. If these sets are open, then $\mathcal{U}$ is an <u>open cover</u>.

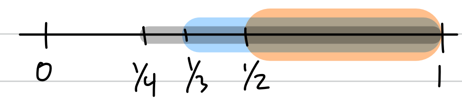
<u>Def</u> A topological space X is <u>compact</u> if every open cover $\mathcal{U}$ has a finite subcover, i.e. a subcollection $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ with $X = U_1 \cup \ldots \cup U_n$.

<u>Ex</u> $\mathbb{R}$ not compact: $\{(n-1, n+1)\}_{n \in \mathbb{Z}}$ is an open cover with no finite subcover.

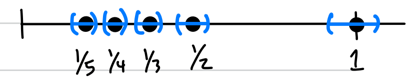


<u>Ex</u> A finite set X is compact (regardless of the topology).
(Given an open cover, for each $x \in X$, choose an open set containing x.)

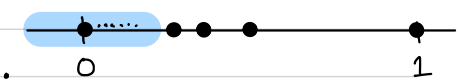
Ex $(0,1]$ not compact: the open cover $\{(1/n, 1]\}_{n\in\mathbb{Z}_+}$ has no finite subcover.



Ex $\{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$ is not compact: the open cover by singletons has no finite subcover.

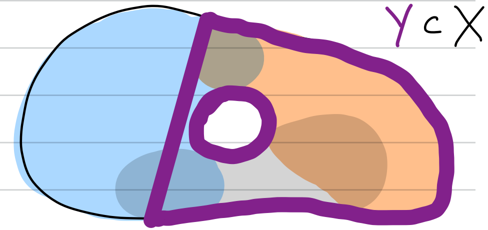


But $\{0\} \cup \{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$ is compact: Given an open cover, note an open set containing 0 contains all but finitely many elements of $\{\frac{1}{n}\}_{n\in\mathbb{Z}_+}$.



Ex We'll see: $X \subset \mathbb{R}^n$ compact $\iff$ X closed and bounded.

We say a collection of sets $\{U_\alpha\}_{\alpha\in J}$ in X covers $Y\subset X$ if $Y\subset \bigcup_\alpha U_\alpha$.

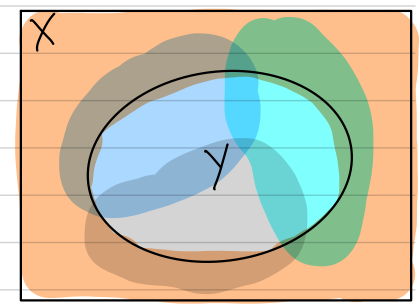


<u>Lemma</u> $Y\subset X$ is compact $\Leftrightarrow$ every cover of Y by open sets in X has a finite subcover

<u>Pf</u> Y compact $\Leftrightarrow$ every cover $\{\underbrace{U_\alpha}_{\text{open in } X}\cap Y\}$ has a finite subcover

$\Leftrightarrow$ every cover $\{\underbrace{U_\alpha}_{\text{open in } X}\}$ has a finite subcover

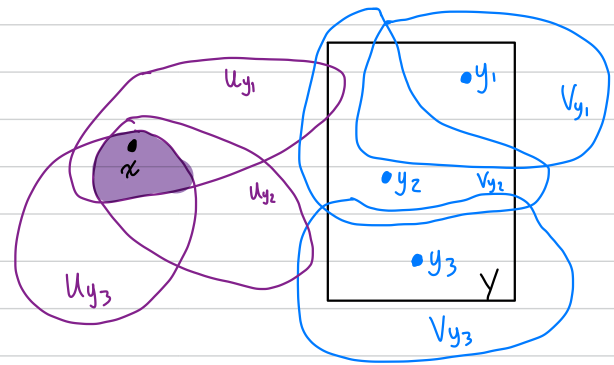
<u>Thm</u> A closed subset Y of a compact space X is compact.



<u>Pf</u> Let $\mathcal{U}$ be a cover of Y by open sets in X.
Then $\mathcal{U}\cup\{X-Y\}$ is an open cover of X.
X compact $\Rightarrow\exists$ finite subcover $\{U_1,\dots,U_n,X-Y\}$ of X.
So $\{U_1,\dots,U_n\}\subset\mathcal{U}$ is a finite subcover of Y.

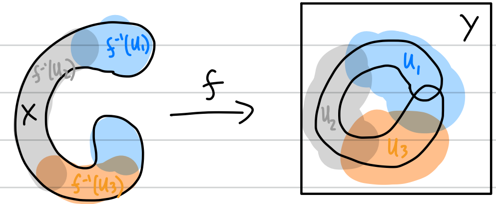
Thm Every compact subspace Y of a Hausdorff space X is closed.

