2.2 EXERCISES

In Problems 1–6, determine whether the given differential equation is separable.

1. $\frac{dy}{dx} - \sin(x + y) = 0$ **2.** $\frac{dy}{dx} = 4y^2 - 3y + 1$

3. $\frac{ds}{dt} = t \ln(s^{2t}) + 8t^2$ **4.** $\frac{dy}{dx} = \frac{ye^{x+y}}{x^2 + 2}$

5. $(xy^2 + 3y^2)dy - 2x\,dx = 0$

6. $s^2 + \frac{ds}{dt} = \frac{s + 1}{st}$

In Problems 7–16, solve the equation.

7. $\frac{dx}{dt} = 3xt^2$ **8.** $x\frac{dy}{dx} = \frac{1}{y^3}$

9. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{1 + x}}$ **10.** $\frac{dx}{dt} = \frac{t}{xe^{t+2x}}$

11. $\frac{dy}{dx} = \frac{\sec^2 y}{1 + x^2}$ **12.** $x\frac{dv}{dx} = \frac{1 - 4v^2}{3v}$

13. $\frac{dx}{dt} - x^3 = x$ **14.** $\frac{dy}{dx} = 3x^2(1 + y^2)^{3/2}$

15. $y^{-1}\,dy + ye^{\cos x} \sin x\,dx = 0$

16. $(x + xy^2)dx + e^{x^2}y\,dy = 0$

In Problems 17–26, solve the initial value problem.

17. $y' = x^3(1 - y)$, $y(0) = 3$

18. $\frac{dy}{dx} = (1 + y^2)\tan x$, $y(0) = \sqrt{3}$

19. $\frac{1}{2}\frac{dy}{dx} = \sqrt{y + 1}\cos x$, $y(\pi) = 0$

20. $x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x + 1)(y + 1)}$, $y(1) = 1$

21. $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y \sin\theta}{y^2 + 1}$, $y(\pi) = 1$

22. $x^2\,dx + 2y\,dy = 0$, $y(0) = 2$

23. $t^{-1}\frac{dy}{dt} = 2\cos^2 y$, $y(0) = \pi/4$

24. $\frac{dy}{dx} = 8x^3e^{-2y}$, $y(1) = 0$

25. $\frac{dy}{dx} = x^2(1 + y)$, $y(0) = 3$

26. $\sqrt{y}\,dx + (1 + x)dy = 0$, $y(0) = 1$

27. Solutions Not Expressible in Terms of Elementary Functions. As discussed in calculus, certain indefinite integrals (antiderivatives) such as $\int e^{x^2}dx$ cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration (integrals with variable upper limit). For example, consider the initial value problem

$$\frac{dy}{dx} = e^{x^2}y^2 , \qquad y(2) = 1 .$$

The differential equation separates if we divide by y^2 and multiply by dx. We integrate the separated equation from $x = 2$ to $x = x_1$ and find

$$\int_{x=2}^{x=x_1} e^{x^2}\,dx = \int_{x=2}^{x=x_1} \frac{dy}{y^2} = -\frac{1}{y}\Big|_{x=2}^{x=x_1} = -\frac{1}{y(x_1)} + \frac{1}{y(2)} .$$

If we let t be the variable of integration and replace x_1 by x and $y(2)$ by 1, then we can express the solution to the initial value problem by

$$y(x) = \left(1 - \int_2^x e^{t^2}dt\right)^{-1} .$$

Use definite integration to find an explicit solution to the initial value problems in parts (a)–(c).

(a) $dy/dx = e^{x^2}$, $y(0) = 0$

(b) $dy/dx = e^{x^2}y^{-2}$, $y(0) = 1$

(c) $dy/dx = \sqrt{1 + \sin x}\,(1 + y^2)$, $y(0) = 1$

(d) Use a numerical integration algorithm (such as Simpson's rule, described in Appendix C) to approximate the solution to part (b) at $x = 0.5$ to three decimal places.

28. Sketch the solution to the initial value problem

$$\frac{dy}{dt} = 2y - 2yt , \qquad y(0) = 3$$

and determine its maximum value.

29. Uniqueness Questions. In Chapter 1 we indicated that in applications most *initial value problems* will have a unique solution. In fact, the existence of unique solutions was so important that we stated an existence and uniqueness theorem, Theorem 1, page 11. The method for separable equations can give us a solution, but it may not give us all the solutions (also see Problem 30). To illustrate this, consider the equation $dy/dx = y^{1/3}$.

(a) Use the method of separation of variables to show that
$$y = \left(\frac{2x}{3} + C\right)^{3/2}$$
is a solution.

(b) Show that the initial value problem $dy/dx = y^{1/3}$ with $y(0) = 0$ is satisfied for $C = 0$ by $y = (2x/3)^{3/2}$ for $x \geq 0$.

(c) Now show that the constant function $y \equiv 0$ also satisfies the initial value problem given in part (b). Hence, this initial value problem does not have a unique solution.

(d) Finally, show that the conditions of Theorem 1 on page 11 are not satisfied.

(The solution $y \equiv 0$ was lost because of the division by zero in the separation process.)

30. As stated in this section, the separation of equation (2) on page 39 requires division by $p(y)$, and this may disguise the fact that the roots of the equation $p(y) = 0$ are actually constant solutions to the differential equation.

(a) To explore this further, separate the equation
$$\frac{dy}{dx} = (x - 3)(y + 1)^{2/3}$$
to derive the solution,
$$y = -1 + (x^2/6 - x + C)^3 .$$

(b) Show that $y \equiv -1$ satisfies the original equation $dy/dx = (x - 3)(y + 1)^{2/3}$.

(c) Show that there is no choice of the constant C that will make the solution in part (a) yield the solution $y \equiv -1$. Thus, we lost the solution $y \equiv -1$ when we divided by $(y + 1)^{2/3}$.

31. Interval of Definition. By looking at an initial value problem $dy/dx = f(x, y)$ with $y(x_0) = y_0$, it is not always possible to determine the domain of the solution $y(x)$ or the interval over which the function $y(x)$ satisfies the differential equation.

(a) Solve the equation $dy/dx = xy^3$.

(b) Give explicitly the solutions to the initial value problem with $y(0) = 1$; $y(0) = 1/2$; $y(0) = 2$.

(c) Determine the domains of the solutions in part (b).

(d) As found in part (c), the domains of the solutions depend on the initial conditions. For the initial value problem $dy/dx = xy^3$ with $y(0) = a$, $a > 0$, show that as a approaches zero from the right the domain approaches the whole real line $(-\infty, \infty)$ and as a approaches $+\infty$ the domain shrinks to a single point.

(e) Sketch the solutions to the initial value problem $dy/dx = xy^3$ with $y(0) = a$ for $a = \pm 1/2, \pm 1$, and ± 2.

32. Analyze the solution $y = \phi(x)$ to the initial value problem
$$\frac{dy}{dx} = y^2 - 3y + 2 , \qquad y(0) = 1.5$$
using approximation methods and then compare with its exact form as follows.

