6.1 EXERCISES

In Problems 1–6, determine the largest interval (a, b) for which Theorem 1 guarantees the existence of a unique solution on (a, b) to the given initial value problem.

1. $xy''' - 3y' + e^x y = x^2 - 1$;
$y(-2) = 1$, $y'(-2) = 0$, $y''(-2) = 2$

2. $y''' - \sqrt{x}y = \sin x$;
$y(\pi) = 0$, $y'(\pi) = 11$, $y''(\pi) = 3$

3. $y''' - y'' + \sqrt{x-1}y = \tan x$;
$y(5) = y'(5) = y''(5) = 1$

4. $x(x+1)y''' - 3xy' + y = 0$;
$y(-1/2) = 1$, $y'(-1/2) = y''(-1/2) = 0$

5. $x\sqrt{x+1}y''' - y' + xy = 0$;
$y(1/2) = y'(1/2) = -1$, $y''(1/2) = 1$

6. $(x^2 - 1)y''' + e^x y = \ln x$;
$y(3/4) = 1$, $y'(3/4) = y''(3/4) = 0$

In Problems 7–14, determine whether the given functions are linearly dependent or linearly independent on the specified interval. Justify your decisions.

7. $\{e^{3x}, e^{5x}, e^{-x}\}$ on $(-\infty, \infty)$

8. $\{x^2, x^2 - 1, 5\}$ on $(-\infty, \infty)$

9. $\{\sin^2 x, \cos^2 x, 1\}$ on $(-\infty, \infty)$

10. $\{\sin x, \cos x, \tan x\}$ on $(-\pi/2, \pi/2)$

11. $\{x^{-1}, x^{1/2}, x\}$ on $(0, \infty)$

12. $\{\cos 2x, \cos^2 x, \sin^2 x\}$ on $(-\infty, \infty)$

13. $\{x, x^2, x^3, x^4\}$ on $(-\infty, \infty)$

14. $\{x, xe^x, 1\}$ on $(-\infty, \infty)$

Using the Wronskian in Problems 15–18, verify that the given functions form a fundamental solution set for the given differential equation and find a general solution.

15. $y''' + 2y'' - 11y' - 12y = 0$;
$\left\{e^{3x}, e^{-x}, e^{-4x}\right\}$

16. $y''' - y'' + 4y' - 4y = 0$;
$\left\{e^x, \cos 2x, \sin 2x\right\}$

17. $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$, $x > 0$;
$\{x, x^2, x^3\}$

18. $y^{(4)} - y = 0$; $\{e^x, e^{-x}, \cos x, \sin x\}$

In Problems 19–22, a particular solution and a fundamental solution set are given for a nonhomogeneous equation and its corresponding homogeneous equation. (a) Find a general solution to the nonhomogeneous equation. (b) Find the solution that satisfies the specified initial conditions.

19. $y''' + y'' + 3y' - 5y = 2 + 6x - 5x^2$;
$y(0) = -1$, $y'(0) = 1$, $y''(0) = -3$;
$y_p = x^2$; $\{e^x, e^{-x}\cos 2x, e^{-x}\sin 2x\}$

20. $xy''' - y'' = -2$; $y(1) = 2$, $y'(1) = -1$,
$y''(1) = -4$; $y_p = x^2$; $\{1, x, x^3\}$

21. $x^3y''' + xy' - y = 3 - \ln x$, $x > 0$;
$y(1) = 3$, $y'(1) = 3$, $y''(1) = 0$;
$y_p = \ln x$; $\{x, x\ln x, x(\ln x)^2\}$

22. $y^{(4)} + 4y = 5\cos x$;
$y(0) = 2$, $y'(0) = 1$, $y''(0) = -1$,
$y'''(0) = -2$; $y_p = \cos x$;
$\{e^x\cos x, e^x\sin x, e^{-x}\cos x, e^{-x}\sin x\}$

23. Let $L[y] := y''' + y' + xy$, $y_1(x) := \sin x$, and $y_2(x) := x$. Verify that $L[y_1](x) = x\sin x$ and $L[y_2](x) = x^2 + 1$. Then use the superposition principle (linearity) to find a solution to the differential equation:
(a) $L[y] = 2x\sin x - x^2 - 1$.
(b) $L[y] = 4x^2 + 4 - 6x\sin x$.

24. Let $L[y] := y''' - xy'' + 4y' - 3xy$, $y_1(x) := \cos 2x$, and $y_2(x) := -1/3$. Verify that $L[y_1](x) = x\cos 2x$ and $L[y_2](x) = x$. Then use the superposition principle (linearity) to find a solution to the differential equation:
(a) $L[y] = 7x\cos 2x - 3x$.
(b) $L[y] = -6x\cos 2x + 11x$.

