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Local exponential stability of competitive neural networks with different time scales

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Abstract

This contribution presents a new method of analyzing the dynamics of a biological relevant neural network with different time scales based on the theory of flow invariance. We are able to show that the resulting stability conditions are less restrictive and more general than with K-monotone theory or singular perturbation theory. The theoretical results are further substantiated by simulation results conducted for analysis and design of these neural networks.

1. Introduction

The collective computational capabilities of neural networks, such as optimization, association and oscillation, rely on the dynamic behaviors of the neural networks. The qualitative analysis of neural networks, including the analysis and stability and oscillations, characterize the dynamics of these networks.

Neural networks dealing with static patterns can be classified into two types according to the ways of presenting the patterns to them (Matsuoka, 1992): (N1) the key pattern is given as an initial state of the network; (N2) the key pattern is given as a constant input to the network.

The Hopfield network is an N1-type network and its connections have to be symmetric in order to guarantee stable convergence of the state (Cohen and Grossberg, 1983). The continuous additive bidirectional associative memory represents the heteroassociative analogue of the Hopfield network, and its global stability was shown in (Kosko, 1992).

N2-type networks are recurrent neural networks in which the recurrent connections are usually inhibited. The constraint of symmetry in connection weights is not valid for them for two reasons. First, confining the network to being symmetric might result in too strict a restriction on the capability of the network. Second the symmetry of the connections by itself does not guarantee the uniqueness of the convergence state because the final output depends not only on the constant input but also on the initial state of the network. Therefore, this network should be designed so that its final state is independent of the initial state. Such a network has the property of absolute stability.

On absolute stability of asymmetric recurrent neural networks, several sufficient conditions can be seen in the literature (Protopopescu et al., 1988; Hirsch, 1989; Michel et al., 1989; Kelly, 1990; Matsuoka, 1992). The results in Protopopescu et al. (1988) and Michel et al. (1989) are too strict particularly in the case that mutual connections between each pair of units have different signs of weights. It is because they evaluate the absolute values of connection weights and neglect their signs.

In Hirsch (1989) was the importance of global stability noted and he obtained a few conditions using Gersgorin's circle theorem. In Kelly (1990) was the contraction mapping technique applied to obtain some sufficient conditions for global stability. In (Matsuoka, 1992) were generalized some of Hirsch's and Kelly's results using a new Lyapunov function. He did not evaluate like most before him the absolute values of connection weights and neglected their signs but established some absolute stability conditions which were more relaxed than the previous results.

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The matrix measure technique has been used to study the stability property of nonlinear dynamical systems and many well-known results have been unified and generalized. Some general sufficient conditions for nonlinear dynamical systems have been obtained and have been applied to a class of assymmetrical recurrent neural networks (Matsuoka, 1992; Fang and Kincaid, 1996).

Dynamic neural networks which contain both feedforward and feedback connections between the neural layers play an important role in visual processing, pattern recognition, neural computing and control. Moreover, biological networks possess synapses whose synaptic weights vary in time. Thus, competitive neural networks with a combined activity and weight dynamics constitute a more important class of neural networks than the N1- and N2-type. Their capability of storing desired patterns as stable equilibrium points requires stability criteria which include the mutual interference between neuron and learning dynamics.

This paper investigates the dynamics of cortical cognitive maps, modeled by a system of competitive differential equations, from a rigorous analytic standpoint. The networks under study model the fast dynamics of the neural activity levels, the short-term memory (STM), and the slow dynamics of the unsupervised synaptic modifications, the long-term memory (LTM).

Such networks may be considered extensions of Grossberg's shunting network (Grossberg, 1976) or Amari's model for primitive neuronal competition (Amari, 1982). These earlier networks are modeled as a pool of mutually inhibitory neurons with fixed synaptic connections. Our results extend the previous studies to systems where the synapses can be modified by external stimuli. Also, the learning algorithm is unsupervised. In Jin and Gupta (1999) the dynamical behavior of discrete-time neural networks is studied using stable dynamic backpropagation algorithms. Two new stable learning concepts, the multiplier and the constrained learning rate methods, are employed. They describe supervised learning algorithms, and evaluate an error function. Generalized dynamic neural networks described in Galicki et al. (1999) are recurrent neural networks with time-dependent weights. The algorithm for learning continuous trajectories is based on a variational formulation of the Pontryagin maximum principle, and is also supervised. A robust local stability condition has been presented in Suykens et al. (2000) for multilayer recurrent neural networks with two hidden layers. The NL_q theory was proposed as a stability theory for multilayer recurrent neural with application to neural control. The Hebbian adaptive bidirectional associative memory in Kosko (1992) adapts according to the Hebbian learning law its weights in an unsupervised mode, and its global stability was proven.

