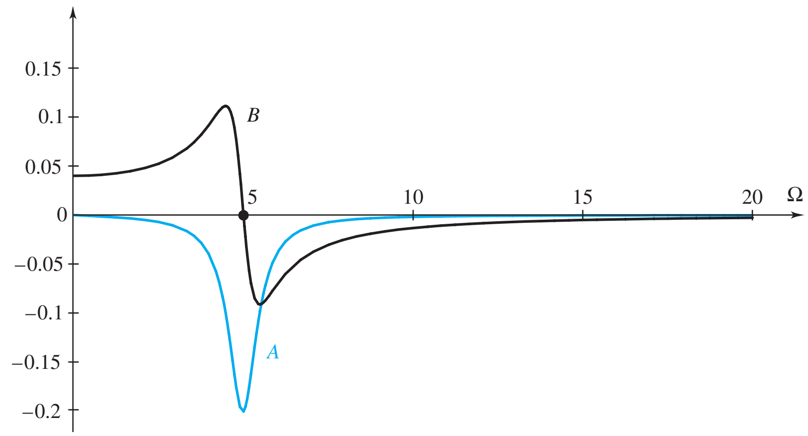

Figure 4.5 Vibration amplitudes around resonance

We find

$$A = \frac{-\Omega}{\Omega^2 + (\Omega^2 - 25)^2}\ , \qquad B = \frac{-\Omega^2 + 25}{\Omega^2 + (\Omega^2 - 25)^2}\ .$$

Figure 4.5 displays A and B as functions of the driving frequency Ω. A resonance clearly occurs around $\Omega \approx 5$. ◆

In most of this chapter, we are going to restrict our attention to differential equations of the form

(6) $$ay'' + by' + cy = f(t)\ ,$$

where $y(t)$ [or $y(x)$, or $x(t)$, etc.] is the unknown function that we seek; a, b, and c are constants; and $f(t)$ [or $f(x)$] is a *known* function. The proper nomenclature for (6) is the *linear, second-order ordinary differential equation with constant coefficients*. In Sections 4.7 and 4.8, we will generalize our focus to equations with nonconstant coefficients, as well as to nonlinear equations. However, (6) is an excellent starting point because we are able to obtain explicit solutions and observe, in concrete form, the theoretical properties that are predicted for more general equations. For motivation of the mathematical procedures and theory for solving (6), we will consistently compare it with the mass–spring paradigm:

$$[\textbf{inertia}] \times y'' + [\textbf{damping}] \times y' + [\textbf{stiffness}] \times y = F_{\text{ext}}\ .$$

4.1 EXERCISES

1. Verify that for $b = 0$ and $F_{\text{ext}}(t) = 0$, equation (3) has a solution of the form

$$y(t) = \cos \omega t, \text{ where } \omega = \sqrt{k/m}\ .$$

2. If $F_{\text{ext}}(t) = 0$, equation (3) becomes

$$my'' + by' + ky = 0\ .$$

For this equation, verify the following:

(a) If $y(t)$ is a solution, so is $cy(t)$, for any constant c.

(b) If $y_1(t)$ and $y_2(t)$ are solutions, so is their sum $y_1(t) + y_2(t)$.

3. Show that if $F_{\text{ext}}(t) = 0$, $m = 1$, $k = 9$, and $b = 6$, then equation (3) has the "critically damped" solutions $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$. What is the limit of these solutions as $t \to \infty$?

4. Verify that $y = \sin 3t + 2\cos 3t$ is a solution to the initial value problem

$$2y'' + 18y = 0 \ ; \quad y(0) = 2 \ , \quad y'(0) = 3 \ .$$

Find the maximum of $|y(t)|$ for $-\infty < t < \infty$.

5. Verify that the exponentially damped sinusoid $y(t) = e^{-3t}\sin(\sqrt{3}\,t)$ is a solution to equation (3) if $F_{ext}(t) = 0$, $m = 1$, $b = 6$, and $k = 12$. What is the limit of this solution as $t \to \infty$?

6. An external force $F(t) = 2\cos 2t$ is applied to a mass–spring system with $m = 1$, $b = 0$, and $k = 4$, which is initially at rest; i.e., $y(0) = 0$, $y'(0) = 0$. Verify that $y(t) = \frac{1}{2}t\sin 2t$ gives the motion of this spring. What will eventually (as t increases) happen to the spring?

In Problems 7–9, find a synchronous solution of the form $A\cos\Omega t + B\sin\Omega t$ to the given forced oscillator equation using the method of Example 4 to solve for A and B.

7. $y'' + 2y' + 4y = 5\sin 3t, \quad \Omega = 3$

8. $y'' + 2y' + 5y = -50\sin 5t, \quad \Omega = 5$

9. $y'' + 2y' + 4y = 6\cos 2t + 8\sin 2t, \quad \Omega = 2$

10. Undamped oscillators that are driven at resonance have unusual (and nonphysical) solutions.

(a) To investigate this, find the synchronous solution $A\cos\Omega t + B\sin\Omega t$ to the generic forced oscillator equation

$$(7) \quad my'' + by' + ky = \cos\Omega t \ .$$

(b) Sketch graphs of the coefficients A and B, as functions of Ω, for $m = 1$, $b = 0.1$, and $k = 25$.

(c) Now set $b = 0$ in your formulas for A and B and resketch the graphs in part (b), with $m = 1$, and $k = 25$. What happens at $\Omega = 5$? Notice that the amplitudes of the synchronous solutions grow without bound as Ω approaches 5.

(d) Show directly, by substituting the form $A\cos\Omega t + B\sin\Omega t$ into equation (7), that when $b = 0$ there are *no* synchronous solutions if $\Omega = \sqrt{k/m}$.

(e) Verify that $(2m\Omega)^{-1}t\sin\Omega t$ solves equation (7) when $b = 0$ and $\Omega = \sqrt{k/m}$. Notice that this *non*synchronous solution grows in time, without bound.

Clearly one cannot neglect damping in analyzing an oscillator forced at resonance, because otherwise the solutions, as shown in part (e), are nonphysical. This behavior will be studied later in this chapter.

4.2 HOMOGENEOUS LINEAR EQUATIONS: THE GENERAL SOLUTION

We begin our study of the linear second-order constant-coefficient differential equation

$$(1) \qquad ay'' + by' + cy = f(t) \quad (a \neq 0)$$

with the special case where the function $f(t)$ is zero:

$$(2) \qquad \boldsymbol{ay'' + by' + cy = 0} \ .$$

This case arises when we consider mass–spring oscillators vibrating freely—that is, without external forces applied. Equation (2) is called the *homogeneous form* of equation (1); $f(t)$ is the "nonhomogeneity" in (1). (This nomenclature is not related to the way we used the term for first-order equations in Section 2.6.)

A look at equation (2) tells us that a solution of (2) must have the property that its second derivative is expressible as a linear combination of its first and zeroth derivatives.† This suggests that we try to find a solution of the form $y = e^{rt}$, since derivatives of e^{rt} are just constants times e^{rt}. If we substitute $y = e^{rt}$ into (2), we obtain

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \ ,$$
$$e^{rt}(ar^2 + br + c) = 0 \ .$$

†The zeroth derivative of a function is the function itself.

We observe that $r = 1$ is a root of the above equation, and dividing the polynomial on the left-hand side of (15) by $r - 1$ leads to the factorization

$$(r-1)(r^2+4r+3) = (r-1)(r+1)(r+3) = 0 \ .$$

Hence, the roots of the auxiliary equation are 1, -1, and -3, and so three solutions of (14) are e^t, e^{-t}, and e^{-3t}. The linear independence of these three exponential functions is proved in Problem 40. A general solution to (14) is then

$$y(t) = c_1e^t + c_2e^{-t} + c_3e^{-3t} \ . \quad (16)$$ ◆

So far we have seen only exponential solutions to the linear second-order constant coefficient equation. You may wonder where the vibratory solutions that govern mass–spring oscillators are. In the next section, it will be seen that they arise when the solutions to the auxiliary equation are complex.

4.2 EXERCISES

In Problems 1–12, find a general solution to the given differential equation.

1. $y'' + 6y' + 9y = 0$
2. $2y'' + 7y' - 4y = 0$
3. $y'' - y' - 2y = 0$
4. $y'' + 5y' + 6y = 0$
5. $y'' - 5y' + 6y = 0$
6. $y'' + 8y' + 16y = 0$
7. $6y'' + y' - 2y = 0$
8. $z'' + z' - z = 0$
9. $4y'' - 4y' + y = 0$
10. $y'' - y' - 11y = 0$
11. $4w'' + 20w' + 25w = 0$
12. $3y'' + 11y' - 7y = 0$

In Problems 13–20, solve the given initial value problem.

