Topological Data Analysis and Persistence Theory

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Lecture 5 : Algebra of Persistence Modules

Outline:
1. Algebraic settings for persistence modules
2. Structure Theorem for persistence modules
3. Isometry Theorem

Please interrupt me!!!

1. Algebraic settings for persistence modules

1.1 Discrete persistence modules

1.1.1 As functors

$$[n] \to \mathrm{Vect}, \qquad \mathbb{Z}_{\geq 0} \longrightarrow \mathrm{Vect}, \qquad \mathbb{Z} \longrightarrow \mathrm{Vect}$$

$[n] = (\{0,1,2,\dots,n\}, \leq)$

1.1.2 As representations of quivers

A quiver is a directed graph

$$A_n: \quad 1 \to 2 \to 3 \to 4 \to \cdots \to n$$

A representation of a quiver consists of a vector space for every vertex and a linear map for every edge.

1.1.3 As graded modules over a graded polynomial ring in one variable

A graded vector space : $V \cong \bigoplus_{j \in \mathbb{Z}} V_j$

(over the field $k = \mathbb{Z}/2\mathbb{Z}$)

Example $k[x]$ the polynomial ring in one variable with coefficients in k.

$k[x]$ has elements $1 + x + x^3 + x^6$, $x^5 + x^{11}$

Let x have degree 1

Then $k[x] \cong \bigoplus_{j=0}^{\infty} V_j$ where $V_j = k$ has basis x^j

$V_j = 0$ if $j < 0$

$V_j = k x^j = \{0, x^j\}$

$1 + x + x^3 + x^6$

degree 0 deg 1 deg 3 deg 6

The graded vector space $\bigoplus_{j \in \mathbb{Z}} V_j$ can be made into a graded $k[x]$-module by letting x act on V_j by specifying a linear map $V_j \longrightarrow V_{j+1}$.

1.2 "Continuous" persistence modules

1.2.1 As functors

$\mathbb{R}_{\geq 0} \to \mathrm{Vect}$ $\qquad$ $\mathbb{R} \to \mathrm{Vect}$

1.2.2 As graded modules over a graded polynomial ring in one variable with real coefficients

A $\mathbb{R}$-graded vector space : $\quad V \cong \bigoplus_{a \in \mathbb{R}} V_a$

Example Let U_0 be the monoid $(\mathbb{R}_{\geq 0}, +, 0)$

Consider the monoid ring $k[U_0]$:

it has elements $1 + x^{1/2} + x^{\sqrt{2}} + x^{\pi}$

$k[U_0]$ is $\mathbb{R}$-graded : $k[U_0] \cong \bigoplus_{a \in \mathbb{R}} k[U_0]_a$

$k[U_0]_a = 0$ if $a < 0$ $\qquad$ $k[U_0]_a \cong k$ with basis x^a if $a \geq 0$.

The $\mathbb{R}$-graded vector space $V \cong \bigoplus_{a \in \mathbb{R}} V_a$ may be given the structure of a graded $k[U_0]$-module by specifying an action of $k[U_0]$ on V s.t. if $m \in V_a$ then $x^s \cdot m \in V_{a+s}$.

1.2.3 As sheaves and cosheaves

$(\mathbb{R}, \leq)$ has the Alexandrov topology whose open sets are the "up-sets" $[a,\infty)$ and (a,∞)

Let $\mathrm{Open}(\mathbb{R},\leq)$ be the category of these open sets and morphisms given by inclusion.

Lemma Any functor $M : \mathbb{R} \to \mathrm{Vect}$ may be extended to a functor $\hat{M} : \mathrm{Open}(\mathbb{R},\leq)^{op} \longrightarrow \mathrm{Vect}$

$$\hat{M}[a,\infty) = M(a) \qquad \hat{M}(a,\infty) = \lim_{b\in(a,\infty)} M(b)$$

Proposition $\hat{M}$ is a sheaf.

Dually, $\mathrm{Open}(\mathbb{R},\geq)$ consists of "down-sets" $(-\infty,a]$, $(-\infty,a)$ and inclusions; M extends to $\check{M} : \mathrm{Open}(\mathbb{R},\geq) \to \mathrm{Vect}$; and $\check{M}$ is a cosheaf.