(a) Sketch the direction field of the differential equation and use it to guess the value of $\lim_{x\to\infty} \phi(x)$.

(b) Use Euler's method with a step size of 0.1 to find an approximation of $\phi(1)$.

(c) Find a formula for $\phi(x)$ and graph $\phi(x)$ on the direction field from part (a).

(d) What is the exact value of $\phi(1)$? Compare with your approximation in part (b).

(e) Using the exact solution obtained in part (c), determine $\lim_{x\to\infty} \phi(x)$ and compare with your guess in part (a).

33. Mixing. Suppose a brine containing 0.3 kilogram (kg) of salt per liter (L) runs into a tank initially filled with 400 L of water containing 2 kg of salt. If the brine enters at 10 L/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. Find the mass of salt in the tank after 10 min (see Figure 2.4). [*Hint:* Let A denote the number of kilograms of salt in the tank at t min after the process begins and use the fact that

rate of increase in A = rate of input − rate of exit.

A further discussion of mixing problems is given in Section 3.2.]

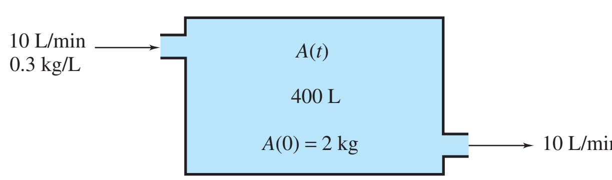

Figure 2.4 Schematic representation of a mixing problem

34. Newton's Law of Cooling. According to Newton's law of cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M - T$. This gives the differential equation

$$dT/dt = k(M - T) \; .$$

(a) Solve the differential equation for T.

(b) A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 min, the thermometer reads 80°F. What is the reading after 20 min?

(Further applications of Newton's law of cooling appear in Section 3.3.)

35. Blood plasma is stored at 40°F. Before the plasma can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 min for the plasma to warm to 90°F. Assume Newton's law of cooling (Problem 34) applies. How long will it take for the plasma to warm to 90°F if the oven temperature is set at **(a)** 100°F, **(b)** 140°F, and **(c)** 80°F?

36. A pot of boiling water at 100°C is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 min, the water temperature has decreased to 80°C, and another 5 min later it has dropped to 65°C. Assuming Newton's law of cooling (Problem 34) applies, determine the (constant) temperature of the kitchen.

37. Compound Interest. If $P(t)$ is the amount of dollars in a savings bank account that pays a yearly interest rate of $r\%$ *compounded continuously,* then

$$\frac{dP}{dt} = \frac{r}{100}P \; , \qquad t \text{ in years.}$$

Assume the interest is 5% annually, $P(0) = \$1000$, and no monies are withdrawn.

(a) How much will be in the account after 2 yr?

(b) When will the account reach $4000?

(c) If $1000 is added to the account every 12 months, how much will be in the account after $3\frac{1}{2}$ yr?

38. Free Fall. In Section 2.1, we discussed a model for an object falling toward Earth. Assuming that only air resistance and gravity are acting on the object, we found that the velocity v must satisfy the equation

$$m\frac{dv}{dt} = mg - bv \; ,$$

where m is the mass, g is the acceleration due to gravity, and $b > 0$ is a constant (see Figure 2.1). If $m = 100$ kg, $g = 9.8$ m/sec^2, $b = 5$ kg/sec, and $v(0) = 10$ m/sec, solve for $v(t)$. What is the limiting (i.e., terminal) velocity of the object?

39. Grand Prix Race. Driver A had been leading archrival B for a while by a *steady* 3 miles. Only 2 miles from the finish, driver A ran out of gas and decelerated thereafter at a rate proportional to the square of his remaining speed. One mile later, driver A's speed was exactly halved. If driver B's speed remained constant, who won the race?

40. The *atmospheric pressure* (force per unit area) on a surface at an altitude z is due to the weight of the column of air situated above the surface. Therefore, the difference in air pressure p between the top and bottom of a cylindrical volume element of height Δz and cross-section area A equals the weight of the air enclosed (density ρ times volume $V = A\Delta z$ times gravity g), per unit area:

$$p(z + \Delta z) - p(z) = -\frac{\rho(z)(A\Delta z)g}{A} = -\rho(z)g\Delta z.$$

Let $\Delta z \to 0$ to derive the differential equation $dp/dz = -\rho g$. To analyze this further we must postulate a formula that relates pressure and density. The *perfect gas law* relates pressure, volume, mass m, and absolute temperature T according to $pV = mRT/M$, where R is the universal gas constant and M is the molar mass of the air. Therefore, density and pressure are related by $\rho := m/V = Mp/RT$.

(a) Derive the equation $\dfrac{dp}{dz} = -\dfrac{Mg}{RT}p$ and solve it for the "isothermal" case where T is constant to obtain the *barometric pressure equation* $p(z) = p(z_0)\exp[-Mg(z-z_0)/RT]$.

(b) If the temperature also varies with altitude $T = T(z)$, derive the solution

$$p(z) = p(z_0)\exp\left\{-\frac{Mg}{R}\int_{z_0}^{z}\frac{d\zeta}{T(\zeta)}\right\}.$$

(c) Suppose an engineer measures the barometric pressure at the top of a building to be 99,000 Pa (pascals), and 101,000 Pa at the base $(z = z_0)$. If the absolute temperature varies as $T(z) = 288 - 0.0065(z - z_0)$, determine the height of the building. Take $R = 8.31$ N-m/mol-K, $M = 0.029$ kg/mol, and $g = 9.8$ m/sec^2. (An amusing story concerning this problem can be found at http://www.snopes.com/college/exam/barometer.asp)

In Example 3 we had no difficulty expressing the integral for the integrating factor $\mu(x) = e^{\int 1\,dx} = e^x$. Clearly, situations will arise where this integral, too, cannot be expressed with elementary functions. In such cases we must again resort to a numerical procedure such as Euler's method (Section 1.4) or to a "nested loop" implementation of Simpson's rule. You are invited to explore such a possibility in Problem 27.

Because we have established explicit formulas for the solutions to *linear* first-order differential equations, we get as a dividend a direct proof of the following theorem.

Existence and Uniqueness of Solution

Theorem 1. Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

(15) $$\frac{dy}{dx} + P(x)y = Q(x)\ , \qquad y(x_0) = y_0\ .$$

In fact, the solution is given by (8) for a suitable value of C.

The essentials of the proof of Theorem 1 are contained in the deliberations leading to equation (8); Problem 34 provides the details. This theorem differs from Theorem 1 on page 11 in that for the *linear* initial value problem (15), we have the existence and uniqueness of the solution on the *whole* interval (a, b), rather than on some smaller unspecified interval about x_0.

The theory of linear differential equations is an important branch of mathematics not only because these equations occur in applications but also because of the elegant structure associated with them. For example, first-order linear equations always have a general solution given by equation (8). Some further properties of first-order linear equations are described in Problems 28 and 36. Higher-order linear equations are treated in Chapters 4, 6, and 8.

2.3 EXERCISES

In Problems 1–6, determine whether the given equation is separable, linear, neither, or both.