13. $y'' + 2y' - 8y = 0$; $y(0) = 3$, $y'(0) = -12$
14. $y'' + y' = 0$; $y(0) = 2$, $y'(0) = 1$
15. $y'' - 4y' - 5y = 0$; $y(-1) = 3$, $y'(-1) = 9$
16. $y'' - 4y' + 3y = 0$; $y(0) = 1$, $y'(0) = 1/3$
17. $z'' - 2z' - 2z = 0$; $z(0) = 0$, $z'(0) = 3$
18. $y'' - 6y' + 9y = 0$; $y(0) = 2$, $y'(0) = 25/3$
19. $y'' + 2y' + y = 0$; $y(0) = 1$, $y'(0) = -3$
20. $y'' - 4y' + 4y = 0$; $y(1) = 1$, $y'(1) = 1$

21. First-Order Constant-Coefficient Equations.
(a) Substituting $y = e^{rt}$, find the auxiliary equation for the first-order linear equation

$$ay' + by = 0 \ ,$$

where a and b are constants with $a \neq 0$.
(b) Use the result of part (a) to find the general solution.

In Problems 22–25, use the method described in Problem 21 to find a general solution to the given equation.

22. $3y' - 7y = 0$
23. $5y' + 4y = 0$
24. $3z' + 11z = 0$
25. $6w' - 13w = 0$

26. Boundary Value Problems. When the values of a solution to a differential equation are specified at two different points, these conditions are called **boundary conditions.** (In contrast, initial conditions specify the values of a function and its derivative at the same point.) The purpose of this exercise is to show that for boundary value problems there is no existence–uniqueness theorem that is analogous to Theorem 1. Given that every solution to

$$y'' + y = 0 \quad (17)$$

is of the form

$$y(t) = c_1 \cos t + c_2 \sin t \ ,$$

where c_1 and c_2 are arbitrary constants, show that
(a) There is a unique solution to (17) that satisfies the boundary conditions $y(0) = 2$ and $y(\pi/2) = 0$.
(b) There is no solution to (17) that satisfies $y(0) = 2$ and $y(\pi) = 0$.
(c) There are infinitely many solutions to (17) that satisfy $y(0) = 2$ and $y(\pi) = -2$.

In Problems 27–32, use Definition 1 to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$.

27. $y_1(t) = \cos t \sin t$, $y_2(t) = \sin 2t$
28. $y_1(t) = e^{3t}$, $y_2(t) = e^{-4t}$

29. $y_1(t) = te^{2t}$, $y_2(t) = e^{2t}$

30. $y_1(t) = t^2 \cos(\ln t)$, $y_2(t) = t^2 \sin(\ln t)$

31. $y_1(t) = \tan^2 t - \sec^2 t$, $y_2(t) \equiv 3$

32. $y_1(t) \equiv 0$, $y_2(t) = e^t$

33. Explain why two functions are linearly dependent on an interval I if and only if there exist constants c_1 and c_2, *not both zero*, such that

$$c_1 y_1(t) + c_2 y_2(t) = 0 \quad \text{for all } t \text{ in } I .$$

34. Wronskian. For any two differentiable functions y_1 and y_2, the function

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) \tag{18}$$

is called the *Wronskian*† of y_1 and y_2. This function plays a crucial role in proof of Theorem 2.

(a) Show that $W[y_1, y_2]$ can be conveniently expressed as the 2×2 determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} .$$

(b) Let $y_1(t)$, $y_2(t)$ be a pair of solutions to the homogeneous equation $ay'' + by' + cy = 0$ (with $a \neq 0$) on an open interval I. Prove that $y_1(t)$ and $y_2(t)$ are linearly independent on I if and only if their Wronskian is never zero on I. [*Hint:* This is just a reformulation of Lemma 1.]

(c) Show that if $y_1(t)$ and $y_2(t)$ are any two differentiable functions that are linearly dependent on I, then their Wronskian is identically zero on I.

35. Linear Dependence of Three Functions. Three functions $y_1(t)$, $y_2(t)$, and $y_3(t)$ are said to be linearly dependent on an interval I if, on I, at least one of these functions is a linear combination of the remaining two [e.g., if $y_1(t) = c_1 y_2(t) + c_2 y_3(t)$]. Equivalently (compare Problem 33), y_1, y_2, and y_3 are linearly dependent on I if there exist constants C_1, C_2, and C_3, *not all zero*, such that

$$C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t) = 0 \quad \text{for all } t \text{ in } I.$$

Otherwise, we say that these functions are linearly independent on I.

For each of the following, determine whether the given three functions are linearly dependent or linearly independent on $(-\infty, \infty)$:

(a) $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = t^2$.

(b)

(c) $y_1(t) = -3$, $y_2(t) = 5\sin^2 t$, $y_3(t) = \cos^2 t$.

$y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = t^2 e^t$.

(d) $y_1(t) = e^t$, $y_2(t) = e^{-t}$, $y_3(t) = \cosh t$.

36. Using the definition in Problem 35, prove that if r_1, r_2, and r_3 are distinct real numbers, then the functions $e^{r_1 t}$, $e^{r_2 t}$, and $e^{r_3 t}$ are linearly independent on $(-\infty, \infty)$. [*Hint*: Assume to the contrary that, say, $e^{r_1 t} = c_1 e^{r_2 t} + c_2 e^{r_3 t}$ for all t. Divide by $e^{r_2 t}$ to get $e^{(r_1 - r_2)t} = c_1 + c_2 e^{(r_3 - r_2)t}$ and then differentiate to deduce that $e^{(r_1 - r_2)t}$ and $e^{(r_3 - r_2)t}$ are linearly dependent, which is a contradiction. (Why?)]

In Problems 37–41, find three linearly independent solutions (see Problem 35) of the given third-order differential equation and write a general solution as an arbitrary linear combination of these.

37. $y''' + y'' - 6y' + 4y = 0$

38. $y''' - 6y'' - y' + 6y = 0$

39. $z''' + 2z'' - 4z' - 8z = 0$

40. $y''' - 7y'' + 7y' + 15y = 0$

41. $y''' + 3y'' - 4y' - 12y = 0$

42. (True or False): If f_1, f_2, f_3 are three functions defined on $(-\infty, \infty)$ that are *pairwise* linearly independent on $(-\infty, \infty)$, then f_1, f_2, f_3 form a linearly independent set on $(-\infty, \infty)$. Justify your answer.

43. Solve the initial value problem:

$$y''' - y' = 0 ; \quad y(0) = 2 ,$$
$$y'(0) = 3 , \quad y''(0) = -1 .$$

44. Solve the initial value problem:

$$y''' - 2y'' - y' + 2y = 0 ;$$
$$y(0) = 2 , \ y'(0) = 3 , \ y''(0) = 5 .$$

45. By using Newton's method or some other numerical procedure to approximate the roots of the auxiliary equation, find general solutions to the following equations:

(a) $3y''' + 18y'' + 13y' - 19y = 0$.

(b) $y^{\text{iv}} - 5y'' + 5y = 0$.

(c) $y^{\text{v}} - 3y^{\text{iv}} - 5y''' + 15y'' + 4y' - 12y = 0$.

46. One way to define hyperbolic functions is by means of differential equations. Consider the equation $y'' - y = 0$. The *hyperbolic cosine*, $\cosh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 1$ and $y'(0) = 0$. The *hyperbolic sine*, $\sinh t$, is defined as the solution of this equation subject to the initial values: $y(0) = 0$ and $y'(0) = 1$.

(a) Solve these initial value problems to derive explicit formulas for $\cosh t$, and $\sinh t$. Also

†*Historical Footnote:* The Wronskian was named after the Polish mathematician H. Wronski (1778–1863).

show that $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$.

(b) Prove that a general solution of the equation $y'' - y = 0$ is given by $y = c_1 \cosh t + c_2 \sinh t$.

(c) Suppose a, b, and c are given constants for which $ar^2 + br + c = 0$ has two distinct real roots. If the two roots are expressed in the form $\alpha - \beta$ and $\alpha + \beta$, show that a general solution of the equation $ay'' + by' + cy = 0$ is $y = c_1 e^{\alpha t}\cosh(\beta t) + c_2 e^{\alpha t}\sinh(\beta t)$.

(d) Use the result of part (c) to solve the initial value problem: $y'' + y' - 6y = 0$, $y(0) = 2$, $y'(0) = -17/2$.

4.3 AUXILIARY EQUATIONS WITH COMPLEX ROOTS

The *simple harmonic equation* $y'' + y = 0$, so called because of its relation to the fundamental vibration of a musical tone, has as solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$. Notice, however, that the auxiliary equation associated with the harmonic equation is $r^2 + 1 = 0$, which has imaginary roots $r = \pm i$, where i denotes $\sqrt{-1}$.† In the previous section, we expressed the solutions to a linear second-order equation with constant coefficients in terms of exponential functions. It would appear, then, that one might be able to attribute a meaning to the forms e^{it} and e^{-it} and that these "functions" should be related to $\cos t$ and $\sin t$. This matchup is accomplished by Euler's formula, which is discussed in this section.