25. Prove that L defined in (7) is a linear operator by verifying that properties (9) and (10) hold for any n-times differentiable functions $y, y_1, \ldots, y_m$ on (a, b).

26. Existence of Fundamental Solution Sets. By Theorem 1, for each $j = 1, 2, \ldots, n$ there is a unique solution $y_j(x)$ to equation (17) satisfying the initial conditions

$$y_j^{(k)}(x_0) = \begin{cases} 1 , & \text{for } k = j-1 , \\ 0 , & \text{for } k \neq j-1, \quad 0 \le k \le n-1 . \end{cases}$$

(a) Show that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental solution set for (17). [*Hint:* Write out the Wronskian at x_0.]

(b) For given initial values $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, express the solution $y(x)$ to (17) satisfying $y^{(k)}(x_0) = \gamma_k$, $k = 0, \ldots, n-1$, [as in equations (4)] in terms of this fundamental solution set.

27. Show that the set of functions $\{1, x, x^2, \ldots, x^n\}$, where n is a positive integer, is linearly independent on every open interval (a, b). [*Hint:* Use the fact that a polynomial of degree at most n has no more than n zeros unless it is identically zero.]

28. The set of functions

$$\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\},$$

where n is a positive integer, is linearly independent on every interval (a, b). Prove this in the special case $n = 2$ and $(a, b) = (-\infty, \infty)$.

29. (a) Show that if $f_1, \ldots, f_m$ are linearly independent on $(-1, 1)$, then they are linearly independent on $(-\infty, \infty)$.

(b) Give an example to show that if $f_1, \ldots, f_m$ are linearly independent on $(-\infty, \infty)$, then they need not be linearly independent on $(-1, 1)$.

30. To prove Abel's identity (26) for $n = 3$, proceed as follows:

(a) Let $W(x) := W[y_1, y_2, y_3](x)$. Use the product rule for differentiation to show

$$W'(x) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} .$$

(b) Show that the above expression reduces to

$$W'(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} . \tag{32}$$

(c) Since each y_i satisfies (17), show that

$$y_i^{(3)}(x) = -\sum_{k=1}^{3} p_k(x) y_i^{(3-k)}(x) \quad (i = 1, 2, 3) . \tag{33}$$

(d) Substituting the expressions in (33) into (32), show that

$$W'(x) = -p_1(x)W(x) . \tag{34}$$

(e) Deduce Abel's identity by solving the first-order differential equation (34).

31. Reduction of Order. If a nontrivial solution $f(x)$ is known for the *homogeneous* equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 ,$$

the substitution $y(x) = v(x)f(x)$ can be used to reduce the order of the equation, as was shown in Section 4.7 for second-order equations. By completing the following steps, demonstrate the method for the third-order equation

$$y''' - 2y'' - 5y' + 6y = 0 , \tag{35}$$

given that $f(x) = e^x$ is a solution.

(a) Set $y(x) = v(x)e^x$ and compute y', y'', and y'''.

(b) Substitute your expressions from (a) into (35) to obtain a *second-order* equation in $w := v'$.

(c) Solve the second-order equation in part (b) for w and integrate to find v. Determine two linearly independent choices for v, say, v_1 and v_2.

(d) By part (c), the functions $y_1(x) = v_1(x)e^x$ and $y_2(x) = v_2(x)e^x$ are two solutions to (35). Verify that the three solutions e^x, $y_1(x)$, and $y_2(x)$ are linearly independent on $(-\infty, \infty)$.

32. Given that the function $f(x) = x$ is a solution to $y''' - x^2y' + xy = 0$, show that the substitution $y(x) = v(x)f(x) = v(x)x$ reduces this equation to $xw'' + 3w' - x^3w = 0$, where $w = v'$.

33. Use the reduction of order method described in Problem 31 to find three linearly independent solutions to $y''' - 2y'' + y' - 2y = 0$, given that $f(x) = e^{2x}$ is a solution.

34. Constructing Differential Equations. Given three functions $f_1(x), f_2(x), f_3(x)$ that are each three times differentiable and whose Wronskian is never zero on (a, b), show that the equation

$$\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & y \\ f_1'(x) & f_2'(x) & f_3'(x) & y' \\ f_1''(x) & f_2''(x) & f_3''(x) & y'' \\ f_1'''(x) & f_2'''(x) & f_3'''(x) & y''' \end{vmatrix} = 0$$

is a third-order linear differential equation for which $\{f_1, f_2, f_3\}$ is a fundamental solution set. What is the coefficient of y''' in this equation?

35. Use the result of Problem 34 to construct a third-order differential equation for which $\{x, \sin x, \cos x\}$ is a fundamental solution set.