1. $\dfrac{dx}{dt} + xt = e^x$

2. $x^2\dfrac{dy}{dx} + \sin x - y = 0$

3. $3t = e^t\dfrac{dy}{dt} + y\ln t$

4. $(t^2 + 1)\dfrac{dy}{dt} = yt - y$

5. $3r = \dfrac{dr}{d\theta} - \theta^3$

6. $x\dfrac{dx}{dt} + t^2x = \sin t$

In Problems 7–16, obtain the general solution to the equation.

7. $\dfrac{dy}{dx} = \dfrac{y}{x} + 2x + 1$

8. $\dfrac{dy}{dx} - y - e^{3x} = 0$

9. $x\dfrac{dy}{dx} + 2y = x^{-3}$

10. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

11. $(t + y + 1)dt - dy = 0$

12. $\dfrac{dy}{dx} = x^2e^{-4x} - 4y$

13. $y\dfrac{dx}{dy} + 2x = 5y^3$

14. $x\dfrac{dy}{dx} + 3(y + x^2) = \dfrac{\sin x}{x}$

15. $(x^2 + 1)\dfrac{dy}{dx} + xy - x = 0$

16. $(1 - x^2)\dfrac{dy}{dx} - x^2y = (1 + x)\sqrt{1 - x^2}$

In Problems 17–22, solve the initial value problem.

17. $\frac{dy}{dx} - \frac{y}{x} = xe^x$, $\quad y(1) = e - 1$

18. $\frac{dy}{dx} + 4y - e^{-x} = 0$, $\quad y(0) = \frac{4}{3}$

19. $t^2\frac{dx}{dt} + 3tx = t^4 \ln t + 1, \quad x(1) = 0$

20. $\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x$, $\quad y(1) = 1$

21. $\cos x\frac{dy}{dx} + y \sin x = 2x \cos^2 x$,

$$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$$

22. $\sin x \frac{dy}{dx} + y \cos x = x \sin x$, $\quad y\left(\frac{\pi}{2}\right) = 2$

23. Radioactive Decay. In Example 2 assume that the rate at which RA_1 decays into RA_2 is $40e^{-20t}$ kg/sec and the decay constant for RA_2 is $k = 5$/sec. Find the mass $y(t)$ of RA_2 for $t \geq 0$ if initially $y(0) = 10$ kg.

24. In Example 2 the decay constant for isotope RA_1 was 10/sec, which expresses itself in the exponent of the rate term $50e^{-10t}$ kg/sec. When the decay constant for RA_2 is $k = 2$/sec, we see that in formula (14) for y the term $(185/4)e^{-2t}$ eventually dominates (has greater magnitude for t large).

(a) Redo Example 2 taking $k = 20$/sec. Now which term in the solution eventually dominates?

(b) Redo Example 2 taking $k = 10$/sec.

25. (a) Using definite integration, show that the solution to the initial value problem

$$\frac{dy}{dx} + 2xy = 1 , \quad y(2) = 1 ,$$

can be expressed as

$$y(x) = e^{-x^2}\left(e^4 + \int_2^x e^{t^2}\, dt\right) .$$

(b) Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution at $x = 3$.

26. Use numerical integration (such as Simpson's rule, Appendix C) to approximate the solution, at $x = 1$, to the initial value problem

$$\frac{dy}{dx} + \frac{\sin 2x}{2(1 + \sin^2 x)}y = 1 , \quad y(0) = 0 .$$

Ensure your approximation is accurate to three decimal places.

27. Consider the initial value problem

$$\frac{dy}{dx} + \sqrt{1 + \sin^2 x}\, y = x , \quad y(0) = 2 .$$

(a) Using definite integration, show that the integrating factor for the differential equation can be written as

$$\mu(x) = \exp\left(\int_0^x \sqrt{1 + \sin^2 t}\, dt\right)$$

and that the solution to the initial value problem is

$$y(x) = \frac{1}{\mu(x)}\int_0^x \mu(s)\, s\, ds + \frac{2}{\mu(x)} .$$

(b) Obtain an approximation to the solution at $x = 1$ by using numerical integration (such as Simpson's rule, Appendix C) in a nested loop to estimate values of $\mu(x)$ and, thereby, the value of

$$\int_0^1 \mu(s)\, s\, ds .$$

[*Hint:* First, use Simpson's rule to approximate $\mu(x)$ at $x = 0.1, 0.2, \ldots, 1$. Then use these values and apply Simpson's rule again to approximate $\int_0^1 \mu(s)\, s\, ds$.]

(c) Use Euler's method (Section 1.4) to approximate the solution at $x = 1$, with step sizes $h = 0.1$ and 0.05.

[A direct comparison of the merits of the two numerical schemes in parts (b) and (c) is very complicated, since it should take into account the number of functional evaluations in each algorithm as well as the inherent accuracies.]

28. Constant Multiples of Solutions.

(a) Show that $y = e^{-x}$ is a solution of the linear equation

$$\frac{dy}{dx} + y = 0 , \tag{16}$$

and $y = x^{-1}$ is a solution of the nonlinear equation

$$\frac{dy}{dx} + y^2 = 0 . \tag{17}$$

(b) Show that for any constant C, the function Ce^{-x} is a solution of equation (16), while Cx^{-1} is a solution of equation (17) only when $C = 0$ or 1.

(c) Show that for any linear equation of the form

$$\frac{dy}{dx} + P(x)y = 0 ,$$

if $\hat{y}(x)$ is a solution, then for any constant C the function $C\hat{y}(x)$ is also a solution.

29. Use your ingenuity to solve the equation

$$\frac{dy}{dx} = \frac{1}{e^{4y} + 2x} .$$

[*Hint:* The roles of the independent and dependent variables may be reversed.]

30. Bernoulli Equations. The equation

$$\frac{dy}{dx} + 2y = xy^{-2} \tag{18}$$

is an example of a Bernoulli equation. (Further discussion of Bernoulli equations is in Section 2.6.)

(a) Show that the substitution $v = y^3$ reduces equation (18) to the equation

$$\frac{dv}{dx} + 6v = 3x . \tag{19}$$

(b) Solve equation (19) for v. Then make the substitution $v = y^3$ to obtain the solution to equation (18).

31. Discontinuous Coefficients. As we will see in Chapter 3, occasions arise when the coefficient $P(x)$ in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a "reasonable" solution. For example, consider the initial value problem

$$\frac{dy}{dx} + P(x)y = x , \qquad y(0) = 1 ,$$

where

$$P(x) := \begin{cases} 1 , & 0 \le x \le 2 , \\ 3 , & x > 2 . \end{cases}$$

(a) Find the general solution for $0 \le x \le 2$.

(b) Choose the constant in the solution of part (a) so that the initial condition is satisfied.

(c) Find the general solution for $x > 2$.

(d) Now choose the constant in the general solution from part (c) so that the solution from part (b) and the solution from part (c) agree at $x = 2$. By patching the two solutions together, we can obtain a continuous function that satisfies the differential equation except at $x = 2$, where its derivative is undefined.