In this paper we apply the theory of flow invariance on large-scale neural networks, which have two types of state variables (LTM and STM) describing the slow unsupervised and the fast dynamics of the system. We will give the mathematical conditions for showing when the STM and LTM trajectories are bounded. Our design is more general than that given in Lemmon and Kumar (1989) since it is not required to assume a high gain approximation and it does not treat the two dynamics seperately. It also does not require the excitatory region to comprise only one neuron or that the weights have to be symmetric as in Kosko (1992).

We consider a laterally inhibited network with a deterministic signal Hebbian learning law (Hebb, 1949) that is similar to the spatiotemporal system of Amari (1983). The general neural network equations describing the temporal evolution of the STM and LTM states for the jth neuron of an N-neuron network are:

$$\text{STM}: \quad \epsilon \dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j \sum_{i=1}^{p} m_{ij} y_i, \tag{1.1}$$

$$\text{LTM}: \quad \dot{m}_{ij} = -m_{ij} + y_i f(x_j), \tag{1.2}$$

where x_j is the current activity level, a_j is the time constant of the neuron, B_j is the contribution of the external stimulus term, $f(x_i)$ is the neuron's output, y_i is the external stimulus, and m_{ij} is the synaptic efficiency. ϵ is the fast time-scale associated with the STM state. D_{ij} represents a synaptic connection parameter between the ith neuron and the jth neuron. We assume here, that the recurrent neural network consists of both feedforward and feedback connections between the layers and neurons forming complicated dynamics.

The neural network is modeled by a system of deterministic equations with a time-dependent input vector rather than a source emitting input signals with a prescribed probability distribution.[1] By introducing the dynamic variable $S_j = \mathbf{y}^{\mathrm{T}} \mathbf{m}_j$, we get a state space representation of the LTM and STM equations of the system:

$$\epsilon \dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j S_j, \tag{1.3}$$

$$\dot{S}_j = -S_j + |\mathbf{y}|^2 f(x_j). \tag{1.4}$$

The input stimuli are assumed to be normalized vectors of unit magnitude $|\mathbf{y}|^2 = 1$. This system is subject to our analysis considerations to show that the LTM and STM trajectories are bounded.

[1] Our interest is to store patterns as equilibrium points in the N-dimensional space. In fact, in Amari (1982) is demonstrated the formation of stable one-dimensional cortical maps under the aspect of topological correspondence and under the restriction of a constant probability of the input signal.

2. Equilibrium and local asymptotic stability analysis of neuro-synaptic systems

In this section, we present a new condition for the uniqueness and global exponential stability for neuro-synaptic systems which improves the previous stability results. The existence and uniqueness of the equilibrium is given based on flow-invariance while the global exponential stability is shown by a strict Lyapunov function.

The theory of flow-invariance gives a qualitative interpretation of the dynamics of a system, taking into account the invariance of the flow of the system. In other words a trajectory gets trapped in an invariant set.

Before stating the stability results based on the concept of flow-invariance, we define several important notions used in nonlinear analysis.

Definition 1. Let F: $R^N \to R^N$ be a Lipschitz continuous map and let S be a subset of R^N. We say that S is *flow-invariant* with respect to the system of differential equations

$$x'(t) = F(x(t)) \tag{S}$$

if any solution $x(t)$ starting in S at $t = 0$ remains in S for all $t \geqslant 0$ as long as $x(t)$ is defined. In dynamical systems terminology, such sets are called positively invariant under the flow generated by (S).

Definition 2. We say that system (S) is *dissipative* in R^N if there exists a precompact (bounded) set $U \subset R^N$ such that for any solution $x(t)$ of (S) there exists $T \geqslant 0$ such that $x(t) \in U$ for all $t \geqslant T$. In other words, all solutions of (S) enter this bounded set U in finite time.