When $b^2 - 4ac < 0$, the roots of the auxiliary equation

(1) $$ar^2 + br + c = 0$$

associated with the homogeneous equation

(2) $$ay'' + by' + cy = 0$$

are the complex conjugate numbers

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta \quad \left(i = \sqrt{-1}\right) ,$$

where α, β are the real numbers

(3) $$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} .$$

As in the previous section, we would like to assert that the functions $e^{r_1 t}$ and $e^{r_2 t}$ are solutions to the equation (2). This is in fact the case, but before we can proceed, we need to address some fundamental questions. For example, if $r_1 = \alpha + i\beta$ is a complex number, what do we mean by the expression $e^{(\alpha + i\beta)t}$? If we assume that the law of exponents applies to complex numbers, then

(4) $$e^{(\alpha+i\beta)t} = e^{\alpha t + i\beta t} = e^{\alpha t} e^{i\beta t} .$$

We now need only clarify the meaning of $e^{i\beta t}$.

For this purpose, let's assume that the Maclaurin series for e^z is the same for complex numbers z as it is for real numbers. Observing that $i^2 = -1$, then for θ real we have

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) . \end{aligned}$$

†Electrical engineers frequently use the symbol j to denote $\sqrt{-1}$.

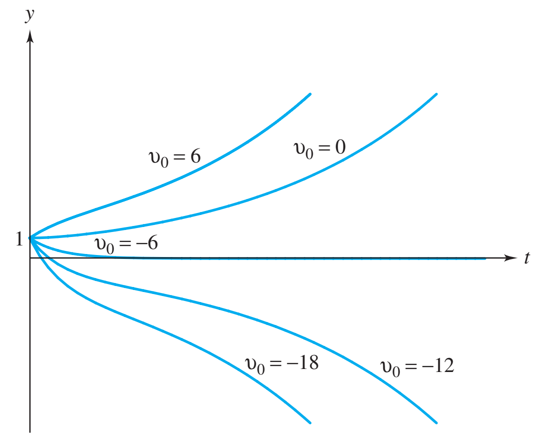

Figure 4.8 Solution graphs for Example 5

and the plots in Figure 4.8, confirm our prediction that all (nonequilibrium) solutions diverge—except for the one with $\upsilon_0 = -6$.

What is the physical significance of this isolated bounded solution? Evidently, if the mass is given an initial inwardly directed velocity of -6, it has barely enough energy to overcome the effect of the spring banishing it to $+\infty$ but not enough energy to cross the equilibrium point (and get pushed to $-\infty$). So it asymptotically approaches the (extremely delicate) equilibrium position $y = 0$. ◆

In Section 4.8, we will see that taking further liberties with the mass–spring interpretation enables us to predict qualitative features of more complicated equations.

Throughout this section we have assumed that the coefficients a, b, and c in the differential equation were real numbers. If we now allow them to be *complex* constants, then the roots r_1, r_2 of the auxiliary equation (1) are, in general, also complex but not necessarily conjugates of each other. When $r_1 \neq r_2$, a general solution to equation (2) still has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} ,$$

but c_1 and c_2 are now arbitrary complex-valued constants, and we have to resort to the clumsy calculations of Example 1.

We also remark that a complex differential equation can be regarded as a system of two real differential equations since we can always work separately with its real and imaginary parts. Systems are discussed in Chapters 5 and 9.

4.3 EXERCISES

In Problems 1–8, the auxiliary equation for the given differential equation has complex roots. Find a general solution.

1. $y'' + y = 0$

2. $y'' + 9y = 0$

3. $y'' - 10y' + 26y = 0$

4. $z'' - 6z' + 10z = 0$

5. $y'' - 4y' + 7y = 0$

6. $w'' + 4w' + 6w = 0$

7. $4y'' + 4y' + 6y = 0$

8. $4y'' - 4y' + 26y = 0$

In Problems 9–20, find a general solution.

9. $y'' + 4y' + 8y = 0$

10. $y'' - 8y' + 7y = 0$

11. $z'' + 10z' + 25z = 0$

12. $u'' + 7u = 0$

13. $y'' + 2y' + 5y = 0$

14. $y'' - 2y' + 26y = 0$

15. $y'' + 10y' + 41y = 0$

16. $y'' - 3y' - 11y = 0$

17. $y'' - y' + 7y = 0$

18. $2y'' + 13y' - 7y = 0$

19. $y''' + y'' + 3y' - 5y = 0$

20. $y''' - y'' + 2y = 0$

In Problems 21–27, solve the given initial value problem.

21. $y'' + 2y' + 2y = 0$; $y(0) = 2$, $y'(0) = 1$

22. $y'' + 2y' + 17y = 0$; $y(0) = 1$, $y'(0) = -1$

23. $w'' - 4w' + 2w = 0$; $w(0) = 0$, $w'(0) = 1$

24. $y'' + 9y = 0$; $y(0) = 1$, $y'(0) = 1$

25. $y'' - 2y' + 2y = 0$; $y(\pi) = e^{\pi}$, $y'(\pi) = 0$

26. $y'' - 2y' + y = 0$; $y(0) = 1$, $y'(0) = -2$

27. $y''' - 4y'' + 7y' - 6y = 0$; $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$

28. To see the effect of changing the parameter b in the initial value problem

$$y'' + by' + 4y = 0 \; ; \qquad y(0) = 1 \; , \; y'(0) = 0 \; ,$$

solve the problem for $b = 5$, 4, and 2 and sketch the solutions.

29. Find a general solution to the following higher-order equations.

(a) $y''' - y'' + y' + 3y = 0$

(b) $y''' + 2y'' + 5y' - 26y = 0$

(c) $y^{\text{iv}} + 13y'' + 36y = 0$

30. Using the representation for $e^{(\alpha + i\beta)t}$ in (6), verify the differentiation formula (7).

31. Using the mass–spring analogy, predict the behavior as $t \to +\infty$ of the solution to the given initial value problem. Then confirm your prediction by actually solving the problem.

(a) $y'' + 16y = 0$; $y(0) = 2$, $y'(0) = 0$

(b) $y'' + 100y' + y = 0$; $y(0) = 1$, $y'(0) = 0$

(c) $y'' - 6y' + 8y = 0$; $y(0) = 1$, $y'(0) = 0$

(d) $y'' + 2y' - 3y = 0$; $y(0) = -2$, $y'(0) = 0$

(e) $y'' - y' - 6y = 0$; $y(0) = 1$, $y'(0) = 1$

32. Vibrating Spring without Damping. A vibrating spring without damping can be modeled by the initial value problem (11) in Example 3 by taking $b = 0$.

(a) If $m = 10$ kg, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec, find the equation of motion for this undamped vibrating spring.

(b) When the equation of motion is of the form displayed in (9), the motion is said to be **oscillatory** with **frequency** $\beta/2\pi$. Find the frequency of oscillation for the spring system of part (a).

33. Vibrating Spring with Damping. Using the model for a vibrating spring with damping discussed in Example 3:

(a) Find the equation of motion for the vibrating spring with damping if $m = 10$ kg, $b = 60$ kg/sec, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec.

(b) Find the frequency of oscillation for the spring system of part (a). [*Hint:* See the definition of frequency given in Problem 32(b).]

(c) Compare the results of Problems 32 and 33 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution?

34. *RLC* Series Circuit. In the study of an electrical circuit consisting of a resistor, capacitor, inductor, and an electromotive force (see Figure 4.9), we are led to an initial value problem of the form

$$\textbf{(20)} \qquad L\frac{dI}{dt} + RI + \frac{q}{C} = E(t) \; ;$$
$$q(0) = q_0 \; ,$$
$$I(0) = I_0 \; ,$$

where L is the inductance in henrys, R is the resistance in ohms, C is the capacitance in farads, $E(t)$ is the electromotive force in volts, $q(t)$ is the charge in coulombs on the capacitor at time t, and $I = dq/dt$ is the current in amperes. Find the current at time t if the charge on the capacitor is initially zero, the initial current is zero, $L = 10$ H, $R = 20\ \Omega$, $C = (6260)^{-1}$ F, and $E(t) = 100$ V. [*Hint:* Differentiate both sides of the differential equation in (20) to obtain a homogeneous linear second-order equation for $I(t)$. Then use (20) to determine dI/dt at $t = 0$.]

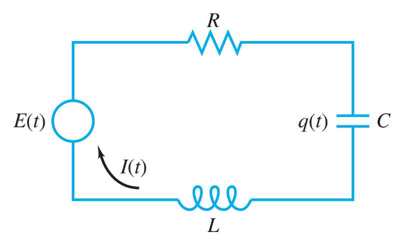

Figure 4.9 *RLC* series circuit

35. Swinging Door. The motion of a swinging door with an adjustment screw that controls the amount of friction on the hinges is governed by the initial value problem

$$I\theta'' + b\theta' + k\theta = 0 \; ; \quad \theta(0) = \theta_0 \; , \quad \theta'(0) = v_0 \; ,$$

where θ is the angle that the door is open, I is the moment of inertia of the door about its hinges, $b > 0$ is a damping constant that varies with the amount of friction on the door, $k > 0$ is the spring constant associated with the swinging door, θ_0 is the initial angle that the door is opened, and v_0 is the initial angular velocity imparted to the door (see Figure 4.10). If I and k are fixed, determine for which values of b the door will *not* continually swing back and forth when closing.