Pf We'll show $X-Y$ is open.
Let $x \in X-Y$. For each $y \in Y$ $\exists$ disjoint opens $V_y \ni y$, $U_y \ni x$.
Y compact $\Rightarrow$ Y has finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$.
Note $Y \subset V_{y_1} \cup \dots \cup V_{y_n}$ is disjoint from the open set $U_{y_1} \cap \dots \cap U_{y_n} \ni x$.
Hence $X-Y$ is open.

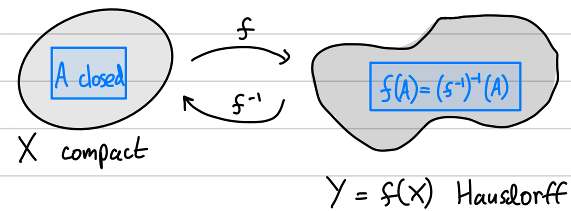


Thm If $f: X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

Pf Let $\mathcal{U}$ be a cover of $f(X)$ by open sets in Y.
Then $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of X.
X compact $\Rightarrow$ $\exists$ finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$ of X.
So $U_1, \dots, U_n$ is a finite subcover of $f(X)$.



Thm If $f: X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then f is a homeomorphism.



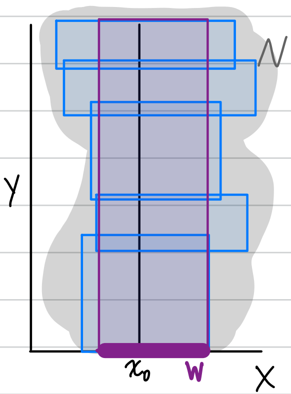
Pf To see that $f^{-1}: Y \to X$ is continuous, note

A closed in $X \Rightarrow A$ compact $\Rightarrow f(A)$ compact $\Rightarrow f(A) = (f^{-1})^{-1}(A)$ closed.

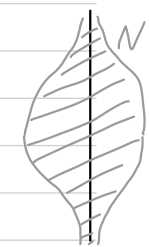
since X compact   since Y Hausdorff

Thm Finite products of compact spaces are compact.
(Tychonoff theorem: Arbitrary products of compact spaces are compact.
One proof in Munkres uses Zorn's lemma, another the well-ordering theorem.)

Tube lemma X space, Y compact. Let $x_0 \times Y \subset N \underset{\text{open}}{\subset} X \times Y$.
Then $\exists\, x_0 \in W \underset{\text{open}}{\subset} X$ with $W \times Y \subset N$.

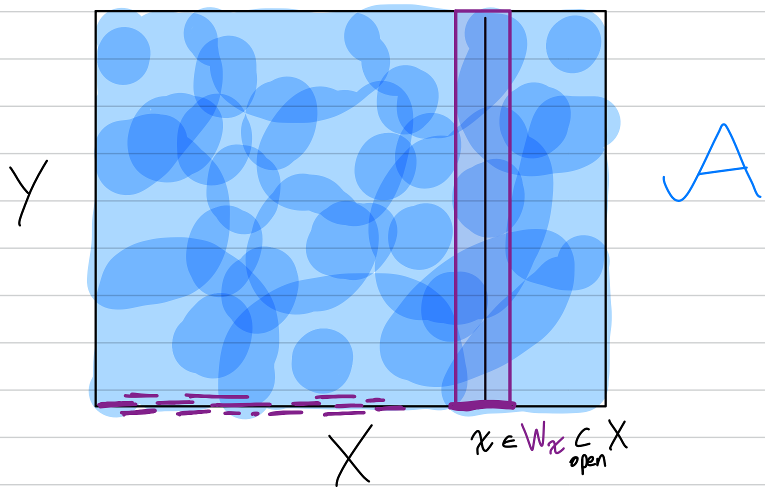


Not:



Pf Cover $\underbrace{x_0 \times Y}_{\cong Y \text{ compact}}$ with basic opens $\{U \times V\}$, each $U \times V \subset N$.
$\exists$ finite subcover $U_1 \times V_1, \dots, U_n \times V_n$ with $x_0 \in U_i \ \forall i$.
Let $W = U_1 \cap \dots \cap U_n$.

