Topological Data Analysis and Persistence Theory

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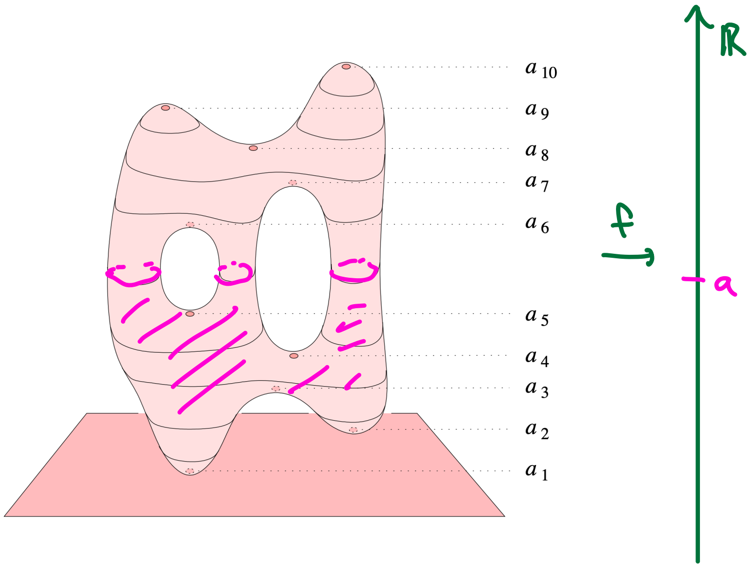
Lecture 7: Multiparameter Persistent Homology via Generalized Morse Theory

joint work with

Mike Catanzaro

Please interrupt me!!!

§1. Diagrams of Spaces and Vector Spaces



$f : X \to \mathbb{R}$

Sublevel sets: $a \in \mathbb{R}$

$F_a := f^{-1}(-\infty, a]$

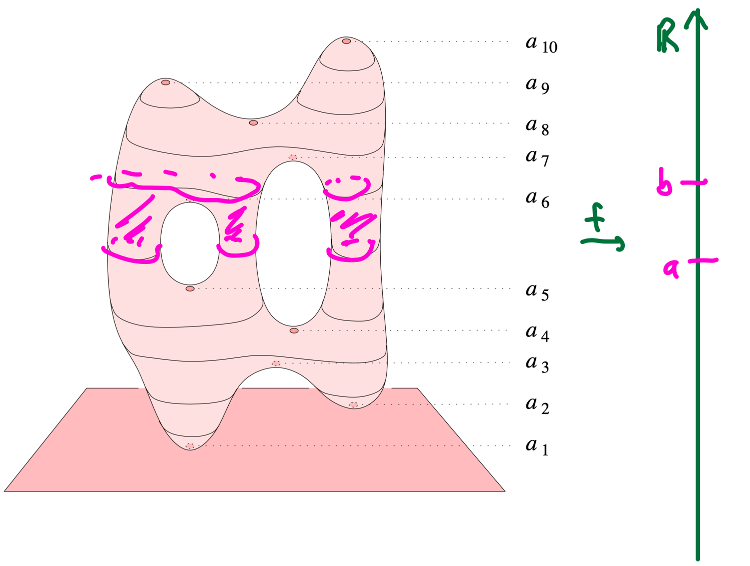
$a \le b \qquad F_a \hookrightarrow F_b$