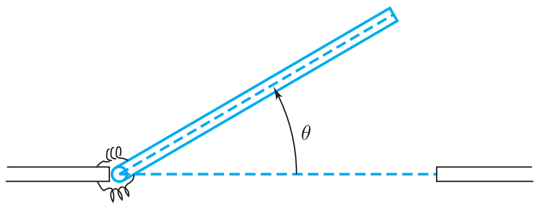

Figure 4.10 Top view of swinging door

36. Although the real general solution form (9) is convenient, it is also possible to use the form

$$\text{(21)} \qquad d_1 e^{(\alpha + i\beta)t} + d_2 e^{(\alpha - i\beta)t}$$

to solve initial value problems, as illustrated in Example 1. The coefficients d_1 and d_2 are complex constants.

(a) Use the form (21) to solve Problem 21. Verify that your form is equivalent to the one derived using (9).

(b) Show that, in general, d_1 and d_2 in (21) must be complex conjugates in order that the solution be real.

37. The auxiliary equations for the following differential equations have repeated complex roots. Adapt the "repeated root" procedure of Section 4.2 to find their general solutions:

(a) $y^{\text{iv}} + 2y'' + y = 0$.

(b) $y^{\text{iv}} + 4y''' + 12y'' + 16y' + 16y = 0$. [*Hint:* The auxiliary equation is $(r^2 + 2r + 4)^2 = 0$.]

38. Prove the sum of angle formula for the sine function by following these steps. Fix x.

(a) Let $f(t) := \sin(x + t)$. Show that $f''(t) + f(t) = 0$, $f(0) = \sin x$, and $f'(0) = \cos x$.

(b) Use the auxiliary equation technique to solve the initial value problem $y'' + y = 0$, $y(0) = \sin x$, and $y'(0) = \cos x$.

(c) By uniqueness, the solution in part (b) is the same as $f(t)$ from part (a). Write this equality; this should be the standard sum of angle formula for $\sin(x + t)$.

4.4 NONHOMOGENEOUS EQUATIONS: THE METHOD OF UNDETERMINED COEFFICIENTS

In this section we employ "judicious guessing" to derive a simple procedure for finding a solution to a *non*homogeneous linear equation with constant coefficients

$$\text{(1)} \qquad ay'' + by' + cy = f(t) \ ,$$

when the nonhomogeneity $f(t)$ is a single term of a special type. Our experience in Section 4.3 indicates that (1) will have an infinite number of solutions. For the moment we are content to find one, particular, solution. To motivate the procedure, let's first look at a few instructive examples.

Example 1 Find a particular solution to

$$\text{(2)} \qquad y'' + 3y' + 2y = 3t \ .$$

Solution We need to find a function $y(t)$ such that the combination $y'' + 3y' + 2y$ is a linear function of t—namely, $3t$. Now what kind of function y "ends up" as a linear function after having its zeroth, first, and second derivatives combined? One immediate answer is: *another linear function*. So we might try $y_1(t) = At$ and attempt to match up $y_1'' + 3y_1' + 2y_1$ with $3t$.

Perhaps you can see that this won't work: $y_1 = At$, $y_1' = A$ and $y_1'' = 0$ gives us

$$y_1'' + 3y_1' + 2y_1 = 3A + 2At \ ,$$

4.4 EXERCISES

In Problems 1–8, decide whether or not the method of undetermined coefficients can be applied to find a particular solution of the given equation.

1. $y'' + 2y' - y = t^{-1}e^t$

2. $5y'' - 3y' + 2y = t^3 \cos 4t$

3. $2y''(x) - 6y'(x) + y(x) = (\sin x)/e^{4x}$

4. $x'' + 5x' - 3x = 3^t$

5. $2\omega''(x) - 3\omega(x) = 4x \sin^2 x + 4x \cos^2 x$

6. $y''(\theta) + 3y'(\theta) - y(\theta) = \sec \theta$

7. $ty'' - y' + 2y = \sin 3t$

8. $8z'(x) - 2z(x) = 3x^{100}e^{4x} \cos 25x$

In Problems 9–26, find a particular solution to the differential equation.

9. $y'' + 2y' - y = 10$

10. $y'' + 3y = -9$

11. $y''(x) + y(x) = 2^x$

12. $2x' + x = 3t^2$

13. $y'' - y' + 9y = 3 \sin 3t$

14. $2z'' + z = 9e^{2t}$

15. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = xe^x$

16. $\theta''(t) - \theta(t) = t \sin t$

17. $y'' - 2y' + y = 8e^t$

18. $y'' + 4y = 8 \sin 2t$

19. $4y'' + 11y' - 3y = -2te^{-3t}$

20. $y'' + 4y = 16t \sin 2t$

21. $x''(t) - 4x'(t) + 4x(t) = te^{2t}$

22. $x''(t) - 2x'(t) + x(t) = 24t^2e^t$

23. $y''(\theta) - 7y'(\theta) = \theta^2$

24. $y''(x) + y(x) = 4x \cos x$

25. $y'' + 2y' + 4y = 111e^{2t} \cos 3t$

26. $y'' + 2y' + 2y = 4te^{-t} \cos t$

In Problems 27–32, determine the form of a particular solution for the differential equation. (Do not evaluate coefficients.)

27. $y'' + 9y = 4t^3 \sin 3t$

28. $y'' + 3y' - 7y = t^4e^t$

28. $y'' + 3y' - 7y = t^4e^t$

29. $y'' - 6y' + 9y = 5t^6e^{3t}$

30. $y'' - 2y' + y = 7e^t \cos t$

31. $y'' + 2y' + 2y = 8t^3e^{-t} \sin t$

32. $y'' - y' - 12y = 2t^6e^{-3t}$

In Problems 33–36, use the method of undetermined coefficients to find a particular solution to the given higher-order equation.

33. $y''' - y'' + y = \sin t$

34. $2y''' + 3y'' + y' - 4y = e^{-t}$

35. $y''' + y'' - 2y = te^t$

36. $y^{(4)} - 3y'' - 8y = \sin t$

4.5 THE SUPERPOSITION PRINCIPLE AND UNDETERMINED COEFFICIENTS REVISITED

The next theorem describes the superposition principle, a very simple observation which nonetheless endows the solution set for our equations with a powerful structure. It extends the applicability of the method of undetermined coefficients and enables us to solve initial value problems for nonhomogeneous differential equations.

Superposition Principle

Theorem 3. Let y_1 be a solution to the differential equation

$$ay'' + by' + cy = f_1(t) ,$$

and y_2 be a solution to

$$ay'' + by' + cy = f_2(t) .$$

Then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t) .$$

Example 5 Write down the form of a particular solution to the equation

$$y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t \ .$$

Solution The roots of the associated homogeneous equation $y'' + 2y' + 2y = 0$ were identified in Example 3 as $-1 \pm i$. Application of (14) dictates the form

$$y_p(t) = t(A_3t^3 + A_2t^2 + A_1t + A_0)e^{-t}\cos t + t(B_3t^3 + B_2t^2 + B_1t + B_0)e^{-t}\sin t \ . \ \blacklozenge$$

The method of undetermined coefficients applies to higher-order linear differential equations with constant coefficients. Details will be provided in Chapter 6, but the following example should be clear.

Example 6 Write down the form of a particular solution to the equation

$$y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t} \ .$$

Solution The auxiliary equation for the associated homogeneous is $r^3 + 2r^2 + r = r(r+1)^2 = 0$, with a double root $r = -1$ and a single root $r = 0$. Term by term, the nonhomogeneities call for the forms

$$A_0e^{-t}\cos t + B_0e^{-t}\sin t \qquad (\text{for } 5e^{-t}\sin t) \ ,$$
$$tA_0 \qquad (\text{for } 3) \ ,$$
$$t^2(A_1t + A_0)e^{-t} \qquad (\text{for } 7te^{-t}) \ .$$

(If -1 were a *triple* root, we would need $t^3(A_1t + A_0)e^{-t}$ for $7te^{-t}$.) Of course, we have to rename the coefficients, so the general form is

$$y_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t + tC + t^2(Dt + E)e^{-t} \ . \ \blacklozenge$$

4.5 EXERCISES

1. Given that $y_1(t) = (1/4)\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, use the superposition principle to find solutions to the following:

(a) $y'' + 2y' + 4y = t + \cos 2t$.

(b) $y'' + 2y' + 4y = 2t - 3\cos 2t$.

(c) $y'' + 2y' + 4y = 11t - 12\cos 2t$.