2. The Structure Theorem for Persistence Modules

2.1 Using Linear Algebra

Consider the persistence module

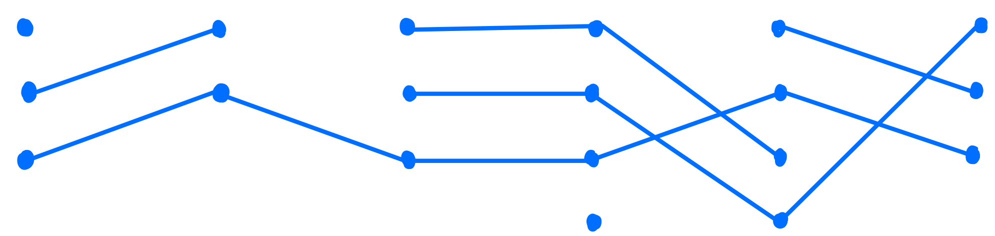
$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} M_n$$

where each M_i is a finite-dimensional vector space.

Structure Theorem There exist bases of the vector spaces M_i such that each linear map is determined by a matching of basis vectors.

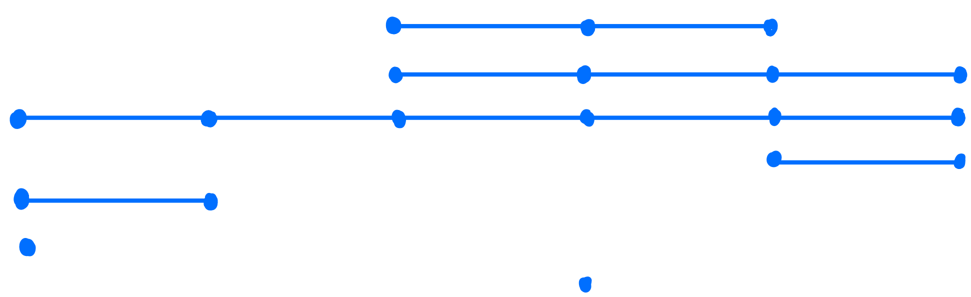
One can prove this using linear algebra.

Example M_0 M_1 M_2 M_3 M_4 M_5



Changing the positions of the vertices, we have:

M_0 M_1 M_2 M_3 M_4 M_5



That is, a persistence module is determined, up to isomorphism, by a collection of intervals, called a barcode.

2.2 Using interval modules

Let vect be the full subcategory of Vect consisting of finite dimensional vector spaces.

Consider $[n] \to \text{vect}$ or $\mathbb{R} \to \text{vect}$

Interval modules

Let I be an interval in $[n]$ or an interval in $\mathbb{R}$.

The interval module with support I, denoted Int_I, is given by $(\text{Int}_I)_a = \begin{cases} k & \text{if } a \in I \\ 0 & \text{if } a \notin I \end{cases}$

and linear maps given by identity maps whenever possible.

Example $\text{Int}_{[2,4]} : [5] \to \text{vect}$

$$\begin{array}{ccccccccccc} 0 & \to & 0 & \to & k & \xrightarrow{1} & k & \xrightarrow{1} & k & \to & 0 \\ 0 & & 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

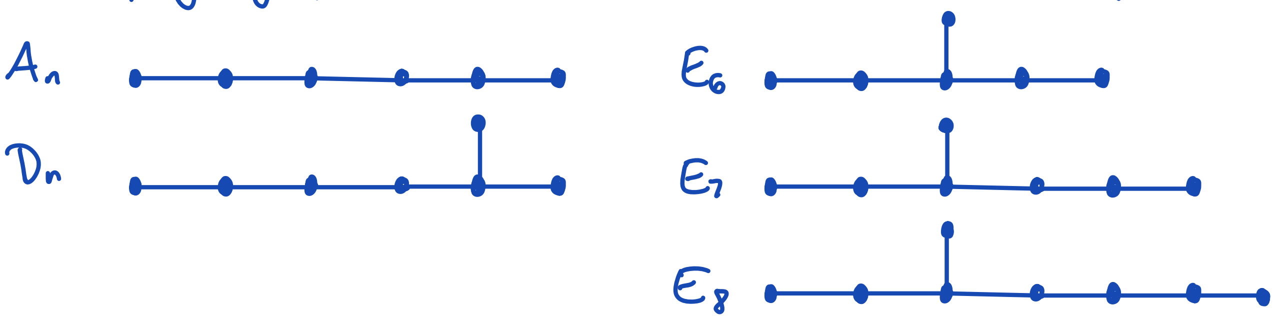
Structure Theorem Let $M : [n] \to \text{vect}$, $M : \mathbb{Z}_{\geq 0} \to \text{vect}$, $M : \mathbb{Z} \to \text{vect}$ or $M : \mathbb{R} \to \text{vect}$.