If (S) is dissipative then all solutions of (S) are defined for $t \geqslant 0$, and there exists a compact set $A \subset U$ which attracts all solutions of (S). The set A is invariant under the flow of (S) and it is called the *global attractor* of (S) in R^N.

After we have introduced the definitions, we are ready to state the stability results based on the concept of flow-invariance. For simplicity reasons, we will omit ϵ in Eq. (1.3).

Theorem 1. *Consider the system of differential equations*

$$x_j'(t) = -a_j x_j(t) + \sum_{i=1}^{N} D_{ij} f(x_i(t)) + B_j S_j(t), \quad j = 1, \ldots, N, \tag{2.1}$$

$$S_j'(t) = -S_j(t) + f(x_j(t)), \; j = 1, \ldots, N \tag{2.2}$$

and suppose that $a_j > 0$ for all $j = 1, \ldots, N$. Also suppose that f is locally Lipschitz and bounded, that is, there exists a constant $M > 0$ such that $-M \leqslant f(x) \leqslant M$ for all $x \in R$. Let

$$l_j = \frac{M}{a_j}\left(\sum_{i=1}^{N} |D_{ij}| + |B_j|\right) > 0, \; j = 1, \ldots, N. \tag{2.3}$$

Then for any $\epsilon > 0$ and for any initial condition $\{x_j(0), S_j(0)\} \in R^{2N}$ there exists a $T \geqslant 0$ such that

$$S_j(t) \in [-M-\epsilon, M+\epsilon], \quad x_j(t) \in [-l_j - \epsilon, l_j + \epsilon]$$

for all $j = 1, \ldots, N$ and all $t \geqslant T$.

Proof. Since f is locally Lipschitz, system (2.1) and (2.2) enjoys local existence and uniqueness of solutions. Moreover, since f is uniformly bounded, there exist constants $K_1, \ldots, K_5 > 0$ such that

$$|x_j'(t)| \leqslant K_1 + K_2|x_j(t)| + K_3|S_j(t)|,$$
$$|S_j'(t)| \leqslant K_4 + K_5|S_j(t)|,$$

thus all solutions are defined globally (for all $t \geqslant 0$).

Given $\epsilon > 0$, we define

$$\delta_j = \begin{cases} \min\left(\dfrac{a_j\epsilon}{2|B_j|}, \epsilon\right), & B_j \neq 0, \\ \epsilon, & B_j = 0 \end{cases}$$

for $j = 1, \ldots, N$. It follows that $\delta_j > 0$ and $-|B_j|\delta_j + a_j\epsilon \geqslant a_j\epsilon/2$ for all $j = 1, \ldots, N$. Then for $t \geqslant 0$ and for $S_j(t) \leqslant -M - \delta_j$ the following inequality holds:

$$S_j'(t) \geqslant -(-M-\delta_j) + f(x_j(t)) = \delta_j + (f(x_j(t)) + M) \geqslant \delta_j > 0.$$

Similarly, for $t \geqslant 0$ and for $S_j(t) \geqslant M + \delta_j$ we have that

$$S_j'(t) \leqslant -(M+\delta_j) + f(x_j(t)) = -\delta_j + (f(x_j(t)) - M) \leqslant -\delta_j < 0.$$

Therefore, for any $j \in \{1, \ldots, N\}$ there exists a $T_j^s \geqslant 0$ such that

$$S_j(t) \in [-M-\delta_j, M+\delta_j] \subseteq [-M-\epsilon, M+\epsilon] \tag{2.4}$$

for all $t \geqslant T_j^s$. Let $T^s = \max_j T_j^s$, then (2.4) holds for all $j \in \{1, \ldots, N\}$ and for all $t \geqslant T^s$.