2. Given that $y_1(t) = \cos t$ is a solution to

$$y'' - y' + y = \sin t$$

and $y_2(t) = e^{2t}/3$ is a solution to

$$y'' - y' + y = e^{2t} \ ,$$

use the superposition principle to find solutions to the following differential equations:

(a) $y'' - y' + y = 5\sin t$.

(b) $y'' - y' + y = \sin t - 3e^{2t}$.

(c) $y'' - y' + y = 4\sin t + 18e^{2t}$.

In Problems 3–8, a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.

3. $y'' + y' = 1$, $y_p(t) = t$

4. $y'' - y = t$, $y_p(t) = -t$

5. $y'' + 5y' + 6y = 6x^2 + 10x + 2 + 12e^x$, $y_p(x) = e^x + x^2$

6. $\theta'' - \theta' - 2\theta = 1 - 2t$, $\theta_p(t) = t - 1$

7. $y'' = 2y + 2\tan^3 x$, $y_p(x) = \tan x$

8. $y'' = 2y' - y + 2e^x$, $y_p(x) = x^2e^x$

In Problems 9–16 decide whether the method of undetermined coefficients together with superposition can be applied to find a particular solution of the given equation. Do not solve the equation.

9. $y'' - y' + y = (e^t + t)^2$

10. $3y'' + 2y' + 8y = t^2 + 4t - t^2e^t\sin t$

11. $y'' - 6y' - 4y = 4\sin 3t - e^{3t}t^2 + 1/t$

12. $y'' + y' + ty = e^t + 7$

13. $2y'' + 3y' - 4y = 2t + \sin^2 t + 3$

14. $y'' - 2y' + 3y = \cosh t + \sin^3 t$

15. $y'' + e^ty' + y = 7 + 3t$

16. $2y'' - y' + 6y = t^2e^{-t}\sin t - 8t\cos 3t + 10^t$

In Problems 17–22, find a general solution to the differential equation.

17. $y'' - y = -11t + 1$

18. $y'' - 2y' - 3y = 3t^2 - 5$

19. $y''(x) - 3y'(x) + 2y(x) = e^x \sin x$

20. $y''(\theta) + 4y(\theta) = \sin\theta - \cos\theta$

21. $y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta}\cos\theta$

22. $y''(x) + 6y'(x) + 10y(x)$
$= 10x^4 + 24x^3 + 2x^2 - 12x + 18$

In Problems 23–30, find the solution to the initial value problem.

23. $y' - y = 1\ , \quad y(0) = 0$

24. $y'' = 6t\ ; \quad y(0) = 3\ , \quad y'(0) = -1$

25. $z''(x) + z(x) = 2e^{-x}\ ; \quad z(0) = 0\ , \quad z'(0) = 0$

26. $y'' + 9y = 27\ ; \quad y(0) = 4\ , \quad y'(0) = 6$

27. $y''(x) - y'(x) - 2y(x) = \cos x - \sin 2x\ ;$
$y(0) = -7/20\ , \quad y'(0) = 1/5$

28. $y'' + y' - 12y = e^t + e^{2t} - 1\ ;$
$y(0) = 1\ , \quad y'(0) = 3$

29. $y''(\theta) - y(\theta) = \sin\theta - e^{2\theta}\ ;$
$y(0) = 1\ , \quad y'(0) = -1$

30. $y'' + 2y' + y = t^2 + 1 - e^t\ ;$
$y(0) = 0\ , \quad y'(0) = 2$

In Problems 31–36, determine the form of a particular solution for the differential equation. Do not solve.

31. $y'' + y = \sin t + t\cos t + 10^t$

32. $y'' - y = e^{2t} + te^{2t} + t^2e^{2t}$

33. $x'' - x' - 2x = e^t\cos t - t^2 + \cos^3 t$

34. $y'' + 5y' + 6y = \sin t - \cos 2t$

35. $y'' - 4y' + 5y = e^{5t} + t\sin 3t - \cos 3t$

36. $y'' - 4y' + 4y = t^2e^{2t} - e^{2t}$

In Problems 37–40, find a particular solution to the given higher-order equation.

37. $y''' - 2y'' - y' + 2y = 2t^2 + 4t - 9$

38. $y^{(4)} - 5y'' + 4y = 10\cos t - 20\sin t$

39. $y''' + y'' - 2y = te^t + 1$

40. $y^{(4)} - 3y''' + 3y'' - y' = 6t - 20$

41. Discontinuous Forcing Term. In certain physical models, the nonhomogeneous term, or **forcing term,** $g(t)$ in the equation

$$ay'' + by' + cy = g(t)$$

may not be continuous but may have a jump discontinuity. If this occurs, we can still obtain a reasonable solution using the following procedure. Consider the initial value problem

$$y'' + 2y' + 5y = g(t)\ ; \quad y(0) = 0\ , \ y'(0) = 0\ ,$$

where

$$g(t) = \begin{cases} 10 & \text{if } 0 \le t \le 3\pi/2 \\ 0 & \text{if } t > 3\pi/2 \end{cases} .$$

(a) Find a solution to the initial value problem for $0 \le t \le 3\pi/2$.

(b) Find a general solution for $t > 3\pi/2$.

(c) Now choose the constants in the general solution from part (b) so that the solution from part (a) and the solution from part (b) agree, together with their first derivatives, at $t = 3\pi/2$. This gives us a continuously differentiable function that satisfies the differential equation except at $t = 3\pi/2$.

42. Forced Vibrations. As discussed in Section 4.1, a vibrating spring with damping that is under external force can be modeled by

$$my'' + by' + ky = g(t)\ , \tag{15}$$

where $m > 0$ is the mass of the spring system, $b > 0$ is the damping constant, $k > 0$ is the spring constant, $g(t)$ is the force on the system at time t, and $y(t)$ is the displacement from the equilibrium of the spring system at time t. Assume $b^2 < 4mk$.

(a) Determine the form of the equation of motion for the spring system when $g(t) = \sin\beta t$ by finding a general solution to equation (15).

(b) Discuss the long-term behavior of this system. [*Hint:* Consider what happens to the general solution obtained in part (a) as $t \to +\infty$.]

43. A mass–spring system is driven by a sinusoidal external force $g(t) = 5\sin t$. The mass equals 1, the spring constant equals 3, and the damping coefficient equals 4. If the mass is initially located at $y(0) = 1/2$ and at rest, i.e., $y'(0) = 0$, find its equation of motion.

44. A mass–spring system is driven by the external force $g(t) = 2\sin 3t + 10\cos 3t$. The mass equals 1, the spring constant equals 5, and the damping coefficient equals 2. If the mass is initially located at $y(0) = -1$, with initial velocity $y'(0) = 5$, find its equation of motion.

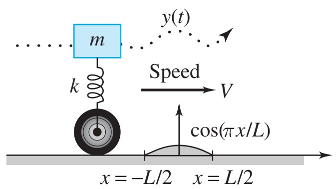

Figure 4.11 Speed bump

45. Speed Bumps. Often bumps like the one depicted in Figure 4.11 are built into roads to discourage speeding. The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with the speed V is given by

$$y(t) = 0 \qquad \text{for } t \le -L/(2V) \ ,$$

$$my'' + ky = \begin{cases} F_0 \cos(\pi V t/L) & \text{for } |t| < L/(2V) \\ 0 & \text{for } t \ge L/(2V). \end{cases}$$

(The absence of a damping term indicates that the car's shock absorbers are not functioning.)

(a) Taking $m = k = 1, L = \pi$, and $F_0 = 1$ in appropriate units, solve this initial value problem. Thereby show that the formula for the oscillatory motion *after the car has traversed the speed bump* is $y(t) = A \sin t$, where the constant A depends on the speed V.

(b) Plot the amplitude $|A|$ of the solution $y(t)$ found in part (a) versus the car's speed V. From the graph, estimate the speed that produces the most violent shaking of the vehicle.

46. Show that the *boundary value problem*

$$y'' + \lambda^2 y = \sin t \ ; \quad y(0) = 0 \ , \quad y(\pi) = 1 \ ,$$

has a solution if and only if $\lambda \ne \pm 1, \pm 2, \pm 3, \ldots$.

47. Find the solution(s) to

$$y'' + 9y = 27 \cos 6t$$

(if it exists) satisfying the *boundary conditions*

(a) $y(0) = -1 \ , \quad y(\pi/6) = 3$.

(b) $y(0) = -1 \ , \quad y(\pi/3) = 5$.

(c) $y(0) = -1 \ , \quad y(\pi/3) = -1$.

48. All that is known concerning a mysterious second-order constant-coefficient differential equation $y'' + py' + qy = g(t)$ is that $t^2 + 1 + e^t \cos t$, $t^2 + 1 + e^t \sin t$, and $t^2 + 1 + e^t \cos t + e^t \sin t$ are solutions.

(a) Determine two linearly independent solutions to the corresponding homogeneous equation.

(b) Find a suitable choice of p, q, and $g(t)$ that enables these solutions.