Then $M \cong \bigoplus_j \text{Int}_{I_j}$ for some collection of intervals I_j.

2.3 Gabriel's Theorem for Quivers

A quiver is of <u>finite type</u> if it has only finitely many isomorphism classes of indecomposable representations.

<u>Gabriel's Theorem</u> A quiver is of finite type iff its underlying graph is one of the ADE Coxeter-Dynkin diagrams:



(the order of the arrows in the quiver doesn't matter).

For A_n, the indecomposable representations are the interval representations $0 \to 0 \leftarrow k \to k \to k \leftarrow k \to 0 \leftarrow 0$

2.4 Using Abstract Algebra

Let $M : \mathbb{Z}_{\geq 0} \longrightarrow \text{vect}$

We may view M as a graded module over the graded polynomial ring $k[x]$.

The structure theorem for M is a special case of the stucture theorem for graded modules over a graded p.i.d.

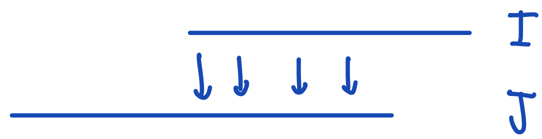
3. Isometry Theorem

3.1 Maps of Interval modules

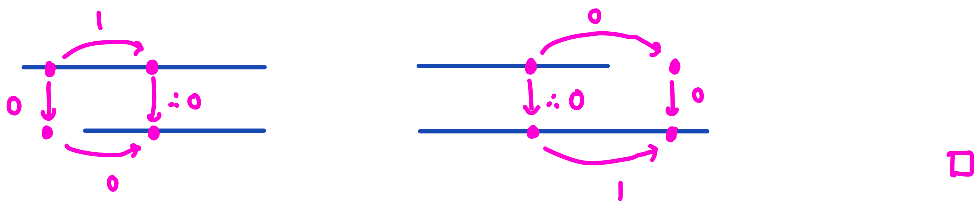
We represent interval modules by their supporting interval and denote them by this interval:

Lemma

Nonzero maps of interval modules are of the following form:

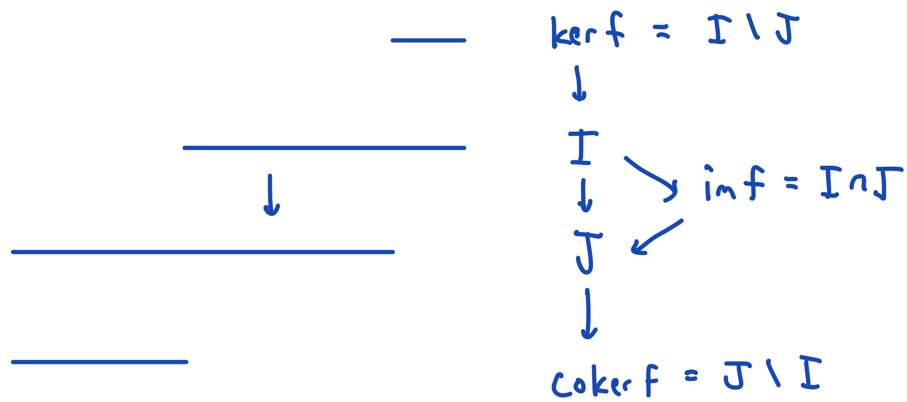


Proof



Up to isomorphism, these maps are given by identity maps on $I \cap J$.