Pf theorem For X, Y compact, let $\mathcal{A}$ be open cover of $X \times Y$. (General case is by induction.)
For each $x \in X$, compact $x \times Y$ covered by $A_1, \dots, A_n \in \mathcal{A}$.
Apply lemma with $N = A_1 \cup \dots \cup A_n$:
Get $x \in W_x \underset{\text{open}}{\subset} X$ s.t. $W_x \times Y$ is covered by finitely many sets in $\mathcal{A}$.
X compact $\Rightarrow$ open cover $\{W_x\}$ of X has finite subcover $W_1, \dots, W_k$.
So $W_1 \times Y, \dots, W_k \times Y$ cover $X \times Y$ and are each covered by finitely many sets in $\mathcal{A}$.



Y
A
X
x ∈ W_x ⊂ X
open

Def A collection $\mathcal{C}$ of subsets of X has the <u>finite intersection property</u> (f.i.p.)
if $\forall \{C_1, \dots, C_n\} \subset \mathcal{C}$, $\quad C_1 \cap \dots \cap C_n \neq \emptyset$.

Ex Nested sequence $\quad C_1 \supset C_2 \supset C_3 \supset \dots$

$C_n = [-1/n, 1/n] \subset [-1,1]$ compact $\qquad\qquad C_n = [n, \infty) \subset \mathbb{R}$ not compact

Thm X topological space. Then X compact $\iff$
every collection of closed sets with f.i.p. has nonempty intersection.

Pf X compact $\iff$
For every collection of open sets, no finite subcover implies not a cover.

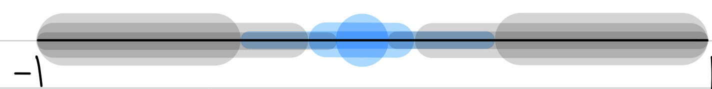
complement $\updownarrow$ $\qquad\qquad \updownarrow \qquad\qquad \updownarrow$

closed sets $\qquad$ f.i.p. $\qquad$ nonempty intersection

Picture $X = [-1,1]$

Open sets $\mathcal{U} = \{[-1, -1/n) \cup (1/n, 1]\}_{n \in \mathbb{Z}_+}$

Closed sets $\mathcal{C} = \{[-1/n, 1/n]\}_{n \in \mathbb{Z}_+}$



Special case of finite intersection property (f.i.p):

$C_1 \supset C_2 \supset C_3 \supset \cdots$ nested sequence of nonempty sets in X

has f.i.p. automatically

X compact $\Rightarrow$ $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

Section 27 Compact subspaces of $\mathbb{R}$

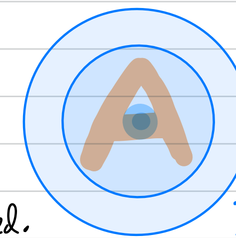
Thm Let X be a totally ordered set with the least upper bound property. Then in the order topology, each closed interval $[a,b]$ is compact.

Pf See book.

Cor Every closed interval $[a,b]$ in $\mathbb{R}$ is compact.

Recall $A \subset \mathbb{R}^m$ is **bounded** if $A \subset B_M(\vec{0}) = \{x \in \mathbb{R}^m \mid \|x\| < M\}$ for some M.

(Equivalently, $d(a,a') \leq N$ for some $N \in \mathbb{R}$.)

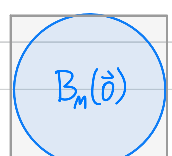


$B_r(\vec{0})$

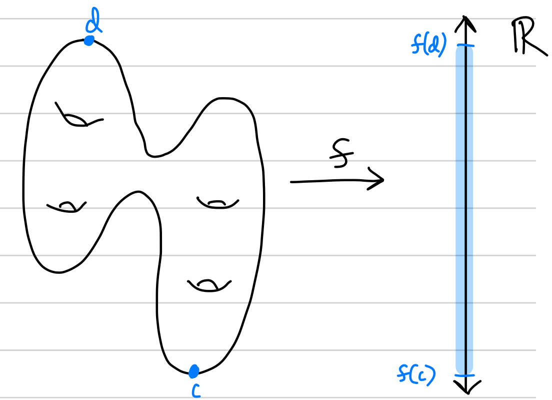
Heine-Borel Thm $A \subset \mathbb{R}^m$ is compact $\iff$ it is closed and bounded.

Pf ($\Rightarrow$) A compact means A closed (since $\mathbb{R}^m$ is Hausdorff).
Also, the open cover $\{B_r(\vec{0})\}_{r>0}$ of A has a finite subcover, so A bounded.

($\Leftarrow$) A bounded means $A \subset B_M(\vec{0}) \subset [-M,M]^m$, which is compact as a finite product of the compact space $[-M,M]$. So A is a closed subset of a compact space, hence compact.