$F : (\mathbb{R}, \le) \longrightarrow \mathrm{Top}$

Let $H = H_j(-, k)$ ↑ field

$\mathbb{R} \xrightarrow{F} \mathrm{Top} \xrightarrow{H} \mathrm{Vect}$

persistence module



$F_{a \le b} = f^{-1}[a, b]$

$[a,b] \subset [c,d] \qquad F_{a\le b} \hookrightarrow F_{c \le d}$

$F : \mathrm{Int}(\mathbb{R}) \longrightarrow \mathrm{Top}$

↑ intervals in $\mathbb{R}$

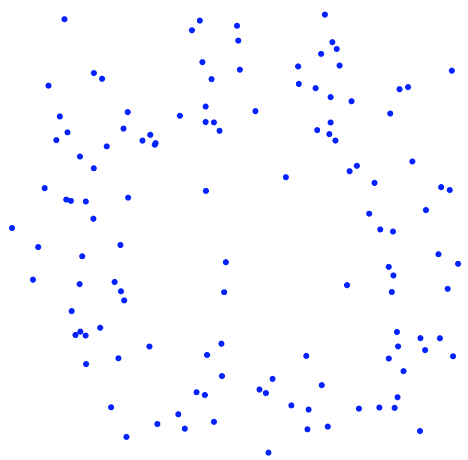
$[a,b] \subset [c,d]$

$c \le a \le b \le d$

$\mathrm{Int}(\mathbb{R}) \xrightarrow{F} \mathrm{Top} \xrightarrow{H} \mathrm{Vect}$

2-parameter persistence module

2. Example from Topological Data Analysis



Given $\{x_i\}_{i=1}^{N} \in \mathbb{R}^2$

$$f_\sigma : \mathbb{R}^2 \to \mathbb{R}$$

$$f_\sigma(x) = \sum_{i=1}^{N} K_\sigma(x - x_i)$$

$$K_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

Vary σ. Get 1-parameter family of functions

$f_\sigma : \mathbb{R}^2 \to \mathbb{R}$.

§3 Generalized Morse Theory / Cerf Theory

Morse Theory : A smooth function $g : M \to \mathbb{R}$

Can be approximated by a Morse function $f : M \to \mathbb{R}$

· singularities are nondegenerate

[· critical value are distinct]

Jean Cerf (1928-) Student of Henri Cartan

$g_t : M \to \mathbb{R}$ one-parameter family of smooth functions

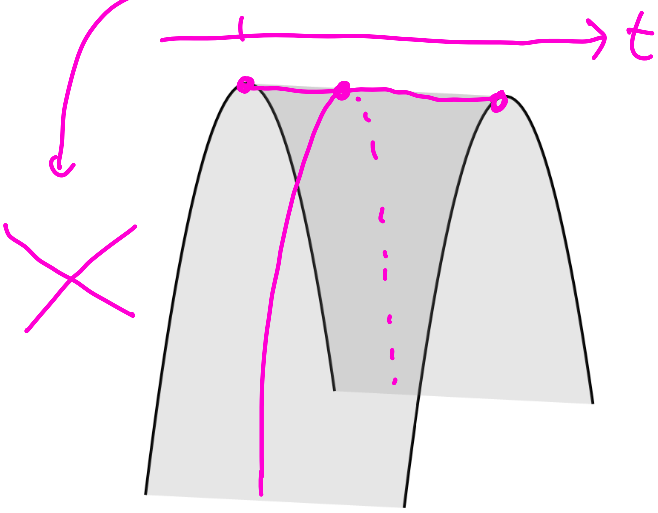
can be approximated by

$f_t : M \to \mathbb{R}$ one-parameter family of smooth functions

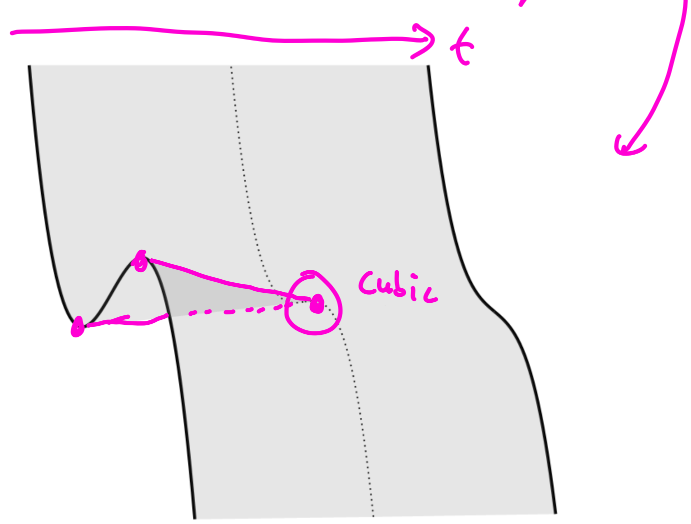
s.t. for all but finitely many t, f_t is Morse

