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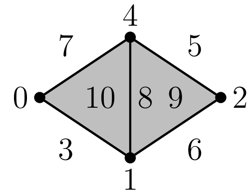
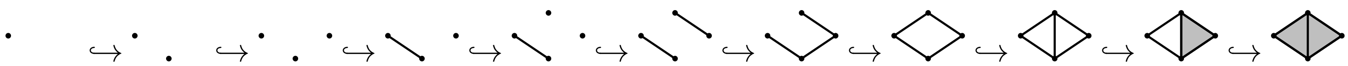
# Topological Data Analysis Worksheet 3

Consider the following filtered simplicial complex.




$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow X_5 \hookrightarrow X_6 \hookrightarrow X_7 \hookrightarrow X_8 \hookrightarrow X_9 \hookrightarrow X_{10}$$



Its persistence diagram for $\tilde{H}_0$ is given by $\{(1,3),(2,6),(4,5)\}$.
Its persistence diagram for $\tilde{H}_1$ is given by $\{(7,10),(8,9)\}$.

**Definition 1.** For $b < d$, let $f_{(b,d)} : \mathbb{R} \to \mathbb{R}$ be the piecewise-linear "tent" function given by

$$f_{(b,d)}(t) = \begin{cases} t-b & \text{if } b \leq t \leq \frac{d+b}{2} \\ d-t & \text{if } \frac{d+b}{2} < t \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Consider a persistence module with persistence diagram $D = \{(b_1,d_1),\ldots,(b_m,d_m)\}$. For $k = 1,2,,3,\ldots$, the *k-th persistence landscape function*, $\lambda_k : \mathbb{R} \to \mathbb{R}$, is given by letting $\lambda_k(t)$ be equal to the $k$th largest value of $\{f_{(b_1,d_1)}(t),\ldots,f_{(b_m,d_m)}(t)\}$. The sequence of these functions is called the *persistence landscape*. The persistence landscape may also be viewed as the function $\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ given by $\lambda(k,t) = \lambda_k(t)$.

**Exercise 1.** Graph the persistence landscapes for $\tilde{H}_0$ and $\tilde{H}_1$.

**Exercise 2.** Compare the persistence landscape functions $\lambda_k$ in the previous exercise with the graded persistence diagrams in Group Work 2 Exercise 3.

**Exercise 3.** Let $I \subset \mathbb{R}$ be an interval. For $\varepsilon \geq 0$, define the $\varepsilon$-*erosion* of $I$, by

$$I^\varepsilon = \{x \in I \mid [x - \varepsilon, x + \varepsilon] \subset I\}.$$

Given an interval $I$ define

$$f_I(t) = \sup\{\varepsilon \geq 0 \mid t \in I^\varepsilon\}.$$

(a) For $I = [b, d)$, give an expression for $I^\varepsilon$.
(b) For $I = [b, d)$, compute $f_I(t)$.
(c) For $I = [b, d)$, graph $y = f_I(t)$.

**Exercise 4.** Let $\mathcal{I} = \{I_j\}_{j \in J}$ where for each $j \in J$, $I_j$ is an interval in $\mathbb{R}$. For $k \in \mathbb{N}$, define

$$\lambda_k(t) = \sup\{\varepsilon \geq 0,\ |\{j \in J,\ t \in I_j^\varepsilon\}| \geq k\}.$$

Call $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ the *persistence landscape* of $\mathcal{I}$. Compare this definition with the one in the first exercise.