Topological Data Analysis and Persistence Theory

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Lecture 9: Mathematics and Persistence

Outline:
1. Algebraic Distances
   joint work with Don Stanley and Jonathan Scott
2. Universal Constructions
   joint work with Alex Elchesen

Please interrupt me!!!

1. Algebraic distances

1.1 Weights and Path metrics on Abelian categories

A weight assigns a value in $[0,\infty]$ to each object.

$w(0) = 0$

Example $\quad \mathrm{Vect}_K \quad w(V) := \dim(V)$

$\qquad\qquad \mathrm{Mod}_R \quad w(M) := \mathrm{pd}(M) + 1$

$0 \to A \to A \oplus C \to C \to 0$

A weight is exact if for each $0 \to A \to B \to C \to 0$

$$w(A) \leq w(B) + w(C), \quad w(B) \leq w(A) + w(C), \quad w(C) \leq w(A) + w(B)$$

Amplitude: $w(A) \leq w(B)$, $\quad$ " $\quad$, $w(C) \leq w(B)$

Primo z: $w(A \oplus C) \leq w(B)$

path metric

Given a weight w we define a path distance

$$d_w(A,B) = \inf_{\underbrace{A \xrightarrow{f_1} A_1 \xleftarrow{f_2} A_2 \to \cdots \xleftarrow{f_n} B}_{\text{zigzag } \gamma}} \underbrace{\sum_i w(\ker f_i) + w(\operatorname{coker} f_i)}_{\mathrm{Cost}_w \gamma}$$

reflexive, symmetric, Δ-ineq, isomorphism invariant

Given a distance d we define a weight $|d|(A) = d(A,0)$.

A weight is stable if $|d_w| = w$.

Proposition $d_w(A,B) \leq w(A) + w(B)$

Say that w lower bounds its path metric if

$$|w(A) - w(B)| \leq d_w(A,B)$$

Theorem (B-Scott-Stanley, 2022) TFAE

- w is exact $\Leftarrow$ w is an amplitude
- w is stable
- w lower bounds its path metric

1.2 Path metrics for Persistence modules

$$M : \underline{P} \longrightarrow \underline{A}$$

Small category

abelian category with a weight w

Assume that the objects of $\underline{P}$ have a measure μ.

$(\mathbb{R}^d, \leq)$ with Lebesgue $(\mathbb{Z}^d, \leq)$ with counting measure

$$(\mu \circ w)(M) = \int_P w(M(p))\, d\mu(p)$$

w exact $\Rightarrow$ $\mu \circ w$ exact

Using $\mu \circ w$ we get a path metric for pers. modules

1.3 Algebraic Wasserstein distance

$1 \le p \le \infty$

$$W_p(M,N) := \inf_{\substack{M \cong \bigoplus_a M^{(a)} \\ N \cong \bigoplus_a N^{(a)}}} \left\| \left(d_{mow}(M^{(a)}, N^{(a)}) \right)_a \right\|_p$$