Remaining cases : • f_t has a birth-death singularity

[· two critical values are equal]



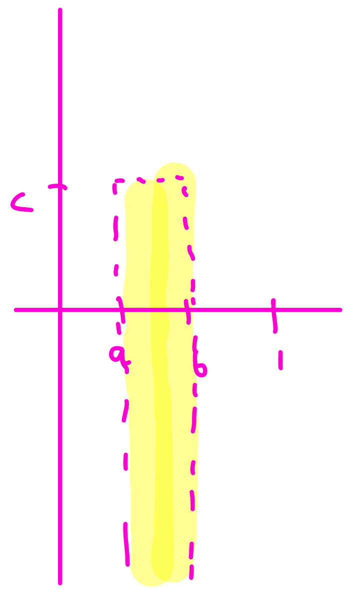
fold singularity



birth-death singularity

§4 Persistence Modules for 1-parameter families of functions

$f_t : X \longrightarrow \mathbb{R} \qquad t \in I = [0,1]$



$f : I \times X \longrightarrow I \times \mathbb{R}$

$(t, x) \longmapsto (t, f_t(x))$

$F : \mathrm{Int}(I) \times \mathbb{R} \longrightarrow \mathrm{Top}$

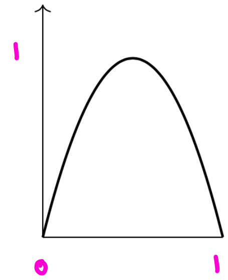
$(a, b, c) \longmapsto \underbrace{f^{-1}\big([a,b] \times (-\infty, c]\big)}_{\text{Subset of } I \times X}$

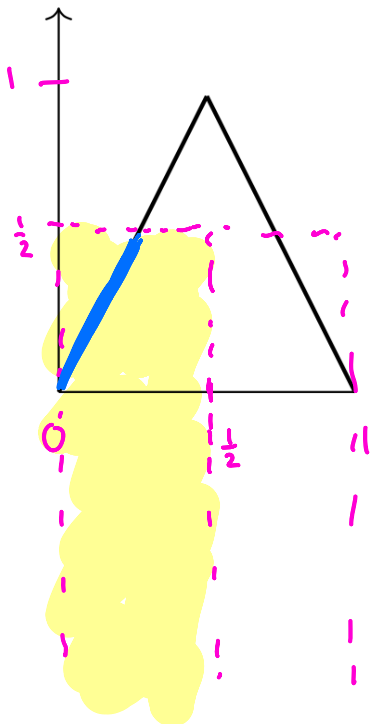
$HF : \mathrm{Int}(I) \times \mathbb{R} \longrightarrow \mathrm{Vect}$

<u>Cerf diagram</u> $\quad \underbrace{\bigcup_{t \in I} \{ (t, f_t(x)) \mid x \text{ is a singularity of } f_t \}}_{\text{Subset of } I \times \mathbb{R}}$

Example $X = *$ $\quad f_t : * \longrightarrow \mathbb{R}$

$\qquad\qquad\qquad\quad * \longmapsto 4t(1-t)$





Subdiagram of F

$$\begin{tikzcd}[column sep=tiny] & F(0,1,1) & \\ & F(0,1,\tfrac{1}{2}) \arrow[u] & \\ F(0,\tfrac{1}{2},\tfrac{1}{2}) \arrow[ur] & & F(\tfrac{1}{2},1,\tfrac{1}{2}) \arrow[ul] \end{tikzcd} \quad \xrightarrow{H_0} \quad \begin{tikzcd}[column sep=tiny] & k & \\ & k^2 \arrow[u, "{[1\ 1]}"] & \\ k \arrow[ur, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & & k \arrow[ul, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] \end{tikzcd}$$