Example 3 Find a general solution to

(29) $$y^{(4)} - y^{(3)} - 3y'' + 5y' - 2y = 0 \ .$$

Solution The auxiliary equation is

$$r^4 - r^3 - 3r^2 + 5r - 2 = (r - 1)^3(r + 2) = 0 \ ,$$

which has roots $r_1 = 1$, $r_2 = 1$, $r_3 = 1$, $r_4 = -2$. Because the root at 1 has multiplicity 3, a general solution is

(30) $$y(x) = C_1e^x + C_2xe^x + C_3x^2e^x + C_4e^{-2x} \ . \ ◆$$

Example 4 Find a general solution to

(31) $$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0 \ ,$$

whose auxiliary equation can be factored as

(32) $$r^4 - 8r^3 + 26r^2 - 40r + 25 = (r^2 - 4r + 5)^2 = 0 \ .$$

Solution The auxiliary equation (32) has repeated complex roots: $r_1 = 2 + i$, $r_2 = 2 + i$, $r_3 = 2 - i$, and $r_4 = 2 - i$. Hence a general solution is

$$y(x) = C_1e^{2x}\cos x + C_2xe^{2x}\cos x + C_3e^{2x}\sin x + C_4xe^{2x}\sin x \ . \ ◆$$

6.2 EXERCISES

In Problems 1–14, find a general solution for the differential equation with x as the independent variable.

1. $y''' + 2y'' - 8y' = 0$

2. $y''' - 3y'' - y' + 3y = 0$

3. $6z''' + 7z'' - z' - 2z = 0$

4. $y''' + 2y'' - 19y' - 20y = 0$

5. $y''' + 3y'' + 28y' + 26y = 0$

6. $y''' - y'' + 2y = 0$

7. $2y''' - y'' - 10y' - 7y = 0$

8. $y''' + 5y'' - 13y' + 7y = 0$

9. $u''' - 9u'' + 27u' - 27u = 0$

10. $y''' + 3y'' - 4y' - 6y = 0$

11. $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$

12. $y''' + 5y'' + 3y' - 9y = 0$

13. $y^{(4)} + 4y'' + 4y = 0$

14. $y^{(4)} + 2y''' + 10y'' + 18y' + 9y = 0$
[*Hint:* $y(x) = \sin 3x$ is a solution.]

In Problems 15–18, find a general solution to the given homogeneous equation.

15. $(D - 1)^2(D + 3)(D^2 + 2D + 5)^2[y] = 0$

16. $(D + 1)^2(D - 6)^3(D + 5)(D^2 + 1) \cdot (D^2 + 4)[y] = 0$

17. $(D + 4)(D - 3)(D + 2)^3(D^2 + 4D + 5)^2 \cdot D^5[y] = 0$

18. $(D - 1)^3(D - 2)(D^2 + D + 1) \cdot (D^2 + 6D + 10)^3[y] = 0$

In Problems 19–21, solve the given initial value problem.

19. $y''' - y'' - 4y' + 4y = 0$;
$y(0) = -4$, $y'(0) = -1$, $y''(0) = -19$

20. $y''' + 7y'' + 14y' + 8y = 0$;
$y(0) = 1$, $y'(0) = -3$, $y''(0) = 13$

21. $y''' - 4y'' + 7y' - 6y = 0$;
$y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$

In Problems 22 and 23, find a general solution for the given linear system using the elimination method of Section 5.2.

22. $d^2x/dt^2 - x + 5y = 0$,
$2x + d^2y/dt^2 + 2y = 0$

23. $d^3x/dt^3 - x + dy/dt + y = 0$,
$dx/dt - x + y = 0$

24. Let $P(r) = a_n r^n + \cdots + a_1 r + a_0$ be a polynomial with real coefficients $a_n, \ldots, a_0$. Prove that if r_1 is a zero of $P(r)$, then so is its complex conjugate $\bar{r}_1$. [*Hint:* Show that $\overline{P(r)} = P(\bar{r})$, where the bar denotes complex conjugation.]

25. Show that the m functions $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are linearly independent on $(-\infty, \infty)$. [*Hint:* Show that these functions are linearly independent if and only if $1, x, \ldots, x^{m-1}$ are linearly independent.]

26. As an alternative proof that the functions $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ are linearly independent on $(-\infty, \infty)$ when $r_1, r_2, \ldots, r_n$ are distinct, assume

$$C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x} = 0 \qquad (33)$$

holds for all x in $(-\infty, \infty)$ and proceed as follows:

(a) Because the r_i's are distinct we can (if necessary) relabel them so that

$$r_1 > r_2 > \cdots > r_n .$$

Divide equation (33) by $e^{r_1 x}$ to obtain

$$C_1 + C_2 \frac{e^{r_2 x}}{e^{r_1 x}} + \cdots + C_n \frac{e^{r_n x}}{e^{r_1 x}} = 0 .$$

Now let $x \to +\infty$ on the left-hand side to obtain $C_1 = 0$.