(e) Sketch the graph of the solution from $x = 0$ to $x = 5$.

32. Discontinuous Forcing Terms. There are occasions when the forcing term $Q(x)$ in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a reasonable solution imitating the procedure discussed in Problem 31. Use this procedure to find the continuous solution to the initial value problem.

$$\frac{dy}{dx} + 2y = Q(x) , \qquad y(0) = 0 ,$$

where

$$Q(x) := \begin{cases} 2 , & 0 \le x \le 3 , \\ -2 , & x > 3 . \end{cases}$$

Sketch the graph of the solution from $x = 0$ to $x = 7$.

33. Singular Points. Those values of x for which $P(x)$ in equation (4) is not defined are called **singular points** of the equation. For example, $x = 0$ is a singular point of the equation $xy' + 2y = 3x$, since when the equation is written in the standard form, $y' + (2/x)y = 3$, we see that $P(x) = 2/x$ is not defined at $x = 0$. On an interval containing a singular point, the questions of the existence and uniqueness of a solution are left unanswered, since Theorem 1 does not apply. To show the possible behavior of solutions near a singular point, consider the following equations.

(a) Show that $xy' + 2y = 3x$ has only one solution defined at $x = 0$. Then show that the initial value problem for this equation with initial condition $y(0) = y_0$ has a unique solution when $y_0 = 0$ and no solution when $y_0 \neq 0$.

(b) Show that $xy' - 2y = 3x$ has an infinite number of solutions defined at $x = 0$. Then show that the initial value problem for this equation with initial condition $y(0) = 0$ has an infinite number of solutions.

34. Existence and Uniqueness. Under the assumptions of Theorem 1, we will prove that equation (8) gives a solution to equation (4) on (a, b). We can then choose the constant C in equation (8) so that the initial value problem (15) is solved.

(a) Show that since $P(x)$ is continuous on (a, b), then $\mu(x)$ defined in (7) is a positive, continuous function satisfying $d\mu/dx = P(x)\mu(x)$ on (a, b).

(b) Since

$$\frac{d}{dx}\int \mu(x)Q(x)\,dx = \mu(x)Q(x) ,$$

verify that y given in equation (8) satisfies equation (4) by differentiating both sides of equation (8).

(c) Show that when we let $\int \mu(x)Q(x)\,dx$ be the antiderivative whose value at x_0 is 0 (i.e., $\int_{x_0}^{x} \mu(t)Q(t)\,dt$) and choose C to be $y_0\,\mu(x_0)$, the initial condition $y(x_0) = y_0$ is satisfied.

(d) Start with the assumption that $y(x)$ is a solution to the initial value problem (15) and argue that the discussion leading to equation (8) implies that $y(x)$ must obey equation (8). Then argue that the initial condition in (15) determines the constant C uniquely.

35. Mixing. Suppose a brine containing 0.2 kg of salt per liter runs into a tank initially filled with 500 L of water containing 5 kg of salt. The brine enters the tank at a rate of 5 L/min. The mixture, kept uniform by stirring, is flowing out at the rate of 5 L/min (see Figure 2.6).

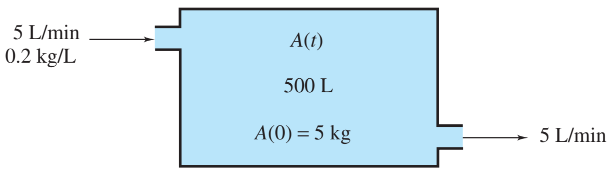

Figure 2.6 Mixing problem with equal flow rates

(a) Find the concentration, in kilograms per liter, of salt in the tank after 10 min. [*Hint:* Let A denote the number of kilograms of salt in the tank at t minutes after the process begins and use the fact that

rate of increase in A = rate of input − rate of exit.

A further discussion of mixing problems is given in Section 3.2.]

(b) After 10 min, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank (see Figure 2.7). What will be the concentration, in kilograms per liter, of salt in the tank 20 min after the leak develops? [*Hint:* Use the method discussed in Problems 31 and 32.]

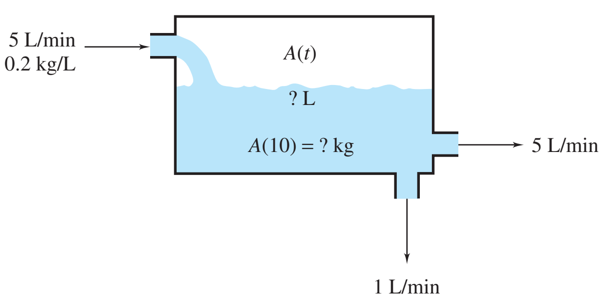

Figure 2.7 Mixing problem with unequal flow rates

36. Variation of Parameters. Here is another procedure for solving linear equations that is particularly useful for higher-order linear equations. This method is called **variation of parameters.** It is based on the idea that just by knowing the *form* of the solution, we can substitute into the given equation and solve for any unknowns. Here we illustrate the method for first-order equations (see Sections 4.6 and 6.4 for the generalization to higher-order equations).

(a) Show that the general solution to

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{20}$$

has the form

$$y(x) = Cy_h(x) + y_p(x) \ ,$$

where y_h ($\not\equiv 0$) is a solution to equation (20) when $Q(x) \equiv 0$, C is a constant, and $y_p(x) = v(x)y_h(x)$ for a suitable function $v(x)$. [*Hint:* Show that we can take $y_h = \mu^{-1}(x)$ and then use equation (8).]

We can in fact determine the unknown function y_h by solving a separable equation. Then direct substitution of vy_h in the original equation will give a simple equation that can be solved for v.

Use this procedure to find the general solution to

$$\frac{dy}{dx} + \frac{3}{x}y = x^2 \ , \qquad x > 0 \ , \tag{21}$$

by completing the following steps:

(b) Find a nontrivial solution y_h to the separable equation

$$\frac{dy}{dx} + \frac{3}{x}y = 0 \ , \qquad x > 0 \ . \tag{22}$$

(c) Assuming (21) has a solution of the form $y_p(x) = v(x)y_h(x)$, substitute this into equation (21), and simplify to obtain $v'(x) = x^2/y_h(x)$.

(d) Now integrate to get $v(x)$.

(e) Verify that $y(x) = Cy_h(x) + v(x)y_h(x)$ is a general solution to (21).

37. Secretion of Hormones. The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-h cycle, then the rate of change of the level of the hormone in the blood may be represented by the initial value problem

$$\frac{dx}{dt} = \alpha - \beta \cos\frac{\pi t}{12} - kx \ , \qquad x(0) = x_0 \ ,$$

where $x(t)$ is the amount of the hormone in the blood at time t, α is the average secretion rate, β is the amount of daily variation in the secretion, and k is a positive constant reflecting the rate at which the body removes the hormone from the blood. If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, solve for $x(t)$.