$$F(a,b,c) = f^{-1}([a,b] \times (-\infty, c])$$

$0 \leq a \leq b \leq 1$

$H_0F : \mathrm{Int}(I) \times \mathbb{R}$

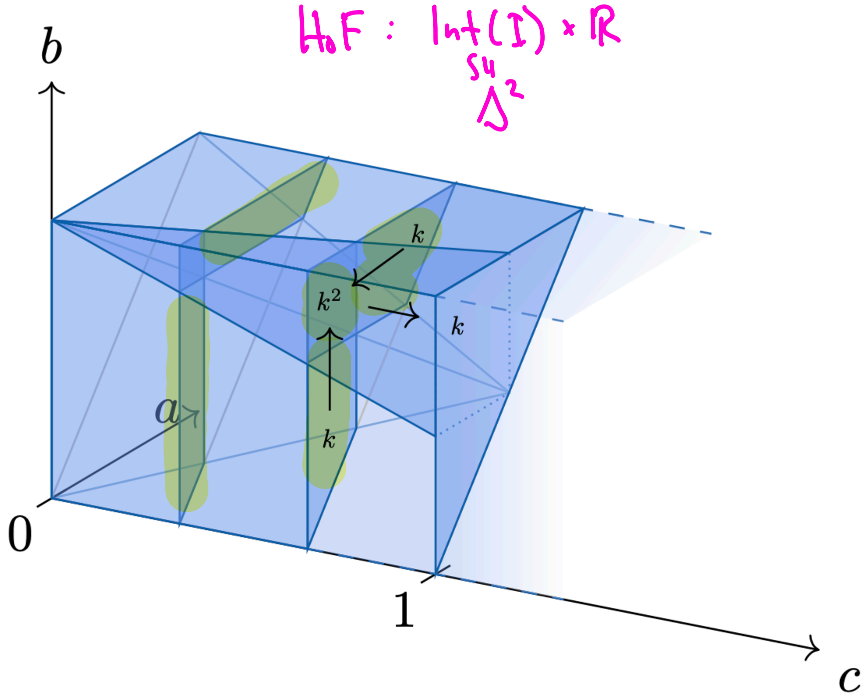
$\subseteq$

Δ^2

$H_*F :$

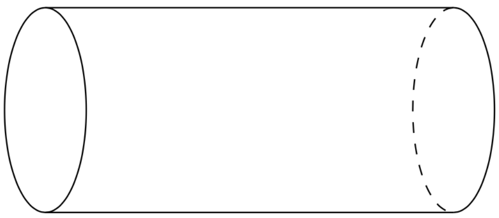
$=$

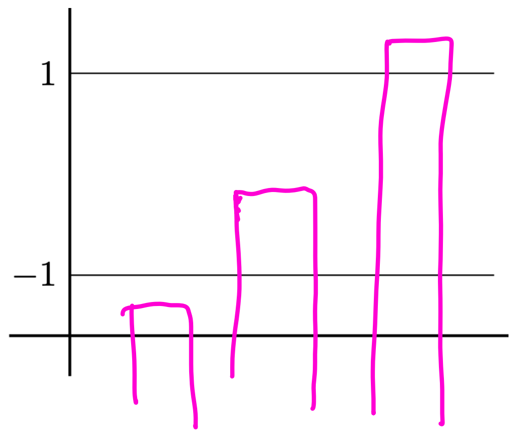
H_0F



Example The cylinder $f_t : S^1 \to \mathbb{R}$

$\theta \longmapsto \sin\theta$