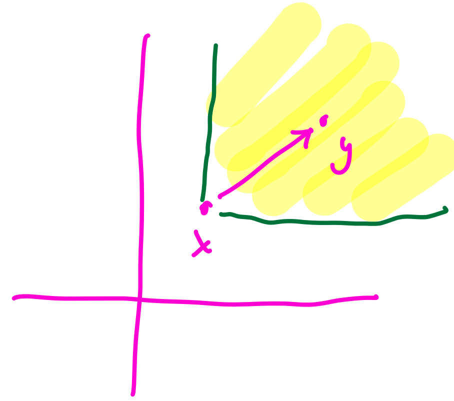
indecomposable objects or zero

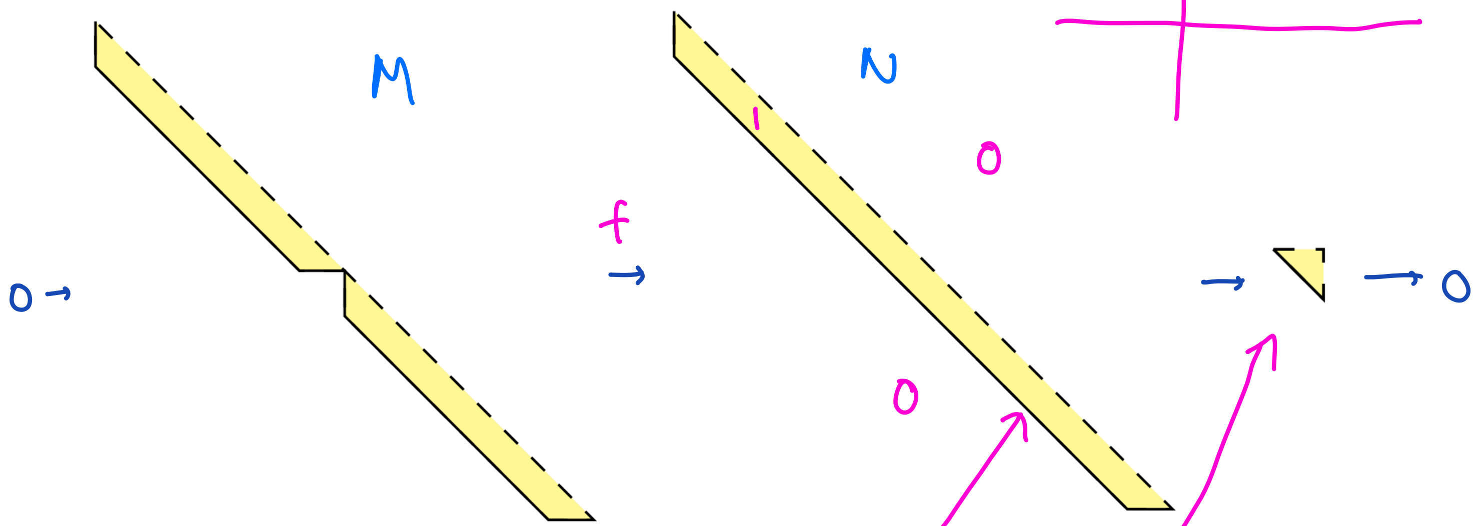
Theorem For one-parameter persistence modules this agrees with the usual definition.

Theorem For one-parameter persistence modules W_1 agrees with the path metric.

1.4 Examples

(a) $M : (\mathbb{R}^2, \leq, \mu) \longrightarrow \underline{\mathrm{Vect}_k}$

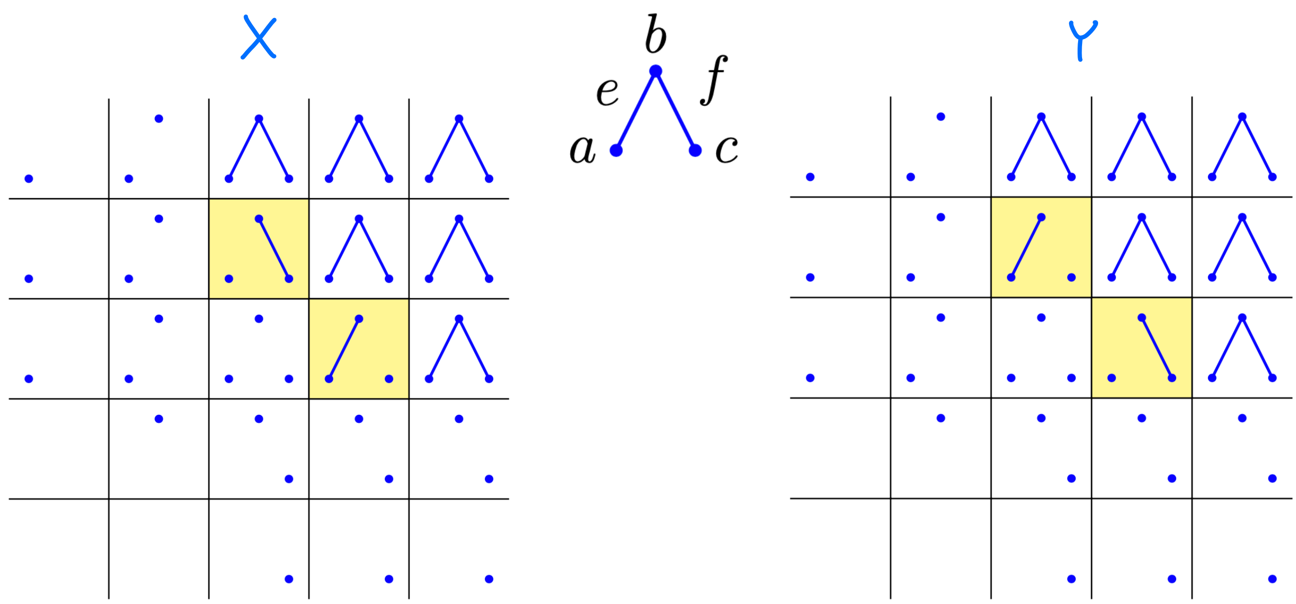




$d_{\mu odim}(M,N) = \mu odim(\operatorname{coker} f) = \text{area}(\quad)$

$W_1(M,N) = \text{area}(\quad)$

(b) $(0<1<2<3<4)^2 \xrightarrow{X} \underline{\text{Spaces}} \xrightarrow{H_0} \underline{\mathrm{Vect}_k}$