$[-M,M]^2$



Extreme value theorem

If $f: X \to \mathbb{R}$ is continuous and X is compact, then $\exists$ $c, d \in X$ with $f(c) \leq f(x) \leq f(d)$ $\forall x \in X$.

<u>Pf</u> X compact, f continuous $\Rightarrow$ $f(X)$ compact
$\Rightarrow$ $f(X)$ closed and bounded (Heine–Borel).
The least upper bound of $f(X)$ must be in $f(X)$:
if not, it would then be a limit point of $f(X)$,
but closed sets contain their limit points.

Similarly for the greatest lower bound.

<u>Def</u> (X, d) metric space $A \subset X$ $d_A : X \to \mathbb{R}$ given by $d_A(x) = \inf \{ d(x,a) \mid a \in A \}$

<u>Remark</u> d_A is continuous (easy, see book)

<u>Def</u> (X, d) metric space $A \subset X$. The <u>diameter</u> of A is given by $\operatorname{diam}(A) = \sup \{ d(a, a'), a, a' \in A \}$.

Lebesgue number lemma Let (X,d) be a compact metric space with open cover $\mathcal{A}$. Then $\exists\, \delta > 0$, called the **Lebesgue number** for $\mathcal{A}$, such that any subset of X with diameter less than δ is contained in some element of $\mathcal{A}$.

Proof If $X \in \mathcal{A}$ then any δ works and we are done. Assume $X \notin \mathcal{A}$.
Choose a finite subset $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X.
For $i = 1, \ldots, n$, let $C_i := X - A_i$.
Define $f : X \to \mathbb{R}$ by $f(x) = \frac{1}{n} \sum_{i=1}^{n} d_{C_i}(x)$.
Claim: $\forall x \in X$ $f(x) > 0$. Let $x \in X$. $\exists\, A_i$ s.t. $x \in A_i$. The $\exists$ ε-ball containing x and contained in A_i. Hence $d_{C_i}(x) \geq \varepsilon$. Therefore $f(x) \geq \frac{\varepsilon}{n}$. ✓
f cont, X cpct $\Rightarrow$ $f(X)$ cpct $\Rightarrow$ $f(X)$ has a minimum value δ.

Show δ works: Let $B \subset X$ with $\operatorname{diam}(B) < \delta$. Let $x_0 \in B$.
Then $B \subset B_\delta(x_0)$. $\delta \leq f(x_0) \leq d_{C_m}(x_0)$ where $d_{C_m}(x_0) = \max\{d_{C_i}(x_0)\}_{i=1}^{m}$
Therefore $B \subset B_\delta(x_0) \subset A_m$. $\square$

Def $f: (X, d_X) \to (Y, d_Y)$ <u>uniformly continuous</u> if given $\varepsilon > 0$ $\exists \delta > 0$ s.t. whenever $x, x' \in X$ with $d_X(x, x') < \delta$, $d_Y(fx, fx') < \varepsilon$.

Theorem $f: (X, d_X) \to (Y, d_Y)$, X cpct $\Rightarrow$ f uniformly continuous.

Proof Let $\varepsilon > 0$. $\{B_{\frac{\varepsilon}{2}}(y)\}_{y \in Y}$ covers Y. $\{f^{-1}(B_{\frac{\varepsilon}{2}}(y))\}_{y \in Y}$ covers X.

Let δ be the Lebesgue number. Then for $x, x' \in X$ w/ $d_X(x, x') < \delta$ $\{x, x'\} \subset f^{-1}(B_{\frac{\varepsilon}{2}}(y_0))$ for some y_0.

ie. $f(x), f(x') \in B_{\frac{\varepsilon}{2}}(y_0)$ so $d_Y(fx, fx') < \varepsilon$. $\square$

Def $x \in X$ is an <u>isolated point</u> of X if $\{x\} \underset{\text{open}}{\subset} X$.

Theorem Let X be a nonempty compact Hausdorff space with no isolated points. Then X is uncountable.

Cor Any closed interval in $\mathbb{R}$ is uncountable. ($[a, b]$ with $a < b$)

<u>Pf of Thm</u> Claim: Given nonempty $U \underset{\text{open}}{\subset} X$ and $x \in X$ $\exists$ nonempty $V \underset{\text{open}}{\subset} U$ s.t. $x \notin \overline{V}$.