Furthermore:

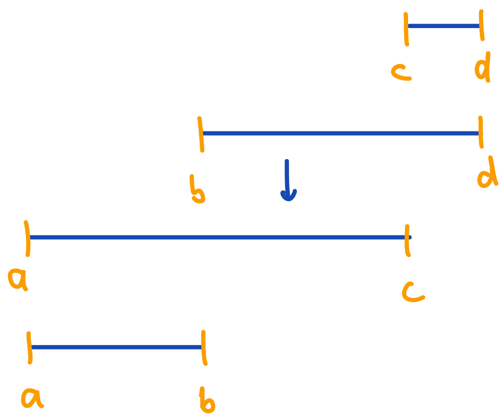


3.2 Isometry Theorem for Interval Modules with a nonzero map between them

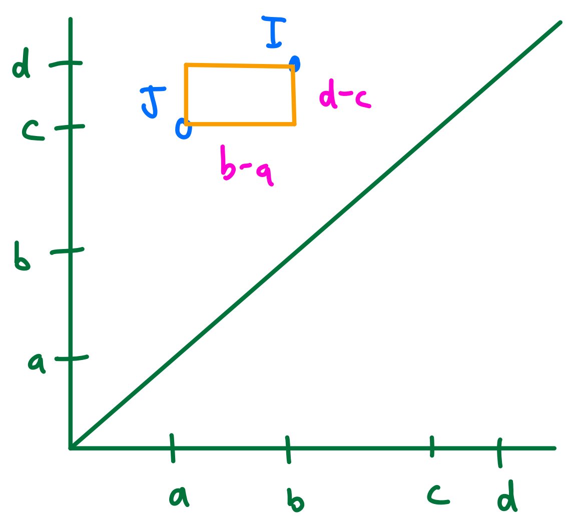
Interleaving distance of I and J

$$d_I(I, J) = \max(\text{length}(I \setminus J), \text{length}(J \setminus I))$$

Now consider the corresponding persistence diagrams



$$\begin{array}{l} \ker f = I \setminus J \\ \downarrow \\ I \\ \downarrow \quad \searrow \text{inf} = I \cap J \\ J \quad \swarrow \\ \downarrow \\ \operatorname{coker} f = J \setminus I \end{array}$$



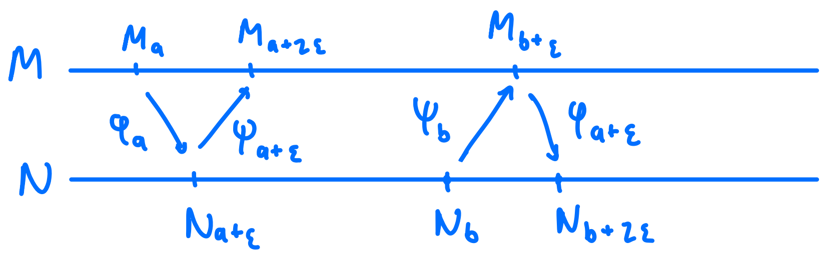
$d_I(I, J)$

$= \max(d-c,\ b-a)$

$= W_\infty(\mathrm{Dgm}\, I, \mathrm{Dgm}\, J)$

3.2 Induced Matchings

Let $\varepsilon \geq 0$. Recall, <u>ε-interleaving</u>:



All maps commute

An <u>ε-matching</u> of $M \cong \bigoplus_{j \in J} I_j$ and $N \cong \bigoplus_{k \in J} I_k$ is a matching of I and J so that matched interval modules are ε-interleaved and unmatched interval modules are ε-interleaved with the 0 module.

By taking direct sums of these ε-interleavings we obtain

<u>Converse Algebraic Stability</u>

If there exists an ε-matching between M and N then they are ε-interleaved.

<u>Corollary</u> $d_I(M, N) \leq W_\infty(\mathrm{Dgm}\, M, \mathrm{Dgm}\, N)$

Induced Matching Theorem / Algebraic Stability

Given an ε-interleaving between M and N
there is an induced ε-matching between M and N.

Corollary $W_\infty(\mathrm{Dgm}\, M, \mathrm{Dgm}\, N) \leq d_I(M, N)$

Combining the results above:

Isometry Theorem M and N are ε-interleaved iff they have an ε-matching.

Corollary $W_\infty(\mathrm{Dgm}\, M, \mathrm{Dgm}\, N) = d_I(M, N)$