4.6 VARIATION OF PARAMETERS

We have seen that the method of undetermined coefficients is a simple procedure for determining a particular solution when the equation has constant coefficients and the nonhomogeneous term is of a special type. Here we present a more general method, called **variation of parameters,**[†] for finding a particular solution.

Consider the nonhomogeneous linear second-order equation

$$ay'' + by' + cy = f(t) \tag{1}$$

and let $\{y_1(t), y_2(t)\}$ be two linearly independent solutions for the corresponding homogeneous equation

$$ay'' + by' + cy = 0 \ .$$

Then we know that a general solution to this homogeneous equation is given by

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t) \ , \tag{2}$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation,

[†] *Historical Footnote:* The method of variation of parameters was invented by Joseph Lagrange in 1774

4.6 EXERCISES

In Problems 1–8, find a general solution to the differential equation using the method of variation of parameters.

1. $y'' + y = \sec t$
2. $y'' + 4y = \tan 2t$
3. $y'' + 2y' + y = e^{-t}$
4. $y'' - 2y' + y = t^{-1}e^{t}$
5. $y'' + 9y = \sec^2(3t)$
6. $y''(\theta) + 16y(\theta) = \sec 4\theta$
7. $y'' + 4y' + 4y = e^{-2t}\ln t$
8. $y'' + 4y = \csc^2(2t)$

In Problems 9 and 10, find a particular solution first by undetermined coefficients, and then by variation of parameters. Which method was quicker?

9. $y'' - y = 2t + 4$
10. $2x''(t) - 2x'(t) - 4x(t) = 2e^{2t}$

In Problems 11–18, find a general solution to the differential equation.

11. $y'' + y = \tan^2 t$
12. $y'' + y = \tan t + e^{3t} - 1$
13. $v'' + 4v = \sec^4(2t)$
14. $y''(\theta) + y(\theta) = \sec^3 \theta$
15. $y'' + y = 3\sec t - t^2 + 1$
16. $y'' + 5y' + 6y = 18t^2$
17. $\frac{1}{2}y'' + 2y = \tan 2t - \frac{1}{2}e^{t}$
18. $y'' - 6y' + 9y = t^{-3}e^{3t}$
19. Express the solution to the initial value problem
$$y'' - y = \frac{1}{t}\ , \qquad y(1) = 0\ ,\ y'(1) = -2\ ,$$
using definite integrals. Using numerical integration (Appendix C) to approximate the integrals, find an approximation for $y(2)$ to two decimal places.
20. Use the method of variation of parameters to show that
$$y(t) = c_1 \cos t + c_2 \sin t + \int_0^t f(s)\sin(t - s)\,ds$$
is a general solution to the differential equation
$$y'' + y = f(t)\ ,$$
where $f(t)$ is a continuous function on $(-\infty,\ \infty)$. [*Hint:* Use the trigonometric identity $\sin(t - s) = \sin t\cos s - \sin s\cos t$.]
21. Suppose y satisfies the equation $y'' + 10y' + 25y = e^{t^3}$ subject to $y(0) = 1$ and $y'(0) = -5$. Estimate $y(0.2)$ to within ± 0.0001 by numerically approximating the integrals in the variation of parameters formula.

4.7 VARIABLE-COEFFICIENT EQUATIONS

The techniques of Sections 4.2 and 4.3 have explicitly demonstrated that solutions to a linear homogeneous constant-coefficient differential equation,

$$ay'' + by' + cy = 0\ , \tag{1}$$

are defined and satisfy the equation over the whole interval $(-\infty, +\infty)$. After all, such solutions are combinations of exponentials, sinusoids, and polynomials.

The variation of parameters formula of Section 4.6 extended this to nonhomogeneous constant-coefficient problems,

$$ay'' + by' + cy = f(t)\ , \tag{2}$$

yielding solutions valid *over all intervals where $f(t)$ is continuous* (ensuring that the integrals in (10) of Section 4.6 containing $f(t)$ exist and are differentiable). We could hardly hope for more; indeed, it is debatable what *meaning* the differential equation (2) would have at a point where $f(t)$ is undefined, or discontinuous.

Taking integration constants to be zero yields $\ln v' = 2\ln(\tan t)$ or $v' = \tan^2 t$, and $v = \tan t - t$. Therefore, a second solution to (19) is $y_2 = (\tan t - t)\cos t = \sin t - t\cos t$. We conclude that a general solution is $c_1\cos t + c_2(\sin t - t\cos t)$. ◆

In this section we have seen that the *theory* for variable-coefficient equations differs only slightly from the constant-coefficient case (in that solution domains are restricted to intervals), but explicit solutions can be hard to come by. In the next section, we will supplement our exposition by describing some nonrigorous procedures that sometimes can be used to predict qualitative features of the solutions.

4.7 EXERCISES

In Problems 1 through 4, use Theorem 5 to discuss the existence and uniqueness of a solution to the differential equation that satisfies the initial conditions $y(1) = Y_0$, $y'(1) = Y_1$, where Y_0 and Y_1 are real constants.

1. $t(t-3)y'' + 2ty' - y = t^2$

2. $(1 + t^2)y'' + ty' - y = \tan t$

3. $t^2y'' + y = \cos t$

4. $e^ty'' - \dfrac{y'}{t-3} + y = \ln t$

In Problems 5 through 8, determine whether Theorem 5 applies. If it does, then discuss what conclusions can be drawn. If it does not, explain why.

5. $t^2z'' + tz' + z = \cos t \;;\quad z(0) = 1 \;,\quad z'(0) = 0$

6. $y'' + yy' = t^2 - 1 \;;\quad y(0) = 1 \;,\quad y'(0) = -1$

7. $y'' + ty' - t^2y = 0 \;;\quad y(0) = 0 \;,\quad y(1) = 0$

8. $(1-t)y'' + ty' - 2y = \sin t \;;$
$y(0) = 1 \;,\; y'(0) = 1$

In Problems 9 through 14, find a general solution to the given Cauchy–Euler equation for $t > 0$.

9. $t^2\dfrac{d^2y}{dt^2} + 2t\dfrac{dy}{dt} - 6y = 0$

10. $t^2y''(t) + 7ty'(t) - 7y(t) = 0$

11. $\dfrac{d^2w}{dt^2} + \dfrac{6}{t}\dfrac{dw}{dt} + \dfrac{4}{t^2}w = 0$

12. $t^2\dfrac{d^2z}{dt^2} + 5t\dfrac{dz}{dt} + 4z = 0$

13. $9t^2y''(t) + 15ty'(t) + y(t) = 0$

14. $t^2y''(t) - 3ty'(t) + 4y(t) = 0$

In Problems 15 through 18, find a general solution for $t < 0$.

15. $y''(t) - \dfrac{1}{t}y'(t) + \dfrac{5}{t^2}y(t) = 0$

16. $t^2y''(t) - 3ty'(t) + 6y(t) = 0$

17. $t^2y''(t) + 9ty'(t) + 17y(t) = 0$

18. $t^2y''(t) + 3ty'(t) + 5y(t) = 0$

In Problems 19 and 20, solve the given initial value problem for the Cauchy–Euler equation.

19. $t^2y''(t) - 4ty'(t) + 4y(t) = 0 \;;$
$y(1) = -2 \;,\quad y'(1) = -11$

20. $t^2y''(t) + 7ty'(t) + 5y(t) = 0 \;;$
$y(1) = -1 \;,\quad y'(1) = 13$

In Problems 21 and 22, devise a modification of the method for Cauchy–Euler equations to find a general solution to the given equation.

21. $(t-2)^2y''(t) - 7(t-2)y'(t) + 7y(t) = 0 \;,$
$t > 2$

22. $(t+1)^2y''(t) + 10(t+1)y'(t) + 14y(t) = 0 \;,$
$t > -1$

23. To justify the solution formulas (8) and (9), perform the following analysis.

(a) Show that if the substitution $t = e^x$ is made in the function $y(t)$ and x is regarded as the new independent variable in $Y(x) := y(e^x)$, the chain rule implies the following relationships:

$$t\frac{dy}{dt} = \frac{dY}{dx}\;, \qquad t^2\frac{d^2y}{dt^2} = \frac{d^2Y}{dx^2} - \frac{dY}{dx}\;.$$

(b) Using part (a), show that if the substitution $t = e^x$ is made in the Cauchy–Euler differential equation (6), the result is a constant-coefficient equation for $Y(x) = y(e^x)$, namely,

$$a\frac{d^2Y}{dx^2} + (b-a)\frac{dY}{dx} + cY = f(e^x) \ . \tag{20}$$

(c) Observe that the auxiliary equation (recall Section 4.2) for the homogeneous form of (20) is the same as (7) in this section. If the roots of the former are complex, linearly independent solutions of (20) have the form $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$; if they are equal, linearly independent solutions of (20) have the form e^{rx} and xe^{rx}. Express x in terms of t to derive the corresponding solution forms (8) and (9).

24. Solve the following Cauchy–Euler equations by using the substitution described in Problem 23 to change them to constant coefficient equations, finding their general solutions by the methods of the preceding sections, and restoring the original independent variable t.