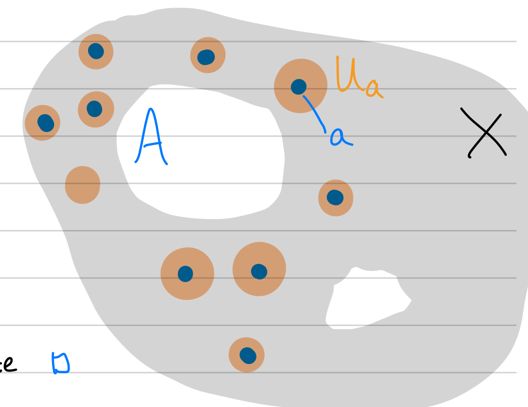
Let $y \in U$ s.t. $y \neq x$. (If $x \notin U$, U nonempty. If $x \in U$, x not isolated.)
$\exists$ disjoint open sets $W_1 \ni x$, $W_2 \ni y$.
Then $V = W_2 \cap U$ is the desired set. ($V \subset U$, $y \in V$, $x \notin \overline{V}$)

We will show $\nexists$ surjective $f: \mathbb{Z}_+ \to X$ (thus X is uncountable)
Let $f: \mathbb{Z}_+ \to X$. Let $x_n = f(n)$.
Apply Claim above to X and x_1 to get nonempty $V_1 \underset{\text{open}}{\subset} X$ s.t. $x_1 \notin \overline{V_1}$.
Given nonempty $V_{n-1} \underset{\text{open}}{\subset} X$, apply Claim to V_{n-1} and x_n to get
nonempty $V_n \underset{\text{open}}{\subset} V_{n-1}$ s.t. $x_n \notin \overline{V_n}$.
Get $\overline{V_1} \supset \overline{V_2} \supset \cdots$ nonempty closed set
X cpct $\Rightarrow$ $\exists$ $x \in \bigcap_{n=1}^{\infty} \overline{V_n}$ $\therefore$ $x \neq x_n$ $\forall n$ $\therefore$ f not onto. $\square$

Section 28: Limit point compactness

Def Topological space X is limit point compact if every infinite set has a limit point.
(Also called Bolzano–Weierstrass property.)

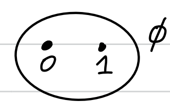
Recall x is a limit point of $A \subset X$ if every nbhd of x intersects A at some point other than x.



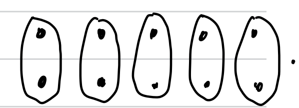
Thm X compact $\Rightarrow$ X limit point compact.

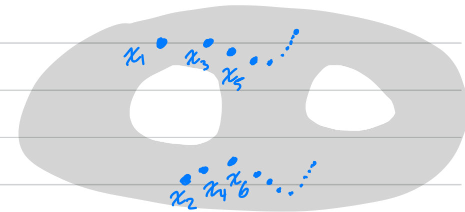
Pf If $A \subset X$ has no limit points, we'll show A finite.
A is closed (contains all its limit points) and hence compact.
$\forall a \in A$, $\exists$ nbhd $U_a \ni a$ s.t. $U_a \cap A = \{a\}$
(otherwise a is a limit point of A).
The open cover $\{U_a\}_{a \in A}$ of A has a finite subcover $\Rightarrow$ A finite $\square$

Ex Converse not true.



$X = \mathbb{Z}_+$ (discrete topology). $Y = \{0,1\}$ with indiscrete topology.
$X \times Y$ is limit point compact since every nonempty subset has a limit point.
$X \times Y$ is not compact since the open cover $\{n \times Y\}_{n \in \mathbb{Z}_+}$ has no finite subcover.





Def X is <u>sequentially compact</u> if every sequence of points in X has a convergent subsequence.

Thm If X is metrizable, then TFAE:

(1) X compact
(2) X limit point compact
(3) X sequentially compact

Pf (1) $\Rightarrow$ (2) above.

(2) $\Rightarrow$ (3). Let $(x_n)_{n \in \mathbb{Z}_+}$ be a sequence in X. Let $A = \{x_1, x_2, x_3, \ldots\}$.
If A finite, then $\exists$ constant (hence convergent) subsequence.
If A infinite, then $\exists$ limit point a
Hence $\forall n \in \mathbb{Z}_+$, $B_{1/n}(a)$ intersects A in infinitely many points. (Otherwise remove finitely many points from $B_{\frac{1}{n}}(x)$ to get open nbhd of a that doesn't intersect A.)
Hence we can choose a convergent subsequence (x_{n_k}) with $x_{n_k} \in B_{1/n_k}(a)$ $\forall k$.