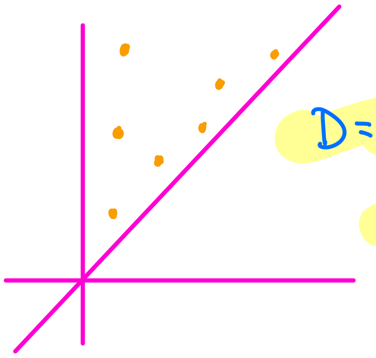
$H_0 X,\ H_0 Y$: equal at each index

$X \overset{f}{\hookrightarrow} X \cup Y \overset{g}{\hookleftarrow} Y$

$$d_{\mathrm{modim}}(H_0X, H_0Y) = \dim(\ker f) + \dim(\ker g)$$
$$= 2 + 2 = 4$$

H_0X, H_0Y indecomposable so $W_p(H_0X, H_0Y) = 4$.

2. Universal Constructions



$D = x_1 + x_2 + \cdots + x_n$

$x_i \in \mathbb{R}^2_<$

$D \in D(\mathbb{R}^2_<)$ free commutative monoid on $\mathbb{R}^2_<$

$\cong$

$D(\mathbb{R}^2_\leq) / D(\mathbb{R}^2_=) =: D(\mathbb{R}^2_\leq, \mathbb{R}^2_=)$

Let Met_{pairs} be the category with objects (X, d, A) morphisms 1-Lipschitz maps

$f : (X, A) \to (Y, B) \quad d_Y(fx, fx') \leq d_X(x, x')$

$\mathrm{Met}_* \subset \mathrm{Met}_{pairs}$ full subcategory with $A = \{x_0\}$

(X, d, x_0)

$1 \leq p \leq \infty$

Let Met^p_* denote Met_* with the symmetric monoidal product given by $X \times Y$ with the metric

$$\| (d_X(x, x'), d_Y(y, y')) \|_p$$

$$\mathrm{Set} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{CMon}$$

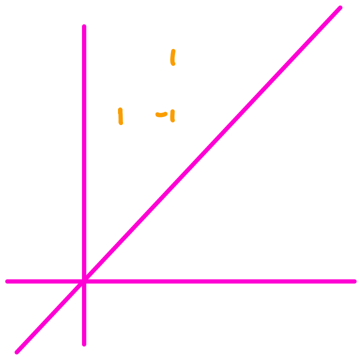
Theorem (B-Elchesen, 2021)

$$\mathrm{Met}_{pairs} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{CMon}(\mathrm{Met}^p_*)$$

$(\mathbb{R}^2_\leq, d, \mathbb{R}^2_=) \mapsto$ Persistence Diagrams with p-Wasserstein distance

Corollary p-Wasserstein distance is the universal (ie largest) p-subadditive metric for persistence diagrams

Distances for Graded Persistence Diagrams



$D_k = X_1 - X_2 + X_3$

Jordan decomposition

$D_k = D_k^+ - D_k^-$

$D_k \in K(\mathbb{R}^2_<) \cong K(\mathbb{R}^2_\le)/K(\mathbb{R}^2_=) =: K(\mathbb{R}^2_\le, \mathbb{R}^2_=)$

free abelian group

Definition (B-Elchesen, 2021) $W_1(D_k, E_k) := W_1(D_k^+ + E_k^-, E_k^+ + D_k^-)$

Theorem (BBE, 2021) $\sum W_1(D_k, E_k)$ is more discriminative than $W_1(D,E)$

$W_1(D,E) \le \sum W_1(D_k, E_k)$ and $\exists K$ s.t. $\sum W_1(D_k, E_k) \le K W_1(D,E)$

Theorem (B-Elchesen, 2021) Given a metric pair (X, d, A)

$\exists$ canonical isometric embeddings:

$$(D(X,A), W_1) \hookrightarrow (K(X,A), W_1) \hookrightarrow (V(X,A), W_1) \hookrightarrow (\hat{V}(X,A), W_1)$$

free comm. monoid pers. diag	free ab gp virtual pers. diag	free vector sp	free Banach space