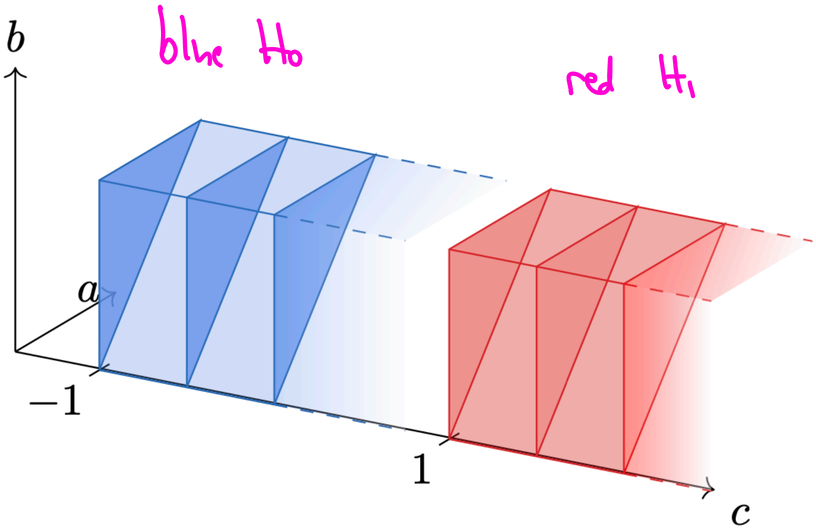


Cerf Diagram

blue H_0

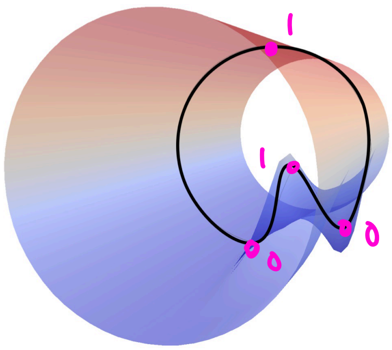
red H_1

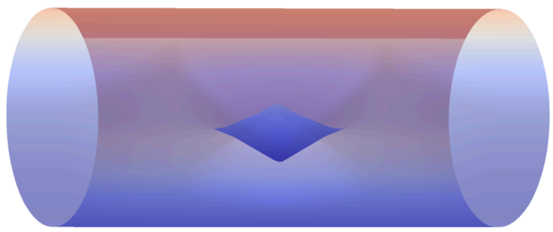
H_*F :

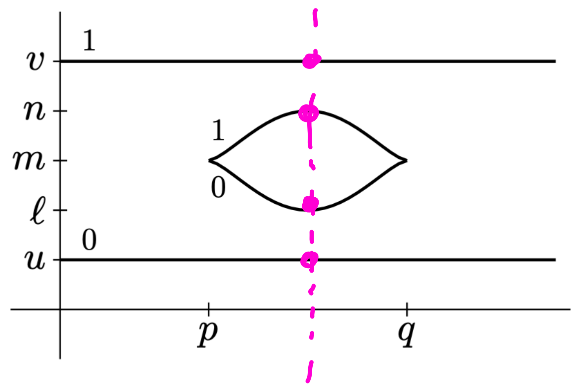


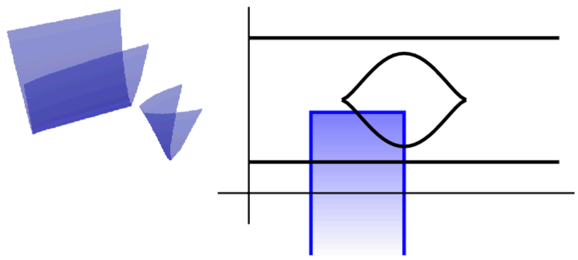
Eliashberg - Mishachev (2002)

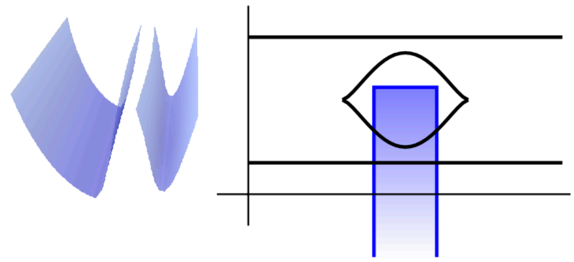
Example The wrinkled cylinder $f_t : S^1 \to \mathbb{R}$