(3) $\Rightarrow$ (1) (See book) Sketch ① X seq cpct $\Rightarrow$ X satisfies Lebesgue number lemma ② X seq cpct $\Rightarrow$ $\forall \varepsilon > 0$, X has finite cover by ε-balls ③ For open cover with Leb # δ, cover X by f. many $\frac{\delta}{3}$-balls. Use to get f. subcover.

$\square$

Section 29: Local compactness (and one-point compactification)

Compact Hausdorff spaces are nice:

Recall: X cpct Hausdorff $A \subset X$ closed $\Leftrightarrow$ compact

X, Y cpct Hausdorff, $f: X \to Y$ continuous

Then f is closed (sends closed sets to closed sets)

and if f is a bijection, then f^{-1} is continuous.

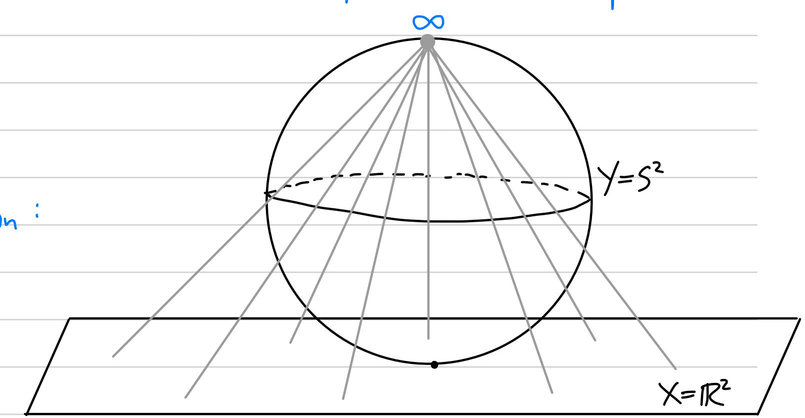
For more general spaces we will consider subsets of compact Hausdorff spaces.

Examples: $n \geq 1$ $(0,1)^n \subset [0,1]^n$

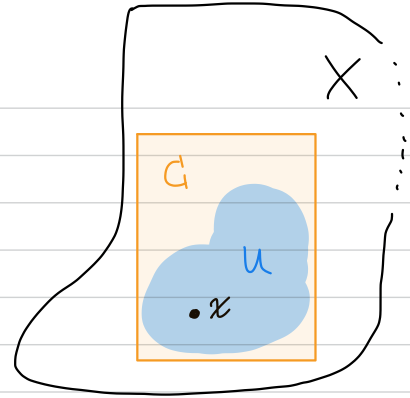
1. $\mathbb{R}^n$: $B_1(0) \subset [-1,1]^n$

$\mathbb{R}^n \subset S^n$

via stereographic projection:

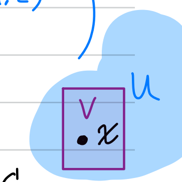


Subspaces of compact spaces are "locally compact":



<u>Def</u> A topological space X is <u>locally compact</u> if $\forall x \in X$, $\exists$ nbhd $U \ni x$ and compact C with $U \subset C \subset X$.

(<u>Rmk</u> If X is Hausdorff, then this definition looks more familiar (see Thm 29.2): X is locally compact $\Longleftrightarrow$ $\forall x \in X$ and nbhd $U \ni x$, $\exists$ nbhd $x \in V \in U$ with $\overline{V}$ compact.)

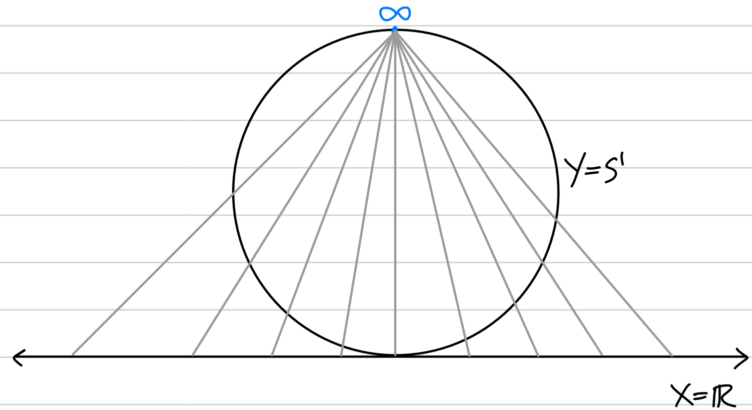


<u>Ex</u> $\mathbb{R}$ is locally compact. $x \in \mathbb{R}$ $U = (x-1, x+1) \subset [x-1, x+1] = C$

<u>Ex</u> $\mathbb{R}^n$ is locally compact. $x \in \mathbb{R}^n$ $U = (x_1-1, x_1+1) \times \dots \times (x_n-1, x_n+1) \subset [x_1-1, x_1+1] \times \dots \times [x_n-1, x_n+1] = C$

<u>Non-Ex</u> $\mathbb{R}^\omega$ not locally compact — basic open sets are not contained in compact sets.

