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Topological Data Analysis Worksheet 1

1. Simplicial complexes

Definition 1. A *k-simplex* is a k-dimensional polytope which is the convex hull of $k+1$ points in Euclidean space in general position (the points $x_1, \ldots, x_{k+1}$ are *affinely independent*, which means that the vectors $x_2 - x_1, \ldots, x_{k+1} - x_1$ are linearly independent). The convex hull of a non-empty subset of these points is called a *face* of the simplex.

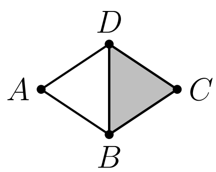
Definition 2. A *simplicial complex* is a set K of simplices such that

(a) the face of any simplex in K is also in K; and
(b) any non-empty intersection of two simplices is a face of both of them.

Definition 3. An *abstract simplicial complex* is a set K of non-empty subsets of a fixed set such that

- if $A \in K$ then all nonempty subsets of A are also in K.

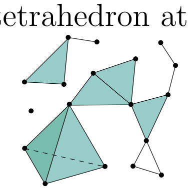
Exercise 1. Write down the corresponding abstract simplicial complex for the following simplicial complex. Order the simplices of the same cardinality using lexicographic order.



2. Betti numbers

Definition 4. The k-th Betti number, β_k, counts the number of k-dimensional features of a simplicial complex.For example, β_0 gives the number of connected components, β_1 gives the number of holes, and β_2 gives the number of voids.

Exercise 2. Give the Betti numbers of the following simplicial complex. Assume that the tetrahedron at the lower left is hollow.



3. Čech complex

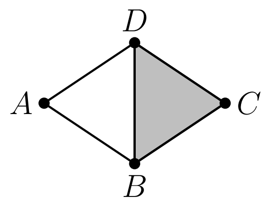
Definition 5. Given a set of points $X = \{X_1, \ldots, X_N\}$ in a metric space and $r \geq 0$, the *r-Čech complex*, $\check{C}_r(X)$, is the abstract simplicial complex on the set X consisting of those subsets $\{X_{i_1}, \ldots, X_{i_m}\}$ such that the intersection of all of the closed balls of radius r centered at the X_{i_j} is nonempty.

Exercise 3. Given the set of points $X = \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$, draw a picture of $\check{C}_{\frac{1}{2}}(X)$.

4. Homology

Definition 6. Given an (abstract) simplicial complex, K, the *k-chains*, C_k, is the vector space over the field $\mathbb{Z}/2\mathbb{Z}$ with basis given by the k-simplices in K. The *boundary map*, $\partial_k : C_k \to C_{k-1}$, is given by the matrix whose entry in the ith row and jth column is 1 if the ith $(k-1)$-simplex is a face of the jth k-simplex, and otherwise 0. The *k-cycles* are the k-chains α such that $\partial_k(\alpha) = 0$. The *k-boundaries* are the k-chains α that are in the image of ∂_{k+1}. The k-cycles and k-boundaries form subspaces Z_k and B_k. It is an exercise to check that $B_k \subset Z_k$. This is the crucial part! We can now define *k-homology* to be the quotient space $H_K = Z_k/B_k$. Then the *k-th Betti number* is given by $\beta_k = \dim(H_k) = \dim(Z_k) - \dim(B_k) = \text{nullity}(\partial_k) - \text{rank}(\partial_{k+1})$.

Exercise 4. Consider the following simplicial complex.



(a) Write down a basis for C_0, C_1 and C_2. Use lexicographic order.
(b) Write down the boundary matrices ∂_1 and ∂_2.
(c) Use these matrices together with $\partial_0 = 0$ and $\partial_3 = 0$ to calculate the Betti numbers β_0, β_1 and β_2.

5. Persistent homology

Definition 7. A *filtered simplicial complex* consists of a nested sequence of simplicial complexes:

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n.$$

Applying homology, we obtain a sequence of vector spaces and linear maps called a *persistence module*:

$$H_k(K_0) \to H_k(K_1) \to H_k(K_2) \to \cdots \to H_k(K_n).$$

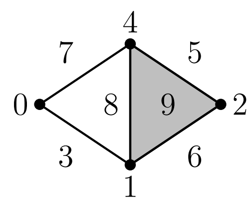
The images of these maps and compositions of these maps are called *persistent homology* vector spaces.

Definition 8. Given a persistence module

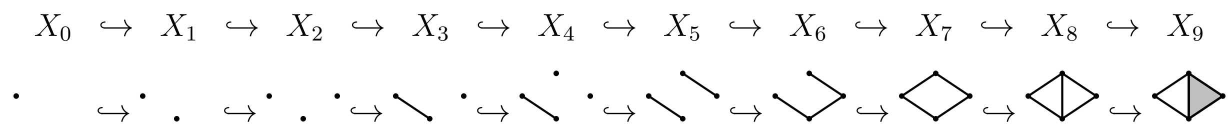
$$M_0 \to M_1 \to M_2 \to \cdots \to M_n, \tag{1}$$

the *rank* function assigns each pair (i, j) with $1 \leq i \leq j \leq n$ the rank of the linear map $M_i \to M_j$ given by concatenating the maps in (1). If $i = j$ then this is the rank of the identity map on M_i which equals the dimension of M_i.

Exercise 5. Consider the following filtered simplicial complex.



That is,



Determine the rank functions for H_0 and H_1.

H_0	0	1	2	3	4	5	6	7	8	9
0										
1	—									
2	—	—								
3	—	—	—							
4	—	—	—	—						
5	—	—	—	—	—					
6	—	—	—	—	—	—				
7	—	—	—	—	—	—	—			
8	—	—	—	—	—	—	—	—		
9	—	—	—	—	—	—	—	—	—	

H_1	0	1	2	3	4	5	6	7	8	9
0										
1	—									
2	—	—								
3	—	—	—							
4	—	—	—	—						
5	—	—	—	—	—					
6	—	—	—	—	—	—				
7	—	—	—	—	—	—	—			
8	—	—	—	—	—	—	—	—		
9	—	—	—	—	—	—	—	—	—	

Exercise 6. Let **Met**, **Simp**, $\mathbf{F}_{\mathbb{R}}$**Simp**, **wGph**, and **wDiGph** denote the categories of metric spaces, simplicial complexes, $\mathbb{R}$-filtered simplicial complexes, weighted simple graphs, and weighted directed simple graphs.

(a) Given $\varepsilon > 0$, use the Vietoris-Rips construction with scale ε to define a functor $\mathrm{VR}_\varepsilon : \mathbf{Met} \to \mathbf{Simp}$.
(b) Define the Vietoris-Rips functor $\mathrm{VR} : \mathbf{Met} \to \mathbf{F}_{\mathbb{R}}\mathbf{Simp}$.
(c) Extend this functor to a functor $\mathrm{VR} : \mathbf{wGph} \to \mathbf{F}_{\mathbb{R}}\mathbf{Simp}$.
(d) Given $\varepsilon > 0$, use the Čech construction with scale ε to define a functor $\check{C}_\varepsilon : \mathbf{Met} \to \mathbf{Simp}$.
(e) Define the Čech functor $\check{C} : \mathbf{Met} \to \mathbf{F}_{\mathbb{R}}\mathbf{Simp}$.
(f) Extend this functor to a functor $\check{C} : \mathbf{wDiGph} \to \mathbf{F}_{\mathbb{R}}\mathbf{Simp}$.