(b) Since $C_1 = 0$, equation (33) becomes

$$C_2 e^{r_2 x} + C_3 e^{r_3 x} + \cdots + C_n e^{r_n x} = 0$$

for all x in $(-\infty, \infty)$. Divide this equation by $e^{r_2 x}$ and let $x \to +\infty$ to conclude that $C_2 = 0$.

(c) Continuing in the manner of (b), argue that all the coefficients, $C_1, C_2, \ldots, C_n$ are zero and hence $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ are linearly independent on $(-\infty, \infty)$.

27. Find a general solution to

$$y^{(4)} + 2y''' - 3y'' - y' + \frac{1}{2}y = 0$$

by using Newton's method (Appendix B) or some other numerical procedure to approximate the roots of the auxiliary equation.

28. Find a general solution to $y''' - 3y' - y = 0$ by using Newton's method or some other numerical procedure to approximate the roots of the auxiliary equation.

29. Find a general solution to

$$y^{(4)} + 2y^{(3)} + 4y'' + 3y' + 2y = 0$$

by using Newton's method to approximate numerically the roots of the auxiliary equation. [*Hint:* To find complex roots, use the Newton recursion formula $z_{n+1} = z_n - f(z_n)/f'(z_n)$ and start with a *complex* initial guess z_0.]

30. **(a)** Derive the form

$$y(x) = A_1 e^x + A_2 e^{-x} + A_3 \cos x + A_4 \sin x$$

for the general solution to the equation $y^{(4)} = y$, from the observation that the fourth roots of unity are $1, -1, i$, and $-i$.

(b) Derive the form

$$y(x) = A_1 e^x + A_2 e^{-x/2} \cos\left(\sqrt{3}x/2\right) + A_3 e^{-x/2} \sin\left(\sqrt{3}x/2\right)$$

for the general solution to the equation $y^{(3)} = y$, from the observation that the cube roots of unity are $1, e^{i2\pi/3}$, and $e^{-i2\pi/3}$.

31. **Higher-Order Cauchy–Euler Equations.** A differential equation that can be expressed in the form

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = 0 ,$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants, is called a homogeneous **Cauchy–Euler** equation. (The second-order case is discussed in Section 4.7.) Use the substitution $y = x^r$ to help determine a fundamental solution set for the following Cauchy–Euler equations:

(a) $x^3 y''' + x^2 y'' - 2xy' + 2y = 0 , \quad x > 0 .$

(b) $x^4 y^{(4)} + 6x^3 y''' + 2x^2 y'' - 4xy' + 4y = 0 ,$ $x > 0 .$

(c) $x^3 y''' - 2x^2 y'' + 13xy' - 13y = 0 ,$ $x > 0$

[*Hint:* $x^{\alpha + i\beta} = e^{(\alpha + i\beta)\ln x} = x^{\alpha}\{\cos(\beta \ln x) + i \sin(\beta \ln x)\}.$]

32. Let $y(x) = Ce^{rx}$, where $C\ (\neq 0)$ and r are real numbers, be a solution to a differential equation. Suppose we cannot determine r exactly but can only approximate it by $\tilde{r}$. Let $\tilde{y}(x) := Ce^{\tilde{r}x}$ and consider the error $|y(x) - \tilde{y}(x)|$.

(a) If r and $\tilde{r}$ are positive, $r \neq \tilde{r}$, show that the error grows exponentially large as x approaches $+\infty$.

(b) If r and $\tilde{r}$ are negative, $r \neq \tilde{r}$, show that the error goes to zero exponentially as x approaches $+\infty$.

33. On a smooth horizontal surface, a mass of m_1 kg is attached to a fixed wall by a spring with spring constant k_1 N/m. Another mass of m_2 kg is attached to the first object by a spring with spring constant k_2 N/m. The objects are aligned horizontally so that the springs are their natural lengths. As we showed in

Section 5.6, this coupled mass–spring system is governed by the system of differential equations

$$\text{(34)} \qquad m_1\frac{d^2x}{dt^2} + (k_1 + k_2)x - k_2 y = 0 \ ,$$

$$\text{(35)} \qquad m_2\frac{d^2y}{dt^2} - k_2 x + k_2 y = 0 \ .$$

Let's assume that $m_1 = m_2 = 1, k_1 = 3$, and $k_2 = 2$. If both objects are displaced 1 m to the right of their equilibrium positions (compare Figure 5.26, page 285) and then released, determine the equations of motion for the objects as follows:

(a) Show that $x(t)$ satisfies the equation

$$\text{(36)} \quad x^{(4)}(t) + 7x''(t) + 6x(t) = 0 \ .$$

(b) Find a general solution $x(t)$ to (36).

(c) Substitute $x(t)$ back into (34) to obtain a general solution for $y(t)$.

(d) Use the initial conditions to determine the solutions, $x(t)$ and $y(t)$, which are the equations of motion.