38. Use the separation of variables technique to derive the solution (7) to the differential equation (6).

39. The temperature T (in units of 100°F) of a university classroom on a cold winter day varies with time t (in hours) as

$$\frac{dT}{dt} = \begin{cases} 1 - T\ , & \text{if heating unit is ON.} \\ -T\ , & \text{if heating unit is OFF.} \end{cases}$$

Suppose $T = 0$ at 9:00 A.M., the heating unit is ON from 9–10 A.M., OFF from 10–11 A.M., ON again from 11 A.M.–noon, and so on for the rest of the day. How warm will the classroom be at noon? At 5:00 P.M.?

2.4 EXACT EQUATIONS

Suppose the mathematical function $F(x,y)$ represents some physical quantity, such as temperature, in a region of the xy-plane. Then the level curves of F, where $F(x,y) = constant$, could be interpreted as isotherms on a weather map, as depicted in Figure 2.8.

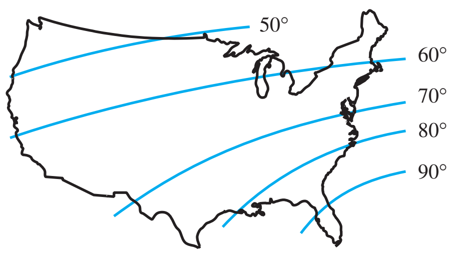

Figure 2.8 Level curves of $F(x, y)$

How does one calculate the slope of the tangent to a level curve? It is accomplished by implicit differentiation: One takes the derivative, with respect to x, of both sides of the equation $F(x,y) = C$, taking into account that y depends on x along the curve:

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}(C) \quad \text{or}$$

$$\textbf{(1)} \qquad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0\ ,$$

and solves for the slope:

$$\textbf{(2)} \qquad \frac{dy}{dx} = f(x,y) = -\frac{\partial F/\partial x}{\partial F/\partial y}\ .$$

The expression obtained by formally multiplying the left-hand member of (1) by dx is known as the *total differential* of F, written dF:

$$dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\ ,$$

and our procedure for obtaining the equation for the slope $f(x,y)$ of the level curve $F(x,y) = C$ can be expressed as setting the total differential $dF = 0$ and solving.

Because equation (2) has the form of a differential equation, we should be able to reverse this logic and come up with a very easy technique for solving some differential equations. After all, any first-order differential equation $dy/dx = f(x, y)$ can be rewritten in the (differential) form

$$\textbf{(3)} \qquad M(x, y)dx + N(x, y)dy = 0$$

Remark. Since we can use either procedure for finding $F(x, y)$, it may be worthwhile to consider each of the integrals $\int M(x, y)\,dx$ and $\int N(x, y)\,dy$. If one is easier to evaluate than the other, this would be sufficient reason for us to use one method over the other. [The skeptical reader should try solving equation (15) by first integrating $M(x, y)$.]

Example 4 Show that

(16) $$(x + 3x^3 \sin y)\,dx + (x^4 \cos y)\,dy = 0$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve (16).

Solution In equation (16), $M = x + 3x^3 \sin y$ and $N = x^4 \cos y$. Because

$$\frac{\partial M}{\partial y} = 3x^3 \cos y \not\equiv 4x^3 \cos y = \frac{\partial N}{\partial x} ,$$

equation (16) is not exact. When we multiply (16) by the factor x^{-1}, we obtain

(17) $$(1 + 3x^2 \sin y)\,dx + (x^3 \cos y)\,dy = 0 \ .$$

For this new equation, $M = 1 + 3x^2 \sin y$ and $N = x^3 \cos y$. If we test for exactness, we now find that

$$\frac{\partial M}{\partial y} = 3x^2 \cos y = \frac{\partial N}{\partial x} ,$$

and hence (17) is exact. Upon solving (17), we find that the solution is given implicitly by $x + x^3 \sin y = C$. Since equations (16) and (17) differ only by a factor of x, then any solution to one will be a solution for the other whenever $x \neq 0$. Hence the solution to equation (16) is given implicitly by $x + x^3 \sin y = C$. ◆

In Section 2.5 we discuss methods for finding factors that, like x^{-1} in Example 4, change inexact equations into exact equations.

2.4 EXERCISES

In Problems 1–8, classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

1. $(x^{10/3} - 2y)\,dx + x\,dy = 0$

2. $(x^2 y + x^4 \cos x)\,dx - x^3\,dy = 0$

3. $\sqrt{-2y - y^2}\,dx + (3 + 2x - x^2)\,dy = 0$

4. $(ye^{xy} + 2x)\,dx + (xe^{xy} - 2y)\,dy = 0$

5. $xy\,dx + dy = 0$

6. $y^2\,dx + (2xy + \cos y)\,dy = 0$

7. $[2x + y\cos(xy)]\,dx + [x\cos(xy) - 2y]\,dy = 0$

8. $\theta\,dr + (3r - \theta - 1)\,d\theta = 0$

In Problems 9–20, determine whether the equation is exact. If it is, then solve it.

9. $(2x + y)\,dx + (x - 2y)\,dy = 0$

10. $(2xy + 3)\,dx + (x^2 - 1)\,dy = 0$

Because (12) is not exact, we compute

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{1-(2xy-1)}{x^2y - x} = \frac{2(1-xy)}{-x(1-xy)} = \frac{-2}{x} .$$

We obtain a function of only x, so an integrating factor for (12) is given by formula (8). That is,

$$\mu(x) = \exp\left(\int \frac{-2}{x}dx\right) = x^{-2} .$$

When we multiply (12) by $\mu = x^{-2}$, we get the exact equation

$$(2 + yx^{-2})dx + (y - x^{-1})dy = 0 .$$

Solving this equation, we ultimately derive the implicit solution

$$2x - yx^{-1} + \frac{y^2}{2} = C . \tag{13}$$

Notice that the solution $x \equiv 0$ was lost in multiplying by $\mu = x^{-2}$. Hence, (13) and $x \equiv 0$ are solutions to equation (12). ◆

There are many differential equations that are not covered by Theorem 3 but for which an integrating factor nevertheless exists. The major difficulty, however, is in finding an explicit formula for these integrating factors, which in general will depend on both x and y.

2.5 EXERCISES

In Problems 1–6, identify the equation as separable, linear, exact, or having an integrating factor that is a function of either x alone or y alone.

1. $(2x + yx^{-1})dx + (xy - 1)dy = 0$

2. $(2y^3 + 2y^2)dx + (3y^2x + 2xy)dy = 0$

3. $(2x + y)dx + (x - 2y)dy = 0$

4. $(y^2 + 2xy)dx - x^2\,dy = 0$

5. $(x^2\sin x + 4y)dx + x\,dy = 0$

6. $(2y^2x - y)dx + x\,dy = 0$

In Problems 7–12, solve the equation.

7. $(2xy)dx + (y^2 - 3x^2)dy = 0$

8. $(3x^2 + y)dx + (x^2y - x)dy = 0$

9. $(x^4 - x + y)dx - x\,dy = 0$

10. $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

11. $(y^2 + 2xy)dx - x^2\,dy = 0$

12. $(2xy^3 + 1)dx + (3x^2y^2 - y^{-1})dy = 0$

In Problems 13 and 14, find an integrating factor of the form x^ny^m and solve the equation.