(a) $t^2y'' + ty' - 9y = 0$
(b) $t^2y'' + 3ty' + 10y = 0$
(c) $t^2y'' + 3ty' + y = t + t^{-1}$
(d) $t^2y'' + ty' + 9y = -\tan(3\ln t)$

25. Let y_1 and y_2 be two functions defined on $(-\infty, \infty)$.

(a) True or False: If y_1 and y_2 are linearly dependent on the interval $[a, b]$, then y_1 and y_2 are linearly dependent on the smaller interval $[c, d] \subset [a, b]$.

(b) True or False: If y_1 and y_2 are linearly dependent on the interval $[a, b]$, then y_1 and y_2 are linearly dependent on the larger interval $[C, D] \supset [a, b]$.

26. Let $y_1(t) = t^3$ and $y_2(t) = |t^3|$. Are y_1 and y_2 linearly independent on the following intervals?

(a) $[0, \infty)$ **(b)** $(-\infty, 0]$ **(c)** $(-\infty, \infty)$
(d) Compute the Wronskian $W[y_1, y_2](t)$ on the interval $(-\infty, \infty)$.

27. Consider the linear equation

$$t^2y'' - 3ty' + 3y = 0 \ , \quad \text{for } -\infty < t < \infty \ . \tag{21}$$

(a) Verify that $y_1(t) := t$ and $y_2(t) := t^3$ are two solutions to (21) on $(-\infty, \infty)$. Furthermore, show that $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$ for $t_0 = 1$.

(b) Prove that $y_1(t)$ and $y_2(t)$ are linearly independent on $(-\infty, \infty)$.

(c) Verify that the function $y_3(t) := |t|^3$ is also a solution to (21) on $(-\infty, \infty)$.

(d) Prove that there is *no* choice of constants c_1, c_2 such that $y_3(t) = c_1y_1(t) + c_2y_2(t)$ for *all* t in $(-\infty, \infty)$. [*Hint:* Argue that the contrary assumption leads to a contradiction.]

(e) From parts (c) and (d), we see that there is at least one solution to (21) on $(-\infty, \infty)$ that is not expressible as a linear combination of the solutions $y_1(t), y_2(t)$. Does this provide a counterexample to the theory in this section? Explain.

28. Let $y_1(t) = t^2$ and $y_2(t) = 2t|t|$. Are y_1 and y_2 linearly independent on the interval:

(a) $[0, \infty)$? **(b)** $(-\infty, 0]$? **(c)** $(-\infty, \infty)$?
(d) Compute the Wronskian $W[y_1, y_2](t)$ on the interval $(-\infty, \infty)$.

29. Prove that if y_1 and y_2 are linearly independent solutions of $y'' + py' + qy = 0$ on (a, b), then they cannot both be zero at the same point t_0 in (a, b).

30. Superposition Principle. Let y_1 be a solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_1(t)$$

on the interval I and let y_2 be a solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = g_2(t)$$

on the same interval. Show that for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution on I to

$$y''(t) + p(t)y'(t) + q(t)y(t) = k_1g_1(t) + k_2g_2(t) \ .$$

31. Determine whether the following functions can be Wronskians on $-1 < t < 1$ for a pair of solutions to some equation $y'' + py' + qy = 0$ (with p and q continuous).

(a) $w(t) = 6e^{4t}$ **(b)** $w(t) = t^3$
(c) $w(t) = (t+1)^{-1}$ **(d)** $w(t) \equiv 0$

32. By completing the following steps, prove that the Wronskian of any two solutions y_1, y_2 to the equation $y'' + py' + qy = 0$ on (a, b) is given by **Abel's formula**[†]

$$\boldsymbol{W[y_1, y_2](t) = C\exp\left\{-\int_{t_0}^{t} p(\tau)d\tau\right\}} \ ,$$

$$t_0 \text{ and } t \text{ in } (a, b) \ ,$$

where the constant C depends on y_1 and y_2.

(a) Show that the Wronskian W satisfies the equation $W' + pW = 0$.

[†] *Historical footnote:* Niels Abel derived this identity in 1827.

(b) Solve the separable equation in part (a).

(c) How does Abel's formula clarify the fact that the Wronskian is either identically zero or never zero on (a, b)?

33. Use Abel's formula (Problem 32) to determine (up to a constant multiple) the Wronskian of two solutions on $(0, \infty)$ to

$$ty'' + (t - 1)y' + 3y = 0 \ .$$

34. All that is known concerning a mysterious differential equation $y'' + p(t)y' + q(t)y = g(t)$ is that the functions t, t^2, and t^3 are solutions.

(a) Determine two linearly independent solutions to the corresponding homogeneous differential equation.

(b) Find the solution to the original equation satisfying the initial conditions $y(2) = 2$, $y'(2) = 5$.

(c) What is $p(t)$? [*Hint:* Use Abel's formula for the Wronskian, Problem 32.]

35. Given that $1 + t$, $1 + 2t$, and $1 + 3t^2$ are solutions to the differential equation $y'' + p(t)y' + q(t)y = g(t)$, find the solution to this equation that satisfies $y(1) = 2$, $y'(1) = 0$.

36. Verify that the given functions y_1 and y_2 are linearly independent solutions of the following differential equation and find the solution that satisfies the given initial conditions.

$$ty'' - (t + 2)y' + 2y = 0 \ ;$$
$$y_1(t) = e^t \ , \quad y_2(t) = t^2 + 2t + 2 \ ;$$
$$y(1) = 0 \ , \quad y'(1) = 1$$

In Problems 37 through 40, use variation of parameters to find a general solution to the differential equation given that the functions y_1 and y_2 are linearly independent solutions to the corresponding homogeneous equation for $t > 0$. Remember to put the equation in standard form.

37. $ty'' - (t + 1)y' + y = t^2$;
$y_1 = e^t$, $\quad y_2 = t + 1$

38. $t^2y'' - 4ty' + 6y = t^3 + 1$;
$y_1 = t^2$, $\quad y_2 = t^3$

39. $ty'' + (5t - 1)y' - 5y = t^2e^{-5t}$;
$y_1 = 5t - 1$, $\quad y_2 = e^{-5t}$

40. $ty'' + (1 - 2t)y' + (t - 1)y = te^t$;
$y_1 = e^t$, $\quad y_2 = e^t \ln t$

In Problems 41 through 43, find general solutions to the nonhomogeneous Cauchy–Euler equations using variation of parameters.

41. $t^2z'' + tz' + 9z = -\tan(3 \ln t)$

42. $t^2y'' + 3ty' + y = t^{-1}$

43. $t^2z'' - tz' + z = t\left(1 + \dfrac{3}{\ln t}\right)$

44. The **Bessel equation** of order one-half

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0 \ , \qquad t > 0$$

has two linearly independent solutions,

$$y_1(t) = t^{-1/2}\cos t \ , \quad y_2(t) = t^{-1/2}\sin t \ .$$

Find a general solution to the nonhomogeneous equation

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = t^{5/2} \ , \qquad t > 0 \ .$$

In Problems 45 through 48, a differential equation and a non-trivial solution f are given. Find a second linearly independent solution using reduction of order.

45. $t^2y'' - 2ty' - 4y = 0$, $\quad t > 0$; $\quad f(t) = t^{-1}$

46. $t^2y'' + 6ty' + 6y = 0$, $\quad t > 0$; $\quad f(t) = t^{-2}$

47. $tx'' - (t + 1)x' + x = 0$, $\quad t > 0$; $\quad f(t) = e^t$

48. $ty'' + (1 - 2t)y' + (t - 1)y = 0$, $\quad t > 0$;
$f(t) = e^t$

49. In quantum mechanics, the study of the Schrödinger equation for the case of a harmonic oscillator leads to a consideration of **Hermite's equation,**

$$y'' - 2ty' + \lambda y = 0 \ ,$$

where λ is a parameter. Use the reduction of order formula to obtain an integral representation of a second linearly independent solution to Hermite's equation for the given value of λ and corresponding solution $f(t)$.

(a) $\lambda = 4$, $\quad f(t) = 1 - 2t^2$

(b) $\lambda = 6$, $\quad f(t) = 3t - 2t^3$

50. Complete the proof of Theorem 8 by solving equation (16).

51. The reduction of order procedure can be used more generally to reduce a homogeneous linear nth-order equation to a homogeneous linear $(n - 1)$th-order equation. For the equation

$$ty''' - ty'' + y' - y = 0 \ ,$$

which has $f(t) = e^t$ as a solution, use the substitution $y(t) = v(t)f(t)$ to reduce this third-order equation to a homogeneous linear second-order equation in the variable $w = v'$.

52. The equation

$$ty''' + (1 - t)y'' + ty' - y = 0$$

has $f(t) = t$ as a solution. Use the substitution $y(t) = v(t)f(t)$ to reduce this third-order equation to a homogeneous linear second-order equation in the variable $w = v'$.

