Example 8 For the initial value problem

$$\textbf{(11)} \qquad 3\frac{dy}{dx} = x^2 - xy^3 \ , \quad y(1) = 6 \ ,$$

does Theorem 1 imply the existence of a unique solution?

Solution Dividing by 3 to conform to the statement of the theorem, we identify $f(x, y)$ as $(x^2 - xy^3)/3$ and $\partial f/\partial y$ as $-xy^2$. Both of these functions are continuous in any rectangle containing the point (1, 6), so the hypotheses of Theorem 1 are satisfied. It then follows from the theorem that the initial value problem (11) has a unique solution in an interval about $x = 1$ of the form $(1 - \delta, 1 + \delta)$, where δ is some positive number. ◆

Example 9 For the initial value problem

$$\textbf{(12)} \qquad \frac{dy}{dx} = 3y^{2/3} \ , \qquad y(2) = 0 \ ,$$

does Theorem 1 imply the existence of a unique solution?

Solution Here $f(x, y) = 3y^{2/3}$ and $\partial f/\partial y = 2y^{-1/3}$. Unfortunately $\partial f/\partial y$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both f and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 and Group Project G of Chapter 2 for the details. ◆

In Example 9 suppose the initial condition is changed to $y(2) = 1$. Then, since f and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2, 1)$ but does not intersect the x-axis—say, $R = \{(x, y): 0 < x < 10, 0 < y < 5\}$—it follows from Theorem 1 that this *new* initial value problem has a unique solution in some interval about $x = 2$.

1.2 EXERCISES

1. **(a)** Show that $y^2 + x - 3 = 0$ is an implicit solution to $dy/dx = -1/(2y)$ on the interval $(-\infty, 3)$.

(b) Show that $xy^3 - xy^3 \sin x = 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{(x \cos x + \sin x - 1)y}{3(x - x \sin x)}$$

on the interval $(0, \pi/2)$.

(c) Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 d^2y/dx^2 = 2y$ on the interval $(0, \infty)$.

2. **(a)** Show that $\phi(x) = x^2$ is an explicit solution to

$$x\frac{dy}{dx} = 2y$$

on the interval $(-\infty, \infty)$.

(b) Show that $\phi(x) = e^x - x$ is an explicit solution to

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

on the interval $(-\infty, \infty)$.

In Problems 3–8, determine whether the given function is a solution to the given differential equation.

3. $x = 2\cos t - 3\sin t \ , \quad x'' + x = 0$

4. $y = \sin x + x^2 \ , \quad \dfrac{d^2y}{dx^2} + y = x^2 + 2$

5. $x = \cos 2t \ , \quad \dfrac{dx}{dt} + tx = \sin 2t$

6. $\theta = 2e^{3t} - e^{2t} \ , \quad \dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$

7. $y = 3\sin 2x + e^{-x} \ , \quad y'' + 4y = 5e^{-x}$

8. $y = e^{2x} - 3e^{-x} \ , \quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

9. $y - \ln y = x^2 + 1\ ,\qquad \dfrac{dy}{dx} = \dfrac{2xy}{y-1}$

10. $x^2 + y^2 = 4\ ,\qquad \dfrac{dy}{dx} = \dfrac{x}{y}$

11. $e^{xy} + y = x - 1\ ,\qquad \dfrac{dy}{dx} = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

12. $x^2 - \sin(x+y) = 1\ ,\qquad \dfrac{dy}{dx} = 2x\sec(x+y) - 1$

13. $\sin y + xy - x^3 = 2\ ,$

$$y'' = \frac{6xy' + (y')^3\sin y - 2(y')^2}{3x^2 - y}$$

14. Show that $\phi(x) = c_1 \sin x + c_2 \cos x$ is a solution to $d^2y/dx^2 + y = 0$ for any choice of the constants c_1 and c_2. Thus, $c_1 \sin x + c_2 \cos x$ is a two-parameter family of solutions to the differential equation.