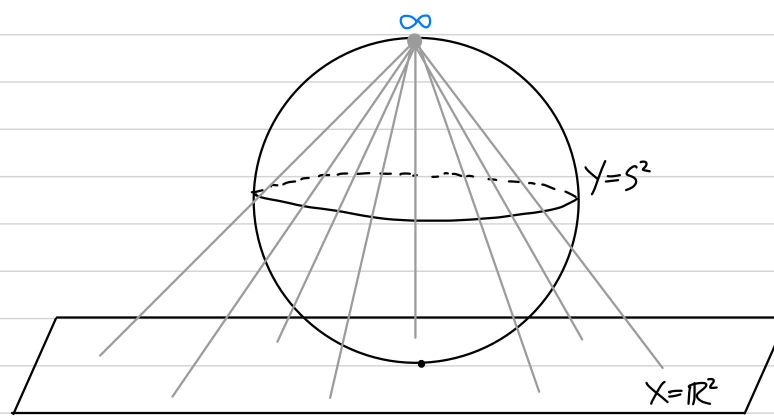
Thm Let X be a topological space.
X is locally compact Hausdorff
$\iff \exists$ a topological space Y s.t.
(1) $X \subset Y$
(2) $Y - X$ is a single point
(3) Y is compact Hausdorff.
Furthermore, for any two such spaces Y, Y', $\exists$ homeomorphism $h: Y \to Y'$ with $h|_X = \mathrm{id}_X$.



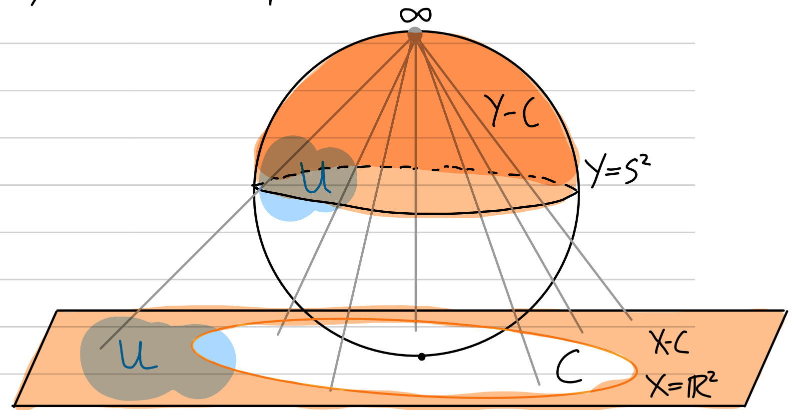
Ex $X = \mathbb{R} \cong (0,1)$
$Y = S^1 \cong [0,1]/_{0\sim 1}$

Ex $X = \mathbb{R}^n$ $\quad Y = S^n$
Special case $X = \mathbb{C}$, $Y = S^2$ Riemann sphere

Y is called the <u>one-point compactification</u> of X.



Recall A closed subspace of a compact space is compact
A compact subspace of a Hausdorff space is closed
Thus for a subspace of a compact Hausdorff space, closed $\Leftrightarrow$ compact.



Pf of thm ($\Rightarrow$) Let $Y = X \cup \{\infty\}$.
Let topology τ for Y consist of:
(a) U, open in X
(b) $Y-C$, C compact in X.

To see that τ is a topology, note:

- $\emptyset$ open in X, $Y = Y - \emptyset$ with $\emptyset$ compact in X
- $U_1 \cap U_2$ open in X

  $(Y-C_1) \cap (Y-C_2) = Y - \underline{(C_1 \cup C_2)}$ finite union of compact sets is compact