13. $(2y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$

14. $(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0$

15. **(a)** Show that if $(\partial N/\partial x - \partial M/\partial y)/(xM - yN)$ depends only on the product xy, that is,

$$\frac{\partial N/\partial x - \partial M/\partial y}{xM - yN} = H(xy) ,$$

then the equation $M(x, y)dx + N(x, y)dy = 0$ has an integrating factor of the form $\mu(xy)$. Give the general formula for $\mu(xy)$.

(b) Use your answer to part (a) to find an implicit solution to

$$(3y + 2xy^2)dx + (x + 2x^2y)dy = 0 ,$$

satisfying the initial condition $y(1) = 1$.

16. **(a)** Prove that $Mdx + Ndy = 0$ has an integrating factor that depends only on the sum $x + y$ if and only if the expression

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N}$$

depends only on $x + y$.

(b) Use part (a) to solve the equation $(3 + y + xy)dx + (3 + x + xy)dy = 0$.

The last equation is homogeneous, so we let $z = v/u$. Then $dv/du = z + u(dz/du)$, and, substituting for v/u, we obtain

$$z + u\frac{dz}{du} = \frac{3 - z}{1 + z} .$$

Separating variables gives

$$\int \frac{z + 1}{z^2 + 2z - 3} dz = -\int \frac{1}{u} du ,$$

$$\frac{1}{2} \ln |z^2 + 2z - 3| = -\ln |u| + C_1 ,$$

from which it follows that

$$z^2 + 2z - 3 = Cu^{-2} .$$

When we substitute back in for z, u, and v, we find

$$(v/u)^2 + 2(v/u) - 3 = Cu^{-2} ,$$

$$v^2 + 2uv - 3u^2 = C ,$$

$$(y + 3)^2 + 2(x - 1)(y + 3) - 3(x - 1)^2 = C .$$

This last equation gives an implicit solution to (16). ◆

2.6 EXERCISES

In Problems 1–8, identify (do not solve) the equation as homogeneous, Bernoulli, linear coefficients, or of the form $y' = G(ax + by)$.

1. $2tx\, dx + (t^2 - x^2)dt = 0$

2. $(y - 4x - 1)^2 dx - dy = 0$

3. $dy/dx + y/x = x^3y^2$

4. $(t + x + 2)dx + (3t - x - 6)dt = 0$

5. $\theta\, dy - y\, d\theta = \sqrt{\theta y}\, d\theta$

6. $(ye^{-2x} + y^3)dx - e^{-2x} dy = 0$

7. $\cos(x + y)dy = \sin(x + y)dx$

8. $(y^3 - \theta y^2)d\theta + 2\theta^2 y\, dy = 0$

Use the method discussed under "Homogeneous Equations" to solve Problems 9–16.

9. $(xy + y^2)dx - x^2 dy = 0$

10. $(3x^2 - y^2)dx + (xy - x^3y^{-1})dy = 0$

11. $(y^2 - xy)dx + x^2 dy = 0$

12. $(x^2 + y^2)dx + 2xy\, dy = 0$

13. $\dfrac{dx}{dt} = \dfrac{x^2 + t\sqrt{t^2 + x^2}}{tx}$

14. $\dfrac{dy}{d\theta} = \dfrac{\theta \sec(y/\theta) + y}{\theta}$

15. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{3xy}$

16. $\dfrac{dy}{dx} = \dfrac{y(\ln y - \ln x + 1)}{x}$

Use the method discussed under "Equations of the Form $dy/dx = G(ax + by)$" to solve Problems 17–20.

17. $dy/dx = \sqrt{x + y} - 1$

18. $dy/dx = (x + y + 2)^2$

19. $dy/dx = (x - y + 5)^2$

20. $dy/dx = \sin(x - y)$

Use the method discussed under "Bernoulli Equations" to solve Problems 21–28.

21. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^2$

22. $\dfrac{dy}{dx} - y = e^{2x}y^3$

23. $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2y^2$

24. $\dfrac{dy}{dx} + \dfrac{y}{x - 2} = 5(x - 2)y^{1/2}$

25. $\dfrac{dx}{dt} + tx^3 + \dfrac{x}{t} = 0$

26. $\dfrac{dy}{dx} + y = e^x y^{-2}$

27. $\dfrac{dr}{d\theta} = \dfrac{r^2 + 2r\theta}{\theta^2}$

28. $\dfrac{dy}{dx} + y^3x + y = 0$

REVIEW PROBLEMS

In Problems 1–30, solve the equation.

1. $\dfrac{dy}{dx} = \dfrac{e^{x+y}}{y-1}$ **2.** $\dfrac{dy}{dx} - 4y = 32x^2$

3. $(x^2 - 2y^{-3})dy + (2xy - 3x^2)dx = 0$

4. $\dfrac{dy}{dx} + \dfrac{3y}{x} = x^2 - 4x + 3$

5. $[\sin(xy) + xy\cos(xy)]\,dx + [1 + x^2\cos(xy)]\,dy = 0$

6. $2xy^3\,dx - (1 - x^2)dy = 0$

7. $t^3y^2\,dt + t^4y^{-6}\,dy = 0$

8. $\dfrac{dy}{dx} + \dfrac{2y}{x} = 2x^2y^2$

9. $(x^2 + y^2)dx + 3xy\,dy = 0$

10. $[1 + (1 + x^2 + 2xy + y^2)^{-1}]\,dx$
$+ [y^{-1/2} + (1 + x^2 + 2xy + y^2)^{-1}]\,dy = 0$

11. $\dfrac{dx}{dt} = 1 + \cos^2(t - x)$

12. $(y^3 + 4e^xy)dx + (2e^x + 3y^2)dy = 0$

13. $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2\sin 2x$

14. $\dfrac{dx}{dt} - \dfrac{x}{t-1} = t^2 + 2$

15. $\dfrac{dy}{dx} = 2 - \sqrt{2x - y + 3}$

16. $\dfrac{dy}{dx} + y\tan x + \sin x = 0$

17. $\dfrac{dy}{d\theta} + 2y = y^2$

18. $\dfrac{dy}{dx} = (2x + y - 1)^2$

19. $(x^2 - 3y^2)dx + 2xy\,dy = 0$

20. $\dfrac{dy}{d\theta} + \dfrac{y}{\theta} = -4\theta y^{-2}$

21. $(y - 2x - 1)dx + (x + y - 4)dy = 0$

22. $(2x - 2y - 8)dx + (x - 3y - 6)dy = 0$

23. $(y - x)dx + (x + y)dy = 0$

24. $\left(\sqrt{y/x} + \cos x\right)dx + \left(\sqrt{x/y} + \sin y\right)dy = 0$

25. $y(x - y - 2)dx + x(y - x + 4)dy = 0$

26. $\dfrac{dy}{dx} + xy = 0$

27. $(3x - y - 5)dx + (x - y + 1)dy = 0$

28. $\dfrac{dy}{dx} = \dfrac{x - y - 1}{x + y + 5}$

29. $(4xy^3 - 9y^2 + 4xy^2)dx$
$+ (3x^2y^2 - 6xy + 2x^2y)dy = 0$

30. $\dfrac{dy}{dx} = (x + y + 1)^2 - (x + y - 1)^2$

In Problems 31–40, solve the initial value problem.