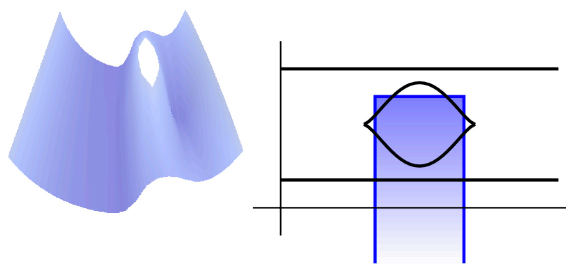


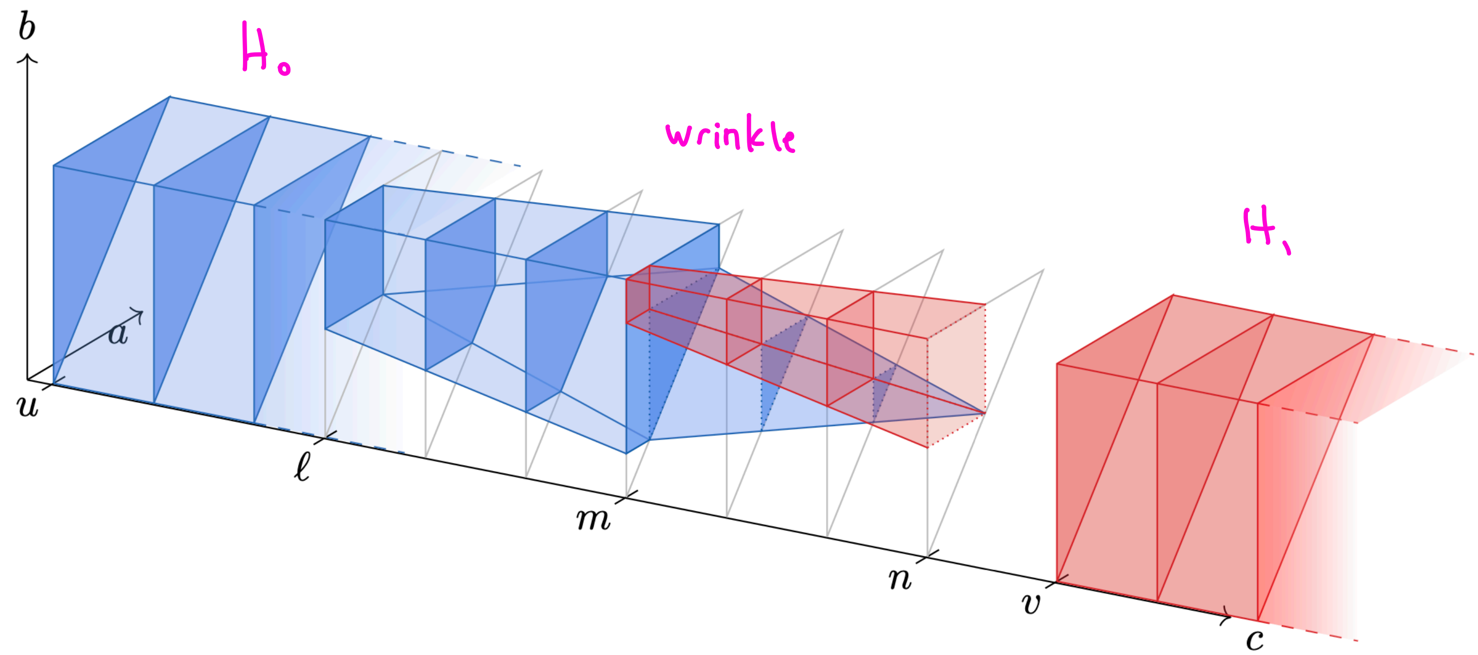






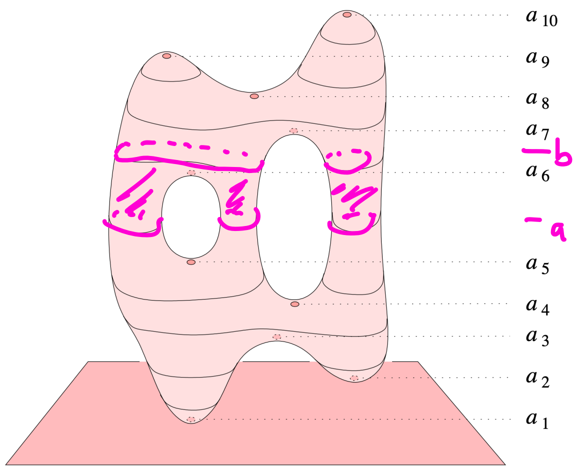






§5 Cobordisms

Easier case from Morse theory

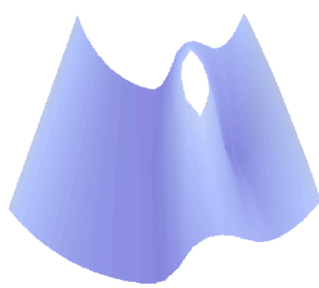


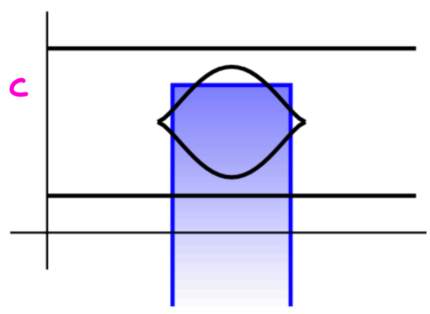
$f : X \to \mathbb{R}$

$f^{-1}(a)$, $f^{-1}(b)$ are manifolds

$\underbrace{f^{-1}([a,b])}_{\text{Manifold with boundary}}$ is a cobordism between $f^{-1}(a)$ and $f^{-1}(b)$

Cerf theory





$f : I \times X \to I \times \mathbb{R}$

$\left.\begin{array}{l} f^{-1}(\{a\} \times (-\infty, c]) \\ f^{-1}(\{b\} \times (-\infty, c]) \end{array}\right\}$ manifolds with boundary

$f^{-1}([a,b] \times (-\infty, c])$ Cobordism