Now we consider $t \geqslant T^s$. For $x_j(t) \leqslant -l_j - \epsilon$, (2.1) and (2.4) imply that

$$x_j'(t) \geqslant a_j(l_j + \epsilon) + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j(-M - \delta_j).$$

Using the definition of l_j given by (2.3), we find that for $t \geqslant T^s$ and $x_j(t) \leqslant -l_j - \epsilon$,

$$x_j'(t) \geqslant a_j l_j + a_j\epsilon - M\left(\sum_{i=1}^{N} |D_{ij}| + |B_j|\right) - |B_j|\delta_j = -|B_j|\delta_j + a_j\epsilon \geqslant \frac{a_j\epsilon}{2} > 0.$$

Similarly, for $t \geq T^s$ and for $x_j(t) \geq l_j + \epsilon$, (2.1) and (2.4) imply that

$$x_j'(t) \leq -a_j(l_j + \epsilon) + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j(M + \delta_j).$$

Using (2.3) again, we find that for $t \geq T^s$ and $x_j(t) \geq l_j + \epsilon$,

$$x_j'(t) \leq -a_j l_j - a_j\epsilon + M\left(\sum_{i=1}^{N} |D_{ij}| + |B_j|\right) + |B_j|\delta_j$$
$$= -a_j\epsilon + |B_j|\delta_j \leq -\frac{a_j\epsilon}{2} < 0.$$

Consequently, for any $j \in \{1, \ldots, N\}$ there exists a $T_j^x \geq T^s \geq 0$ such that

$$x_j(t) \in [l_j - \epsilon, l_j + \epsilon] \tag{2.5}$$

for all $t \geq T_j^x$. Let $T = \max_j T_j^x$, then both (2.4) and (2.5) hold for all $j \in \{1, \ldots, N\}$ and all $t \geq T$.

Corollary 1. *System* (2.1) and (2.2) *is dissipative in* R^{2N} *and therefore it has a compact global attractor*

$$A \subseteq D = \prod_{j=1}^{N} [-l_j, l_j] \times \prod_{j=1}^{N} [-M, M].$$

Corollary 2. *It follows from the proof of Theorem* 1 *that the set D is flow invariant under* (2.1) *and* (2.2). *In other words, D is a positively invariant set of* (2.1) *and* (2.2), *that is, any solution starting in D at* $t = 0$ *remains in D for all* $t \geq 0$.

Theorem 2. *Suppose that* $a_j > 1$ *for* $j \in \{1, \ldots, N\}$ *and suppose that* $f(x)$ *is* C^1 *with* $|f'(x)| \leq k$ *for all x. If*

$$k\left(1 + \max_i \sum_{j=1}^{N} |D_{ji}|\right) < \frac{1}{1 + \max_j \frac{|B_j|}{a_j - 1}}, \tag{2.6}$$

then system (2.1) *and* (2.2) *is globally exponentially stable. That is, there exists a unique equilibrium* $e^* \in D$ *and all solutions converge to* e^* *exponentially fast as* $t \to \infty$.

Proof. Since all solutions of (2.1) and (2.2) are bounded for $t \geq 0$ by Theorem 1, it suffices to prove that every solution is exponentially stable. Existence of e^* follows automatically from Corollary 2. Let $\{x_j(t), S_j(t)\}$ be a solution of (2.1) and (2.2). Consider the variational system associated with $\{S_j(t), x_j(t)\}$:

$$\phi_j'(t) = -a_j\phi_j(t) + \sum_{i=1}^{N} D_{ij} f'(x_i(t))\phi_i(t) + B_j\psi_j(t),$$
$$j = 1, \ldots, N, \tag{2.7}$$

$$\psi_j'(t) = -\psi_j(t) + f'(x_j(t))\phi_j(t), \; j = 1, \ldots, N. \tag{2.8}$$

We will show that (2.6) is a sufficient condition for exponential stability of (2.7) and (2.8). To do so, we rewrite (2.7) and (2.8) in the matrix form:

$$Z'(t) = (A + B(t))Z(t), \tag{2.9}$$

where $Z = (\phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_N, \psi_N)^{\mathrm{T}}$ and matrices A and $B(t)$ are as follows:

$$A = \mathrm{diag}(A_{jj})_{j=1}^{N}, \quad A_{jj} = \begin{pmatrix} -a_j & B_j \\ 0 & -1, \end{pmatrix}$$

$$B(t) = (B_{ij}(t))_{i,j=1}^{N},$$

$$B_{ij}(t) = \begin{cases} \begin{pmatrix} D_{jj} f'(x_j(t)) & 0 \\ f'(x_j(t)) & 0 \end{pmatrix}, & i = j, \\ \begin{pmatrix} D_{ji} f'(x_i(t)) & 0 \\ 0 & 0 \end{pmatrix}, & i \neq j. \end{cases}$$

