MTG 4302/5316: Elements of/Introduction to Topology 1
Final Exam Practice Problems

These problems are candidates for the final exam. Work on them individually and with your classmates. Wherever possible, give direct arguments rather that using general results. *Do not read another person's solutions. Write up your own solutions.* You may ask for help in class and we will have time for review before the final exam.

Chapter 1.

1. Suppose that X_α is nonempty for every $\alpha \in A$. Show that
$$\prod_{\alpha \in A} X_\alpha \neq \emptyset.$$

2. Suppose that $X_\alpha \neq \emptyset$ for all $\alpha \in A$. Let $\beta \in A$ and define
$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$
by $(x_\alpha) \mapsto x_\beta$. Show that π_β is onto for every $\beta \in A$.

3. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto.

Chapter 2.

1. Let $\{X_\alpha\}$ be a collection of topological spaces. Let $\beta \in A$ and let
$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$
be given by $(x_\alpha) \mapsto x_\beta$. Show that π_β is continuous.

2. Let X be a topological space and $A \subset X$. The *closure* of A is defined by $\overline{A} = \cap\{C \mid C \text{ is closed in } X \text{ and } C \supset A\}$.
   (a) Show that $\overline{A}$ is a closed set in X.
   (b) Show that $\overline{\overline{A}} = \overline{A}$.

Chapter 3.

1. Suppose that $f : X \to Y$ is continuous and that $A \subset X$ is connected. Show that $f(A) \subset Y$ is connected.

2. Suppose that $f : X \to Y$ is continuous and that $A \subset X$. Show that if A is compact, then $f(A)$ is compact.

3. Show that there is no continuous function $f : [0, 1] \to [0, 1)$ which is onto.

4. Show that there is no continuous function $f : \mathbb{R} \to \{0, 1\}$ which is onto.

5. Show that if $A \subset \mathbb{R}$ is connected then A is an interval.

6. State the Intermediate Value Theorem.

7. Give an example of a connected space X that is not path connected.

8. A component of a space X is a maximal connected subset. Show that if A is a component, then A is closed.

9. Show that if A and B are components of X, then either $A = B$ or $A \cap B = \emptyset$.

10. Let X be a metric space with metric d. Show that X is a normal space.

11. Let (X, d) be a metric space and let $(X, \mathcal{T})$ be the associated topological space. Show that
   (a) in $(X, \mathcal{T})$ points are closed,
   (b) $(X, \mathcal{T})$ is a Hausdorff space,
   (c) $(X, \mathcal{T})$ is a regular space,
   (d) $(X, \mathcal{T})$ is a normal space.

12. Suppose that X is a compact Hausdorff space. Show that if $A \subset X$ is compact, then A is closed.

Chapter 4.

1. State the Urysohn Lemma.

2. State the Tietze Extension Theorem.

3. Let $\mathbb{R}_\ell$ be the real numbers with the topology having as basis $\{[a, b) \mid a < b \in \mathbb{R}\}$. Show that $\mathbb{R}_\ell$ is a normal space.

Chapter 7.

1. Give examples of complete metric spaces and metric spaces that are not complete.

2. Let X be a metric space and let $f : X \to X$ be a continuous function. Suppose that there is a point $x_0 \in X$ such that $f^n(x_0) \to z$ as $n \to \infty$. Show that $f(z) = z$.

3. State the Contraction Mapping Theorem.

4. Show that every compact metric space is complete.

5. Show that there is a continuous map $f : [0, 1] \to [0, 1] \times [0, 1]$ that is onto.

First Year topology exam problems

Chapter 3.

1. Prove that if X is a compact Hausdorff space, then X is a normal space.

2. Give an example of a topological space X that is Hausdorff but not regular.

3. Give an example of a topological space X that is regular but not normal.

Chapter 4.

1. State and prove Urysohn's Lemma.

2. State and prove the Tietze Extension Theorem.

3. Suppose that X is a normal space with a countable basis. Show that there is a metric ρ on X that generates the topology on X.

Chapter 7.

1. Show that $\mathbb{R}^d$ is a complete metric space.

2. State and prove the Contraction Mapping Theorem.

3. Let X be a complete metric space. A point $x \in X$ is an *isolated point* if $\{x\}$ is open. Suppose that X is countable and non-empty. Show that X has an isolated point.

4. Show that any nonempty complete metric space with no isolated points is uncountable.