(manifold with corners)

Theorem (B-Catanzaro, 2021) Suppose f_t is a generic 1-parameter family of functions on a smooth compact manifold.

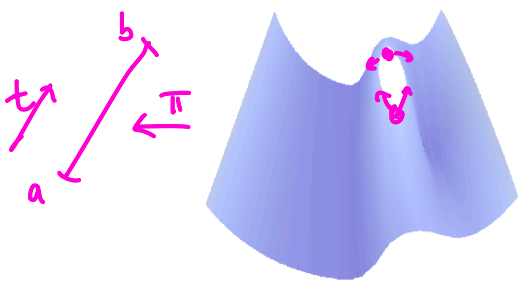
Recall $F(a,b,c) = f^{-1}([a,b] \times (-\infty, c])$. Then

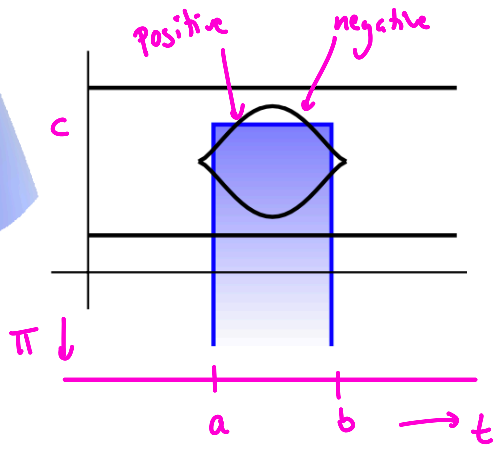
Hessian nondegenerate

- All critical points are Morse
- Gradient is tangent to the boundary

1. $F(a,b,c)$ is a cobordism
2. The projection $\pi : F(a,b,c) \to [a,b]$ is Morse
3. All critical points of π are on the boundary
4. Positive critical points of π are boundary stable

   Negative critical points of π are boundary unstable.





The gradient flow lines from the critical point stay on the boundary.