To show exponential stability of (2.9), we employ the standard stability argument for a nonautonomously perturbed linear system with a constant matrix (Adrianova, 1995): if $\|e^{tA}\| \leq M e^{-\alpha t}$ for $t \geq 0$ with $M, \alpha > 0$ and $\|B(t)\| \leq C_1 < \alpha/M$ for $t \geq 0$ then (2.9) is exponentially asymptotically stable. The result of the theorem is obtained by applying the above criterion to the matrix norm generated by the l^∞ vector norm in R^{2N}. The corresponding matrix norm is

$$\|Q\| = \max_i \sum_{j=1}^{N} |Q_{ij}|. \tag{2.10}$$

The matrix e^{tA} is given by

$$e^{tA} = \mathrm{diag}\begin{pmatrix} e^{-a_j t} & \frac{B_j}{a_j - 1}(e^{-t} - e^{-a_j t}) \\ 0 & e^{-t} \end{pmatrix}_{j=1}^{N},$$

so that

$$\|e^{tA}\| \leq \left(1 + \max_j (1 + \frac{|B_j|}{a_j - 1}\right) e^{-t}, \tag{2.11}$$

thus $M = 1 + \max_j(|B_j|/a_j - 1)$ and $\alpha = 1$. Since $|f'| \leq k$, the corresponding norm of B can be estimated as

$$\|B(t)\| \leq k\left(1 + \max_i \sum_{j=1}^{N} |D_{ji}|\right). \tag{2.12}$$

Finally, we combine estimates (2.10) and (2.11) with the stability criterion to obtain inequality (2.6).

3. Comparisons

In this section, we compare the results obtained by Theorem 2 with those in the literature.

In Adrianova (1995) the convergence to point attractors is proved based on the condition of high gain approximation which means that the output nonlinearity is approximated by a step function. It is also assumed that the synaptic connection parameter D_{ij}

is given by

$$D_{ij} = \begin{cases} \alpha, & i = j, \\ -\beta, & i \neq j. \end{cases}$$

In this paper, we also employ the concept of flow invariance but are able to prove the uniqueness and existence of the equilibrium by only imposing that the output nonlinearity is bounded in R.

The approach proposed in Meyer-Bäse et al. (1996) is based on the theory of singular perturbations, and treats both fast and slow dynamics seperately. It requires certain growth conditions to be satisfied, and four different inequalities to hold. While these imposed conditions are too difficult to test for the N-dimensional case, our approach requires only a simple inequality to hold.

4. Examples

In this section, simulations results are given to verify the theoretical results discussed in previous sections.

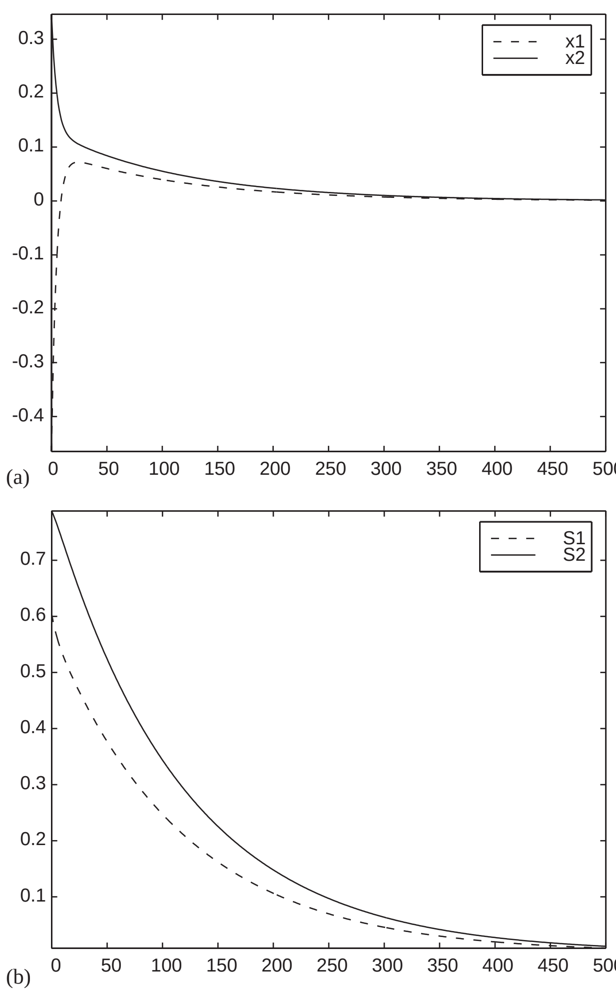

Fig. 1. Time histories of the neural network in Example 1: (a) STM states and (b) LTM states.