34. Suppose the two springs in the coupled mass–spring system discussed in Problem 33 are switched, giving the new data $m_1 = m_2 = 1, k_1 = 2$, and $k_2 = 3$. If both objects are now displaced 1 m to the right of their equilibrium positions and then released, determine the equations of motion of the two objects.

35. Vibrating Beam. In studying the transverse vibrations of a beam, one encounters the homogeneous equation

$$EI\frac{d^4y}{dx^4} - ky = 0 \ ,$$

where $y(x)$ is related to the displacement of the beam at position x, the constant E is Young's modulus, I is the area moment of inertia, and k is a parameter. Assuming E, I, and k are positive constants, find a general solution in terms of sines, cosines, hyperbolic sines, and hyperbolic cosines.

6.3 UNDETERMINED COEFFICIENTS AND THE ANNIHILATOR METHOD

In Sections 4.4 and 4.5 we mastered an easy method for obtaining a particular solution to a nonhomogeneous linear second-order constant coefficient equation,

$$\text{(1)} \qquad L[y] = (aD^2 + bD + c)[y] = f(x) \ ,$$

when the nonhomogeneity $f(x)$ had a particular form (namely, a product of a polynomial, an exponential, and a sinusoid). Roughly speaking, we were motivated by the observation that if a function f, of this type, resulted from operating on y with an operator L of the form $(aD^2 + bD + c)$, then we must have started with a y of the same type. So we solved (1) by postulating a solution form y_p that resembled f, but with *undetermined coefficients,* and we inserted this form into the equation to fix the values of these coefficients. Eventually, we realized that we had to make certain accommodations when f was a solution to the homogeneous equation $L[y] = 0$.

In this section we are going to reexamine the method of undetermined coefficients from another, more rigorous, point of view—partly with the objective of tying up the loose ends in our previous exposition and more importantly with the goal of extending the method to higher-order equations (with constant coefficients). At the outset we'll describe the new point of view that will be adopted for the analysis. Then we illustrate its implications and ultimately derive a simplified set of rules for its implementation: rules that justify and extend the procedures of Section 4.4. The rigorous approach is known as the **annihilator method.**

The first premise of the annihilator method is the observation, gleaned from the analysis of the previous section, that all of the "suitable types" of nonhomogeneities $f(x)$ (products of

6.3 EXERCISES

In Problems 1–4, use the method of undetermined coefficients to determine the form of a particular solution for the given equation.

1. $y''' - 2y'' - 5y' + 6y = e^x + x^2$
2. $y''' + y'' - 5y' + 3y = e^{-x} + \sin x$
3. $y''' + 3y'' - 4y = e^{-2x}$
4. $y''' + y'' - 2y = xe^x + 1$

In Problems 5–10, find a general solution to the given equation.

5. $y''' - 2y'' - 5y' + 6y = e^x + x^2$
6. $y''' + y'' - 5y' + 3y = e^{-x} + \sin x$
7. $y''' + 3y'' - 4y = e^{-2x}$
8. $y''' + y'' - 2y = xe^x + 1$
9. $y''' - 3y'' + 3y' - y = e^x$
10. $y''' + 4y'' + y' - 26y = e^{-3x} \sin 2x + x$

In Problems 11–20, find a differential operator that annihilates the given function.

11. $x^4 - x^2 + 11$
12. $3x^2 - 6x + 1$
13. e^{-7x}
14. e^{5x}
15. $e^{2x} - 6e^x$
16. $x^2 - e^x$
17. $x^2 e^{-x} \sin 2x$
18. $xe^{3x} \cos 5x$
19. $xe^{-2x} + xe^{-5x} \sin 3x$
20. $x^2 e^x - x \sin 4x + x^3$

In Problems 21–30, use the annihilator method to determine the form of a particular solution for the given equation.

21. $u'' - 5u' + 6u = \cos 2x + 1$
22. $y'' + 6y' + 8y = e^{3x} - \sin x$
23. $y'' - 5y' + 6y = e^{3x} - x^2$
24. $\theta'' - \theta = xe^x$
25. $y'' - 6y' + 9y = \sin 2x + x$
26. $y'' + 2y' + y = x^2 - x + 1$
27. $y'' + 2y' + 2y = e^{-x} \cos x + x^2$
28. $y'' - 6y' + 10y = e^{3x} - x$
29. $z''' - 2z'' + z' = x - e^x$
30. $y''' + 2y'' - y' - 2y = e^x - 1$

In Problems 31–33, solve the given initial value problem.