15. Verify that $\phi(x) = 2/(1 - ce^x)$, where c is an arbitrary constant, is a one-parameter family of solutions to

$$\frac{dy}{dx} = \frac{y(y-2)}{2}\ .$$

Graph the solution curves corresponding to $c = 0, \pm 1, \pm 2$ using the same coordinate axes.

16. Verify that $x^2 + cy^2 = 1$, where c is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

and graph several of the solution curves using the same coordinate axes.

17. Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $dy/dx - 3y = -3$ for any choice of the constant C. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as x approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger δ. This illustrates that in Theorem 1 the existence interval can be quite small (if c is small) or quite large (if c is large). Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

20. Determine for which values of m the function $\phi(x) = e^{mx}$ is a solution to the given equation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = 0$

(b) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

21. Determine for which values of m the function $\phi(x) = x^m$ is a solution to the given equation.

(a) $3x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} - 3y = 0$

(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 5y = 0$

22. Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

(a) $y(0) = 2\ ,\quad y'(0) = 1$
(b) $y(1) = 1\ ,\quad y'(1) = 0$

In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.

23. $\dfrac{dy}{dx} = y^4 - x^4\ ,\qquad y(0) = 7$

24. $\dfrac{dy}{dt} - ty = \sin^2 t\ ,\qquad y(\pi) = 5$

25. $3x\dfrac{dx}{dt} + 4t = 0\ ,\qquad x(2) = -\pi$

26. $\dfrac{dx}{dt} + \cos x = \sin t\ ,\qquad x(\pi) = 0$

27. $y\dfrac{dy}{dx} = x\ ,\qquad y(1) = 0$

28. $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}\ ,\qquad y(2) = 1$

29. **(a)** For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions. (See also Group Project G in Chapter 2.)

(b) Does the initial value problem $y' = 3y^{2/3}$, $y(0) = 10^{-7}$, have a unique solution in a neighborhood of $x = 0$?

30. **Implicit Function Theorem.** *Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $G_y(x_0, y_0) \neq 0$, then there exists a differentiable function $y = \phi(x)$, defined in some interval $I = (x_0 - \delta, x_0 + \delta)$, that satisfies $G\big(x, \phi(x)\big) = 0$ for all $x \in I$.*

The implicit function theorem gives conditions under which the relationship $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit function theorem to show that the relationship $x + y + e^{xy} = 0$, given in Example 4, defines y implicitly as a function of x near the point $(0, -1)$.

31. Consider the equation of Example 5,

$$y\frac{dy}{dx} - 4x = 0 \ . \tag{13}$$

(a) Does Theorem 1 imply the existence of a unique solution to (13) that satisfies $y(x_0) = 0$?

(b) Show that when $x_0 \neq 0$, equation (13) can't possibly have a solution in a neighborhood of $x = x_0$ that satisfies $y(x_0) = 0$.

(c) Show that there are two distinct solutions to (13) satisfying $y(0) = 0$ (see Figure 1.4 on page 9).

1.3 DIRECTION FIELDS

The existence and uniqueness theorem discussed in Section 1.2 certainly has great value, but it stops short of telling us anything about the *nature* of the solution to a differential equation. For practical reasons we may need to know the value of the solution at a certain point, or the intervals where the solution is increasing, or the points where the solution attains a maximum value. Certainly, knowing an explicit representation (a formula) for the solution would be a considerable help in answering these questions. However, for many of the differential equations that we are likely to encounter in real-world applications, it will be impossible to find such a formula. Moreover, even if we are lucky enough to obtain an implicit solution, using this relationship to determine an explicit form may be difficult. Thus, we must rely on other methods to analyze or approximate the solution.

One technique that is useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation. Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined. In other words, it gives the direction that a graph of a solution to the equation must have at each point. Consider, for example, the equation

$$\frac{dy}{dx} = x^2 - y \ . \tag{1}$$

The graph of a solution to (1) that passes through the point $(-2, 1)$ must have slope $(-2)^2 - 1 = 3$ at that point, and a solution through $(-1, 1)$ has zero slope at that point.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the