Example 1. Consider a two-neuron system with a nonlinearity of a sigmoidal type and a diagonal synaptic connection matrix **D** with

$$D_{ij} = \begin{cases} \alpha, & i = j, \\ -\beta, & i \neq j. \end{cases}$$

As numerical values we choose $\alpha = 0.5$, $\beta = 0.1$, $A = 20$ and $B = 3$.

The stability conditions imposed in Theorem 2 are fulfilled, and we see from Fig. 1 that the trajectories of the LTM and STM states converge to the zero equilibrium point.

Example 2. The design of a three-neuron system with a nonlinearity of sigmoidal type and a desired equilibrium point at $\mathbf{x} = [5, 5, 5]$ and $\mathbf{S} = [1, 1, 1]$ is demonstrated. The trajectories of the LTM and STM states are shown in Fig. 2.

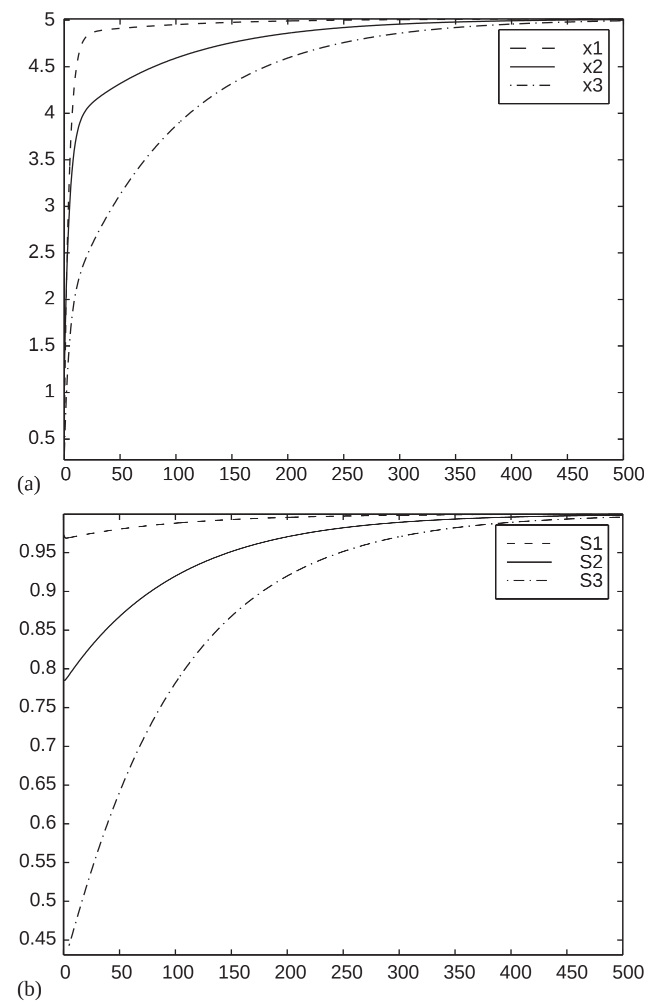

Fig. 2. Time histories of the neural network in Example 2 with $\mathbf{x} = [5, 5, 5]$ and $\mathbf{S} = [1, 1, 1]$ as an equilibrium point: (a) STM states and (b) LTM states.

5. Conclusions

In this paper we prove local exponential stability of competitive neural networks with fast and slow dynamics describing cognitive cortical maps developed by self-organization. Based on the flow invariance technique we can show the conditions that the LTM and STM trajectories are bounded, being at the same time less restrictive and more general than with the K-monotone theory or singular perturbation theory. Numerical simulation results are also given to verify the theoretical results.

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