31. $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$;
$y(0) = -2$, $y'(0) = -8$, $y''(0) = -12$
32. $y''' - 2y'' + 5y' = -24e^{3x}$;
$y(0) = 4$, $y'(0) = -1$, $y''(0) = -5$
33. $y''' - 2y'' - 3y' + 10y$
$= 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34$;
$y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$

34. Use the annihilator method to show that if $a_0 \neq 0$ in equation (4) and $f(x)$ has the form

$$\textbf{(17)} \quad f(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \ ,$$

then

$$y_p(x) = B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0$$

is the form of a particular solution to equation (4).

35. Use the annihilator method to show that if $a_0 = 0$ and $a_1 \neq 0$ in (4) and $f(x)$ has the form given in (17), then equation (4) has a particular solution of the form

$$y_p(x) = x\{B_m x^m + B_{m-1} x^{m-1} + \cdots + B_1 x + B_0\} \ .$$

36. Use the annihilator method to show that if $f(x)$ in (4) has the form $f(x) = Be^{\alpha x}$, then equation (4) has a particular solution of the form $y_p(x) = x^s Be^{\alpha x}$, where s is chosen to be the smallest nonnegative integer such that $x^s e^{\alpha x}$ is not a solution to the corresponding homogeneous equation.

37. Use the annihilator method to show that if $f(x)$ in (4) has the form

$$f(x) = a \cos \beta x + b \sin \beta x \ ,$$

then equation (4) has a particular solution of the form

$$\textbf{(18)} \quad y_p(x) = x^s\{A \cos \beta x + B \sin \beta x\} \ ,$$

where s is chosen to be the smallest nonnegative integer such that $x^s \cos \beta x$ and $x^s \sin \beta x$ are not solutions to the corresponding homogeneous equation.

In Problems 38 and 39, use the elimination method of Section 5.2 to find a general solution to the given system.

38. $x - d^2y/dt^2 = t + 1$,
$dx/dt + dy/dt - 2y = e^t$
39. $d^2x/dt^2 - x + y = 0$,
$x + d^2y/dt^2 - y = e^{3t}$

40. The currents in the electrical network in Figure 6.1 satisfy the system

$$\frac{1}{9}I_1 + 64I_2'' = -2\sin\frac{t}{24} ,$$

$$\frac{1}{64}I_3 + 9I_3'' - 64I_2'' = 0 ,$$

$$I_1 = I_2 + I_3 ,$$

where I_1, I_2, and I_3 are the currents through the different branches of the network. Using the elimination method of Section 5.2, determine the currents if initially $I_1(0) = I_2(0) = I_3(0) = 0$, $I_1'(0) = 73/12$, $I_2'(0) = 3/4$, and $I_3'(0) = 16/3$.

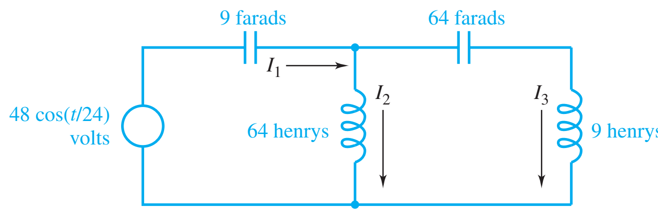

Figure 6.1 An electrical network

6.4 METHOD OF VARIATION OF PARAMETERS

In the previous section, we discussed the method of undetermined coefficients and the annihilator method. These methods work only for linear equations with constant coefficients *and* when the nonhomogeneous term is a solution to some homogeneous linear equation with constant coefficients. In this section we show how the method of **variation of parameters** discussed in Sections 4.6 and 4.7 generalizes to higher-order linear equations with variable coefficients.

Our goal is to find a particular solution to the standard form equation

$$L[y](x) = g(x) , \tag{1}$$

where $L[y] := y^{(n)} + p_1y^{(n-1)} + \cdots + p_ny$ and the coefficient functions $p_1, \ldots, p_n$, as well as g, are continuous on (a, b). The method to be described requires that we already know a fundamental solution set $\{y_1, \ldots, y_n\}$ for the corresponding homogeneous equation

$$L[y](x) = 0 . \tag{2}$$

A general solution to (2) is then

$$y_h(x) = C_1y_1(x) + \cdots + C_ny_n(x) , \tag{3}$$

where $C_1, \ldots, C_n$ are arbitrary constants. In the method of variation of parameters, we assume there exists a particular solution to (1) of the form

$$\boldsymbol{y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x)} \tag{4}$$

and try to determine the functions $v_1, \ldots, v_n$.

There are n unknown functions, so we will need n conditions (equations) to determine them. These conditions are obtained as follows. Differentiating y_p in (4) gives

$$y_p' = (v_1y_1' + \cdots + v_ny_n') + (v_1'y_1 + \cdots + v_n'y_n) . \tag{5}$$

To prevent second derivatives of the unknowns $v_1, \ldots, v_n$ from entering the formula for y_p'', we impose the condition

$$v_1'y_1 + \cdots + v_n'y_n = 0 .$$

6.4 EXERCISES

In Problems 1–6, use the method of variation of parameters to determine a particular solution to the given equation.