53. Isolated Zeros. Let $\phi(t)$ be a solution to $y'' + py' + qy = 0$ on (a, b), where p, q are continuous on (a, b). By completing the following steps, prove that if ϕ is not identically zero, then its zeros in (a, b) are *isolated*, i.e., if $\phi(t_0) = 0$, then there exists a $\delta > 0$ such that $\phi(t) \neq 0$ for $0 < |t - t_0| < \delta$.

(a) Suppose $\phi(t_0) = 0$ and assume to the contrary that for each $n = 1, 2, \ldots$, the function ϕ has a zero at t_n, where $0 < |t_0 - t_n| < 1/n$. Show that this implies $\phi'(t_0) = 0$. [*Hint:* Consider the difference quotient for ϕ at t_0.]

(b) With the assumptions of part (a), we have $\phi(t_0) = \phi'(t_0) = 0$. Conclude from this that ϕ must be identically zero, which is a contradiction. Hence, there is some integer n_0 such that $\phi(t)$ is not zero for $0 < |t - t_0| < 1/n_0$.

54. The reduction of order formula (13) can also be derived from Abels' identity (Problem 32). Let $f(t)$ be a nontrivial solution to (10) and $y(t)$ a second linearly independent solution. Show that

$$\left(\frac{y}{f}\right)' = \frac{W[f, y]}{f^2}$$

and then use Abel's identity for the Wronskian $W[f, y]$ to obtain the reduction of order formula.

4.8 QUALITATIVE CONSIDERATIONS FOR VARIABLE-COEFFICIENT AND NONLINEAR EQUATIONS

There are no techniques for obtaining explicit, closed-form solutions to second-order linear differential equations with variable coefficients (with certain exceptions) or for nonlinear equations. In general, we will have to settle for numerical solutions or power series expansions. So it would be helpful to be able to derive, with simple calculations, some nonrigorous, qualitative conclusions about the behavior of the solutions before we launch the heavy computational machinery. In this section we first display a few examples that illustrate the profound differences that can occur when the equations have variable coefficients or are nonlinear. Then we show how the mass–spring analogy, discussed in Section 4.1, can be exploited to predict some of the attributes of solutions of these more complicated equations.

To begin our discussion we display a linear constant-coefficient, a linear variable-coefficient, and two nonlinear equations.

(a) The equation

$$3y'' + 2y' + 4y = 0 \tag{1}$$

is linear, homogeneous with constant coefficients. We know everything about such equations; the solutions are, at worst, polynomials times exponentials times sinusoids in t, and unique solutions can be found to match any prescribed data $y(a)$, $y'(a)$ at any instant $t = a$. It has the superposition property: If $y_1(t)$ and $y_2(t)$ are solutions, so is $y(t) = c_1y_1(t) + c_2y_2(t)$.

where v_1' and v_2' are determined by the equations

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = f(t)/a \ .$$

Superposition Principle. If y_1 and y_2 are solutions to the equations

$$ay'' + by' + cy = f_1 \quad \text{and} \quad ay'' + by' + cy = f_2 \ ,$$

respectively, then $k_1 y_1 + k_2 y_2$ is a solution to the equation

$$ay'' + by' + cy = k_1 f_1 + k_2 f_2 \ .$$

The superposition principle facilitates finding a particular solution when the nonhomogeneous term is the sum of nonhomogeneities for which particular solutions can be determined.

Cauchy–Euler (Equidimensional) Equations

$$at^2 y'' + bty' + cy = f(t)$$

Substituting $y = t^r$ yields the associated characteristic equation

$$ar^2 + (b - a)r + c = 0$$

for the corresponding *homogeneous* Cauchy–Euler equation. A general solution to the homogeneous equation for $t > 0$ is given by

(i) $c_1 t^{r_1} + c_2 t^{r_2}$, if r_1 and r_2 are distinct real roots;
(ii) $c_1 t^r + c_2 t^r \ln t$, if r is a repeated root;
(iii) $c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t)$, if $\alpha + i\beta$ is a complex root.

A general solution to the nonhomogeneous equation is $y = y_p + y_h$, where y_p is a particular solution and y_h is a general solution to the corresponding homogeneous equation. The method of variation of parameters (but not the method of undetermined coefficients) can be used to find a particular solution.

REVIEW PROBLEMS

In Problems 1–28, find a general solution to the given differential equation.

1. $y'' + 8y' - 9y = 0$
2. $49y'' + 14y' + y = 0$
3. $4y'' - 4y' + 10y = 0$
4. $9y'' - 30y' + 25y = 0$
5. $6y'' - 11y' + 3y = 0$
6. $y'' + 8y' - 14y = 0$
7. $36y'' + 24y' + 5y = 0$
8. $25y'' + 20y' + 4y = 0$
9. $16z'' - 56z' + 49z = 0$
10. $u'' + 11u = 0$
11. $t^2 x''(t) + 5x(t) = 0 \ , \quad t > 0$
12. $2y''' - 3y'' - 12y' + 20y = 0$
13. $y'' + 16y = te^t$
14. $v'' - 4v' + 7v = 0$
15. $3y''' + 10y'' + 9y' + 2y = 0$
16. $y''' + 3y'' + 5y' + 3y = 0$
17. $y''' + 10y' - 11y = 0$
18. $y^{(4)} = 120t$
19. $4y''' + 8y'' - 11y' + 3y = 0$
20. $2y'' - y = t \sin t$
21. $y'' - 3y' + 7y = 7t^2 - e^t$

22. $y'' - 8y' - 33y = 546 \sin t$

23. $y''(\theta) + 16y(\theta) = \tan 4\theta$

24. $10y'' + y' - 3y = t - e^{t/2}$

25. $4y'' - 12y' + 9y = e^{5t} + e^{3t}$

26. $y'' + 6y' + 15y = e^{2t} + 75$

27. $x^2y'' + 2xy' - 2y = 6x^{-2} + 3x \ , \quad x > 0$

28. $y'' = 5x^{-1}y' - 13x^{-2}y \ , \quad x > 0$

In Problems 29–36, find the solution to the given initial value problem.

29. $y'' + 4y' + 7y = 0$;
$y(0) = 1 \ , \quad y'(0) = -2$

30. $y''(\theta) + 2y'(\theta) + y(\theta) = 2 \cos \theta$;
$y(0) = 3 \ , \quad y'(0) = 0$

31. $y'' - 2y' + 10y = 6 \cos 3t - \sin 3t$;
$y(0) = 2 \ , \quad y'(0) = -8$

32. $4y'' - 4y' + 5y = 0$;
$y(0) = 1 \ , \quad y'(0) = -11/2$

33. $y''' - 12y'' + 27y' + 40y = 0$;
$y(0) = -3 \ , \quad y'(0) = -6 \ , \quad y''(0) = -12$

34. $y'' + 5y' - 14y = 0$;
$y(0) = 5 \ , \quad y'(0) = 1$

35. $y''(\theta) + y(\theta) = \sec \theta$; $y(0) = 1 \ , \quad y'(0) = 2$

36. $9y'' + 12y' + 4y = 0$;
$y(0) = -3 \ , \quad y'(0) = 3$

37. Use the mass–spring oscillator analogy to decide whether all solutions to each of the following differential equations are bounded as $t \to +\infty$.

(a) $y'' + t^4y = 0$

(b) $y'' - t^4y = 0$

(c) $y'' + y^7 = 0$

(d) $y'' + y^8 = 0$

(e) $y'' + (3 + \sin t)y = 0$

(f) $y'' + t^2y' + y = 0$

(g) $y'' - t^2y' - y = 0$

38. A 3-kg mass is attached to a spring with stiffness $k = 75$ N/m, as in Figure 4.1, page 153. The mass is displaced $1/4$ m to the left and given a velocity of 1 m/sec to the right. The damping force is negligible. Find the equation of motion of the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium position?

39. A 32-lb weight is attached to a vertical spring, causing it to stretch 6 in. upon coming to rest at equilibrium. The damping constant for the system is 2 lb-sec/ft. An external force $F(t) = 4 \cos 8t$ lb is applied to the weight. Find the steady-state solution for the system. What is its resonant frequency?

TECHNICAL WRITING EXERCISES

1. Compare the two methods—undetermined coefficients and variation of parameters—for determining a particular solution to a nonhomogeneous equation. What are the advantages and disadvantages of each?

2. Consider the differential equation

$$\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + y = 0 \ ,$$

where b is a constant. Describe how the behavior of solutions to this equation changes as b varies.

3. Consider the differential equation

$$\frac{d^2y}{dx^2} + cy = 0 \ ,$$

where c is a constant. Describe how the behavior of solutions to this equation changes as c varies.

4. For students with a background in linear algebra: Compare the theory for linear second-order equations with that for systems of n linear equations in n unknowns whose coefficient matrix has rank $n - 2$. Use the terminology from linear algebra; for example, subspace, basis, dimension, linear transformation, and kernel. Discuss both homogeneous and nonhomogeneous equations.