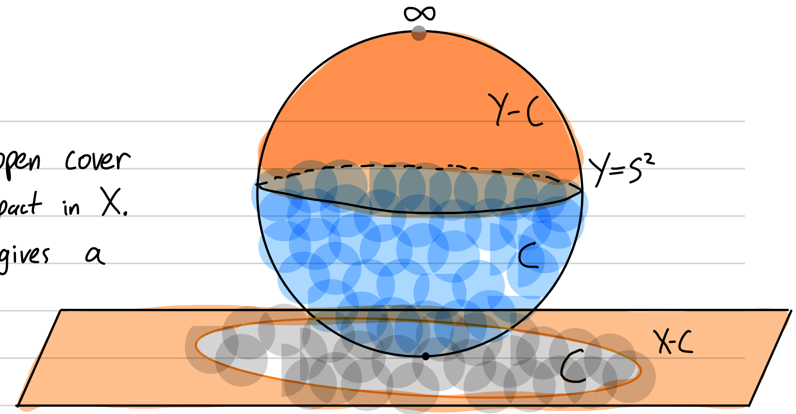
  $U \cap (Y-C) = U \cap \underline{(X-C)}$ open in X C closed in X since X Hausdorff
- $\bigcup_\alpha U_\alpha =: U$ open in X

  $\bigcup_\beta (Y - C_\beta) = Y - \underbrace{\cap_\beta C_\beta}_{=:C}$ arbitrary intersection of compact subspaces of Hausdorff space is compact

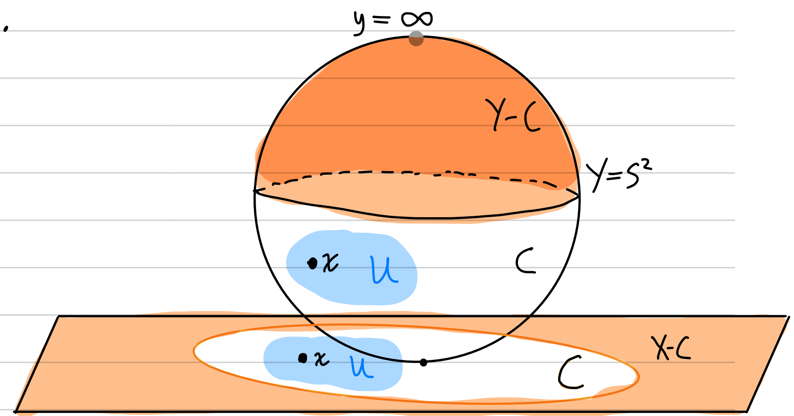
  closed closed

$\bigcup_\alpha U_\alpha \cup \bigcup_\beta (Y - C_\beta) = U \cup (Y-C) = Y - \underline{(C-U)}$ closed subspace of compact C is compact

To see that Y is compact, note if $\mathcal{U}$ is an open cover of Y, then $\mathcal{U}$ has an element $Y-C \ni x$, C compact in X. $\exists$ finite subcover of C, and adding in $Y-C$ gives a finite subcover of Y.



To see that Y is Hausdorff,
note for $x, x' \in X$, can use Hausdorff property of X.
Remaining case is $x \in X$, $y = \infty$.
Since X is locally compact,
choose compact set C containing a nbhd $U \ni x$.
Then $\underset{x}{\underset{\Psi}{U}}$, $\underset{y=\infty}{\underset{\Psi}{X-C}}$ are disjoint opens.



($\Leftarrow$) $X \subset Y = X \cup \{\infty\}$

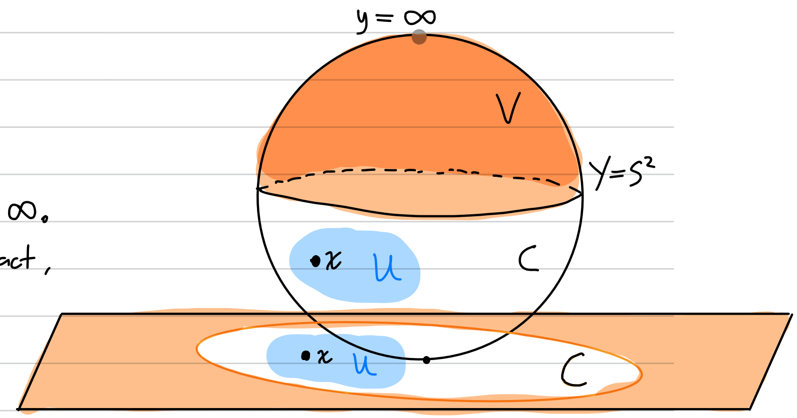
Note Y Hausdorff $\Rightarrow$ X Hausdorff.

Remains to show X locally compact.

For $x \in X$, choose disjoint opens $U \ni x$, $V \ni \infty$.

Let $C = Y - V$. C closed in $Y \Rightarrow C$ compact,

with $U \subset C \subset X$.



It remains to show that Y is unique up to $\cong$.

Define $h: Y \to Y'$ by $h(\infty) = \infty'$, $h(x) = x$ $\forall x \in X$.

One can show h is a homeomorphism (see book). $\square$