1. $y''' - 3y'' + 4y = e^{2x}$
2. $y''' - 2y'' + y' = x$
3. $z''' + 3z'' - 4z = e^{2x}$
4. $y''' - 3y'' + 3y' - y = e^x$
5. $y''' + y' = \tan x\ , \qquad 0 < x < \pi/2$
6. $y''' + y' = \sec\theta\tan\theta\ , \qquad 0 < \theta < \pi/2$
7. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^{-1}\ , \quad x > 0\ ,$$
given that $\{x, x^2, x^3\}$ is a fundamental solution set for the corresponding homogeneous equation.
8. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 2x^2y'' + 3xy' - 3y = x^2\ , \quad x > 0\ ,$$
given that $\{x, x\ln x, x^3\}$ is a fundamental solution set for the corresponding homogeneous equation.
9. Given that $\{e^x, e^{-x}, e^{2x}\}$ is a fundamental solution set for the homogeneous equation corresponding to the equation
$$y''' - 2y'' - y' + 2y = g(x)\ ,$$
determine a formula involving integrals for a particular solution.
10. Given that $\{x, x^{-1}, x^4\}$ is a fundamental solution set for the homogeneous equation corresponding to the equation
$$x^3y''' - x^2y'' - 4xy' + 4y = g(x)\ , \quad x > 0\ ,$$
determine a formula involving integrals for a particular solution.
11. Find a general solution to the Cauchy–Euler equation
$$x^3y''' - 3xy' + 3y = x^4\cos x\ , \qquad x > 0$$
12. Derive the system (7) in the special case when $n = 3$. [*Hint:* To determine the last equation, require that $L[y_p] = g$ and use the fact that y_1, y_2, and y_3 satisfy the corresponding homogeneous equation.]
13. Show that
$$W_k(x) = (-1)^{(n-k)}W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x)\ .$$
14. **Deflection of a Beam Under Axial Force.** A uniform beam under a load and subject to a constant axial force is governed by the differential equation
$$y^{(4)}(x) - k^2y''(x) = q(x)\ , \qquad 0 < x < L,$$
where $y(x)$ is the deflection of the beam, L is the length of the beam, k^2 is proportional to the axial force, and $q(x)$ is proportional to the load (see Figure 6.2).
   (a) Show that a general solution can be written in the form
$$y(x) = C_1 + C_2x + C_3e^{kx} + C_4e^{-kx} + \frac{1}{k^2}\int q(x)x\,dx - \frac{x}{k^2}\int q(x)\,dx + \frac{e^{kx}}{2k^3}\int q(x)e^{-kx}dx - \frac{e^{-kx}}{2k^3}\int q(x)e^{kx}dx\ .$$
   (b) Show that the general solution in part (a) can be rewritten in the form
$$y(x) = c_1 + c_2x + c_3e^{kx} + c_4e^{-kx} + \int_0^x q(s)G(s,x)\,ds\ ,$$
where
$$G(s,x) := \frac{s-x}{k^2} - \frac{\sinh[k(s-x)]}{k^3}\ .$$
   (c) Let $q(x) \equiv 1$. First compute the general solution using the formula in part (a) and then using the formula in part (b). Compare these two general solutions with the general solution
$$y(x) = B_1 + B_2x + B_3e^{kx} + B_4e^{-kx} - \frac{1}{2k^2}x^2\ ,$$
which one would obtain using the method of undetermined coefficients.
   (d) What are some advantages of the formula in part (b)?

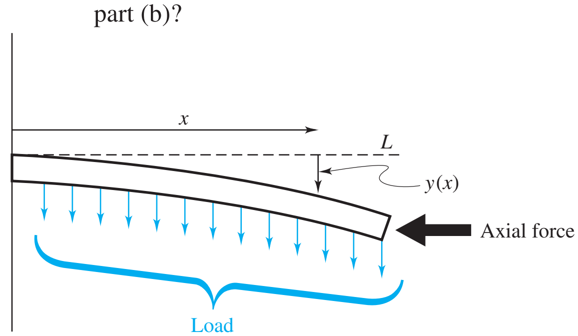

Figure 6.2 Deformation of a beam under axial force and load

where y_p is some particular solution to (1) and y_h is a general solution to the corresponding homogeneous equation. Two useful techniques for finding particular solutions are the **annihilator method** (undetermined coefficients) and the method of **variation of parameters.**

The annihilator method applies to equations of the form

$$L[y] = g(x) \ , \tag{6}$$

where L is a linear differential operator with constant coefficients and the forcing term $g(x)$ is a polynomial, exponential, sine, or cosine, or a linear combination of products of these. Such a function $g(x)$ is annihilated (mapped to zero) by a linear differential operator A that also has constant coefficients. Every solution to the nonhomogeneous equation (6) is then a solution to the homogeneous equation $AL[y] = 0$, and, by comparing the solutions of the latter equation with a general solution to $L[y] = 0$, we can obtain the *form* of a particular solution to (6). These forms have previously been studied in Section 4.4 for the method of undetermined coefficients.