31. $(x^3 - y)\,dx + x\,dy = 0\,, \quad y(1) = 3$

32. $\dfrac{dy}{dx} = \left(\dfrac{x}{y} + \dfrac{y}{x}\right)\,, \quad y(1) = -4$

33. $(t + x + 3)dt + dx = 0\,, \quad x(0) = 1$

34. $\dfrac{dy}{dx} - \dfrac{2y}{x} = x^2\cos x\,, \quad y(\pi) = 2$

35. $(2y^2 + 4x^2)dx - xy\,dy = 0\,, \quad y(1) = -2$

36. $[2\cos(2x + y) - x^2]\,dx$
$+ [\cos(2x + y) + e^y]\,dy = 0\,, \quad y(1) = 0$

37. $(2x - y)dx + (x + y - 3)dy = 0\,, \quad y(0) = 2$

38. $\sqrt{y}\,dx + (x^2 + 4)dy = 0\,, \quad y(0) = 4$

39. $\dfrac{dy}{dx} - \dfrac{2y}{x} = x^{-1}y^{-1}\,, \quad y(1) = 3$

40. $\dfrac{dy}{dx} - 4y = 2xy^2\,, \quad y(0) = -4$

41. Express the solution to the following initial value problem using a definite integral:

$$\frac{dy}{dt} = \frac{1}{1 + t^2} - y\,, \qquad y(2) = 3\,.$$

Then use your expression and numerical integration to estimate $y(3)$ to four decimal places.

Use the method discussed under "Equations with Linear Coefficients" to solve Problems 29–32.

29. $(-3x + y - 1)dx + (x + y + 3)dy = 0$

30. $(x + y - 1)dx + (y - x - 5)dy = 0$

31. $(2x - y)dx + (4x + y - 3)dy = 0$

32. $(2x + y + 4)dx + (x - 2y - 2)dy = 0$

In Problems 33–40, solve the equation given in:

33. Problem 1. **34.** Problem 2.

35. Problem 3. **36.** Problem 4.

37. Problem 5. **38.** Problem 6.

39. Problem 7. **40.** Problem 8.

41. Use the substitution $v = x - y + 2$ to solve equation (8).

42. Use the substitution $y = vx^2$ to solve

$$\frac{dy}{dx} = \frac{2y}{x} + \cos(y/x^2) .$$

43. **(a)** Show that the equation $dy/dx = f(x, y)$ is homogeneous if and only if $f(tx, ty) = f(x, y)$. [*Hint:* Let $t = 1/x$.]

(b) A function $H(x, y)$ is called **homogeneous of order *n*** if $H(tx, ty) = t^n H(x, y)$. Show that the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is homogeneous if $M(x, y)$ and $N(x, y)$ are both homogeneous of the same order.

44. Show that equation (13) reduces to an equation of the form

$$\frac{dy}{dx} = G(ax + by) ,$$

when $a_1b_2 = a_2b_1$. [*Hint:* If $a_1b_2 = a_2b_1$, then $a_2/a_1 = b_2/b_1 = k$, so that $a_2 = ka_1$ and $b_2 = kb_1$.]

45. **Coupled Equations.** In analyzing coupled equations of the form

$$\frac{dy}{dt} = ax + by ,$$

$$\frac{dx}{dt} = \alpha x + \beta y ,$$

where a, b, α, and β are constants, we may wish to determine the relationship between x and y rather than the individual solutions $x(t)$, $y(t)$. For this purpose, divide the first equation by the second to obtain

$$\textbf{(17)} \qquad \frac{dy}{dx} = \frac{ax + by}{\alpha x + \beta y} .$$

This new equation is homogeneous, so we can solve it via the substitution $v = y/x$. We refer to the solutions of (17) as **integral curves.** Determine the integral curves for the system

$$\frac{dy}{dt} = -4x - y ,$$

$$\frac{dx}{dt} = 2x - y .$$

46. **Magnetic Field Lines.** As described in Problem 20 of Exercises 1.3, the magnetic field lines of a dipole satisfy

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - y^2} .$$

Solve this equation and sketch several of these lines.

47. **Riccati Equation.** An equation of the form

$$\textbf{(18)} \qquad \frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

is called a generalized Riccati equation.†

(a) If one solution—say, $u(x)$—of (18) is known, show that the substitution $y = u + 1/v$ reduces (18) to a linear equation in v.

(b) Given that $u(x) = x$ is a solution to

$$\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x} ,$$

use the result of part (a) to find all the other solutions to this equation. (The particular solution $u(x) = x$ can be found by inspection or by using a Taylor series method; see Section 8.1.)

†*Historical Footnote:* Count Jacopo Riccati studied a particular case of this equation in 1724 during his investigation of curves whose radii of curvature depend only on the variable y and not the variable x.

17. **(a)** Find a condition on M and N that is necessary and sufficient for $Mdx + Ndy = 0$ to have an integrating factor that depends only on the product x^2y.

(b) Use part (a) to solve the equation

$$(2x + 2y + 2x^3y + 4x^2y^2)dx + (2x + x^4 + 2x^3y)dy = 0 \ .$$

18. If $xM(x, y) + yN(x, y) \equiv 0$, find the solution to the equation $M(x, y)dx + N(x, y)dy = 0$.

19. **Fluid Flow.** The streamlines associated with a certain fluid flow are represented by the family of curves $y = x - 1 + ke^{-x}$. The velocity potentials of the flow are just the orthogonal trajectories of this family.

(a) Use the method described in Problem 32 of Exercises 2.4 to show that the velocity potentials satisfy $dx + (x - y)dy = 0$.

[*Hint:* First express the family $y = x - 1 + ke^{-x}$ in the form $F(x, y) = k$.]

(b) Find the velocity potentials by solving the equation obtained in part (a).

20. Verify that when the linear differential equation $[P(x)y - Q(x)]dx + dy = 0$ is multiplied by $\mu(x) = e^{\int P(x)dx}$, the result is exact.

2.6 SUBSTITUTIONS AND TRANSFORMATIONS

When the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is not a separable, exact, or linear equation, it may still be possible to transform it into one that we know how to solve. This was in fact our approach in Section 2.5, where we used an integrating factor to transform our original equation into an exact equation.

In this section we study four types of equations that can be transformed into either a separable or linear equation by means of a suitable substitution or transformation.

Substitution Procedure

(a) Identify the type of equation and determine the appropriate substitution or transformation.
(b) Rewrite the original equation in terms of new variables.
(c) Solve the transformed equation.
(d) Express the solution in terms of the original variables.