The method of variation of parameters is more general in that it applies to arbitrary equations of the form (1). The idea is, starting with a fundamental solution set $\{y_1, \ldots, y_n\}$ for (2), to determine functions $v_1, \ldots, v_n$ such that

$$y_p(x) = v_1(x)y_1(x) + \cdots + v_n(x)y_n(x) \tag{7}$$

satisfies (1). This method leads to the formula

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{g(x)W_k(x)}{W[y_1, \ldots, y_n](x)}\, dx \ , \tag{8}$$

where

$$W_k(x) = (-1)^{n-k} W[y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n](x) \ , \qquad k = 1, \ldots, n \ .$$

REVIEW PROBLEMS

1. Determine the intervals for which Theorem 1 on page 318 guarantees the existence of a solution in that interval.

(a) $y^{(4)} - (\ln x)y'' + xy' + 2y = \cos 3x$

(b) $(x^2 - 1)y''' + (\sin x)y'' + \sqrt{x + 4}\, y' + e^x y = x^2 + 3$

2. Determine whether the given functions are linearly dependent or linearly independent on the interval $(0, \infty)$.

(a) $\{e^{2x},\ x^2e^{2x},\ e^{-x}\}$

(b) $\{e^x \sin 2x,\ xe^x \sin 2x,\ e^x,\ xe^x\}$

(c) $\{2e^{2x} - e^x,\ e^{2x} + 1,\ e^{2x} - 3,\ e^x + 1\}$

3. Show that the set of functions $\{\sin x,\ x \sin x,\ x^2 \sin x,\ x^3 \sin x\}$ is linearly independent on $(-\infty, \infty)$.

4. Find a general solution for the given differential equation.

(a) $y^{(4)} + 2y''' - 4y'' - 2y' + 3y = 0$

(b) $y''' + 3y'' - 5y' + y = 0$

(c) $y^{(5)} - y^{(4)} + 2y''' - 2y'' + y' - y = 0$

(d) $y''' - 2y'' - y' + 2y = e^x + x$

5. Find a general solution for the homogeneous linear differential equation with constant coefficients whose auxiliary equation is

(a) $(r + 5)^2(r - 2)^3(r^2 + 1)^2 = 0$.

(b) $r^4(r - 1)^2(r^2 + 2r + 4)^2 = 0$.

6. Given that $y_p = \sin(x^2)$ is a particular solution to

$$y^{(4)} + y = (16x^4 - 11)\sin(x^2) - 48x^2\cos(x^2)$$

on $(0, \infty)$, find a general solution.

7. Find a differential operator that annihilates the given function.

(a) $x^2 - 2x + 5$ **(b)** $e^{3x} + x - 1$

(c) $x \sin 2x$ **(d)** $x^2e^{-2x} \cos 3x$

(e) $x^2 - 2x + xe^{-x} + \sin 2x - \cos 3x$

8. Use the annihilator method to determine the form of a particular solution for the given equation.

(a) $y'' + 6y' + 5y = e^{-x} + x^2 - 1$
(b) $y''' + 2y'' - 19y' - 20y = xe^{-x}$
(c) $y^{(4)} + 6y'' + 9y = x^2 - \sin 3x$
(d) $y''' - y'' + 2y = x \sin x$

9. Find a general solution to the Cauchy–Euler equation

$$x^3y''' - 2x^2y'' - 5xy' + 5y = x^{-2} ,$$
$$x > 0 ,$$

given that $\{x, x^5, x^{-1}\}$ is a fundamental solution set to the corresponding homogeneous equation.

10. Find a general solution to the given Cauchy–Euler equation.

(a) $4x^3y''' + 8x^2y'' - xy' + y = 0 , \quad x > 0$
(b) $x^3y''' + 2x^2y'' + 2xy' + 4y = 0 , \quad x > 0$

TECHNICAL WRITING EXERCISES

1. Describe the differences and similarities between second-order and higher-order linear differential equations. Include in your comparisons both theoretical results and the methods of solution. For example, what complications arise in solving higher-order equations that are not present for the second-order case?

2. Explain the relationship between the method of undetermined coefficients and the annihilator method. What difficulties would you encounter in applying the annihilator method if the linear equation did not have constant coefficients?

3. For students with a background in linear algebra: Compare the theory for kth-order linear differential equations with that for systems of n linear equations in n unknowns whose coefficient matrix has rank $n - k$. Use the terminology from linear algebra; for example, subspaces, basis, dimension, linear transformation, and kernel. Discuss both homogeneous and nonhomogeneous equations.