Homogeneous Equations

Homogeneous Equation

Definition 4. If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is **homogeneous.**

11. $(\cos x \cos y + 2x)dx - (\sin x \sin y + 2y)dy = 0$

12. $(e^x \sin y - 3x^2)dx + (e^x \cos y + y^{-2/3}/3)dy = 0$

13. $(t/y)dy + (1 + \ln y)dt = 0$

14. $e^t(y - t)dt + (1 + e^t)dy = 0$

15. $\cos\theta\, dr - (r \sin\theta - e^\theta)d\theta = 0$

16. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0$

17. $(1/y)dx - (3y - x/y^2)dy = 0$

18. $[2x + y^2 - \cos(x + y)]dx$
$+ [2xy - \cos(x + y) - e^y]dy = 0$

19. $\left(2x + \dfrac{y}{1 + x^2y^2}\right)dx + \left(\dfrac{x}{1 + x^2y^2} - 2y\right)dy = 0$

20. $\left[\dfrac{2}{\sqrt{1 - x^2}} + y\cos(xy)\right]dx$
$+ [x\cos(xy) - y^{-1/3}]dy = 0$

In Problems 21–26, solve the initial value problem.

21. $(1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0$,
$y(1) = \pi$

22. $(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0$,
$y(1) = 1$

23. $(e^ty + te^ty)dt + (te^t + 2)dy = 0$, $y(0) = -1$

24. $(e^tx + 1)dt + (e^t - 1)dx = 0$, $x(1) = 1$

25. $(y^2 \sin x)dx + (1/x - y/x)dy = 0$, $y(\pi) = 1$

26. $(\tan y - 2)dx + (x\sec^2 y + 1/y)dy = 0$,
$y(0) = 1$

27. For each of the following equations, find the most general function $M(x, y)$ so that the equation is exact.
(a) $M(x, y)dx + (\sec^2 y - x/y)dy = 0$
(b) $M(x, y)dx + (\sin x \cos y - xy - e^{-y})dy = 0$

28. For each of the following equations, find the most general function $N(x, y)$ so that the equation is exact.
(a) $[y\cos(xy) + e^x]dx + N(x, y)dy = 0$
(b) $(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0$

29. Consider the equation
$$(y^2 + 2xy)dx - x^2\, dy = 0 \ .$$
(a) Show that this equation is not exact.
(b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.
(c) Use the solution of the resulting exact equation to solve the original equation.
(d) Were any solutions lost in the process?

30. Consider the equation
$$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0 \ .$$
(a) Show that the equation is not exact.
(b) Multiply the equation by x^ny^m and determine values for n and m that make the resulting equation exact.
(c) Use the solution of the resulting exact equation to solve the original equation.

31. Argue that in the proof of Theorem 2 the function g can be taken as
$$g(y) = \int_{y_0}^{y} N(x, t)dt - \int_{y_0}^{y}\left[\frac{\partial}{\partial t}\int_{x_0}^{x} M(s, t)ds\right]dt \ ,$$
which can be expressed as
$$g(y) = \int_{y_0}^{y} N(x, t)dt - \int_{x_0}^{x} M(s, y)ds + \int_{x_0}^{x} M(s, y_0)ds \ .$$
This leads ultimately to the representation
$$F(x, y) = \int_{y_0}^{y} N(x, t)dt + \int_{x_0}^{x} M(s, y_0)ds \ . \quad (18)$$
Evaluate this formula directly with $x_0 = 0$, $y_0 = 0$ to rework
(a) Example 1.
(b) Example 2.
(c) Example 3.

32. Orthogonal Trajectories. A geometric problem occurring often in engineering is that of finding a family of curves (orthogonal trajectories) that intersects a given family of curves orthogonally at each point. For example, we may be given the lines of force of an electric field and want to find the equation for the equipotential curves. Consider the family of curves described by $F(x, y) = k$, where k is a parameter. Recall from the discussion of equation (2) that for each curve in the family, the slope is given by

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \bigg/ \frac{\partial F}{\partial y} \; .$$

(a) Recall that the slope of a curve that is orthogonal (perpendicular) to a given curve is just the negative reciprocal of the slope of the given curve. Using this fact, show that the curves orthogonal to the family $F(x, y) = k$ satisfy the differential equation

$$\frac{\partial F}{\partial y}(x, y)dx - \frac{\partial F}{\partial x}(x, y)dy = 0 \; .$$

(b) Using the preceding differential equation, show that the orthogonal trajectories to the family of circles $x^2 + y^2 = k$ are just straight lines through the origin (see Figure 2.10).

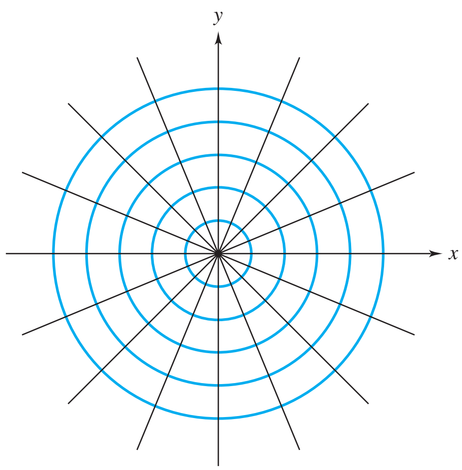

Figure 2.10 Orthogonal trajectories for concentric circles are lines through the center

(c) Show that the orthogonal trajectories to the family of hyperbolas $xy = k$ are the hyperbolas $x^2 - y^2 = k$ (see Figure 2.11).

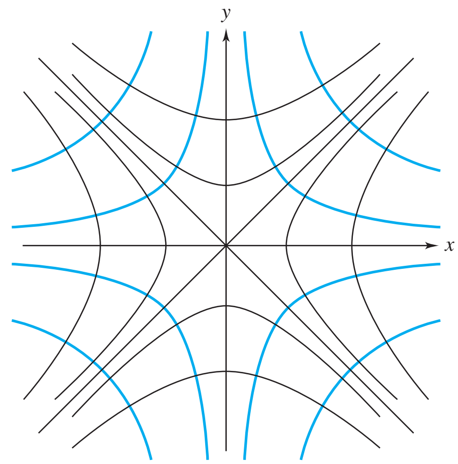

Figure 2.11 Families of orthogonal hyperbolas

33. Use the method in Problem 32 to find the orthogonal trajectories for each of the given families of curves, where k is a parameter.

(a) $2x^2 + y^2 = k$
(b) $y = kx^4$
(c) $y = e^{kx}$
(d) $y^2 = kx$

[*Hint:* First express the family in the form $F(x, y) = k$.]

34. Use the method described in Problem 32 to show that the orthogonal trajectories to the family of curves $x^2 + y^2 = kx$, k a parameter, satisfy

$$(2yx^{-1})dx + (y^2x^{-2} - 1)dy = 0 \; .$$

Find the orthogonal trajectories by solving the above equation. Sketch the family of curves, along with their orthogonal trajectories. [*Hint:* Try multiplying the equation by $x^m y^n$ as in Problem 30.]

35. Using condition (5), show that the right-hand side of (10) is independent of x by showing that its partial derivative with respect to x is zero. [*Hint:* Since the partial derivatives of M are continuous, Leibniz's theorem allows you to interchange the operations of integration and differentiation.]

36. Verify that $F(x, y)$ as defined by (9) and (10) satisfies conditions (4).