7.2 EXERCISES

In Problems 1–12, use Definition 1 to determine the Laplace transform of the given function.

1. t

2. t^2

3. e^{6t}

4. te^{3t}

5. $\cos 2t$

6. $\cos bt$, b a constant

7. $e^{2t}\cos 3t$

8. $e^{-t}\sin 2t$

9. $f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & 2 < t \end{cases}$

10. $f(t) = \begin{cases} 1 - t, & 0 < t < 1, \\ 0, & 1 < t \end{cases}$

11. $f(t) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & \pi < t \end{cases}$

12. $f(t) = \begin{cases} e^{2t}, & 0 < t < 3, \\ 1, & 3 < t \end{cases}$

In Problems 13–20, use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

13. $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$

14. $\mathcal{L}\{5 - e^{2t} + 6t^2\}$

15. $\mathcal{L}\{t^3 - te^t + e^{4t}\cos t\}$

16. $\mathcal{L}\{t^2 - 3t - 2e^{-t}\sin 3t\}$

17. $\mathcal{L}\{e^{3t}\sin 6t - t^3 + e^t\}$

18. $\mathcal{L}\{t^4 - t^2 - t + \sin\sqrt{2}t\}$

19. $\mathcal{L}\{t^4e^{5t} - e^t\cos\sqrt{7}t\}$

20. $\mathcal{L}\{e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}\}$

In Problems 21–28, determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0, 10]$ and sketch the graph of $f(t)$.

21. $f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ (t-2)^2, & 1 < t \le 10 \end{cases}$

22. $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t, & 2 \le t \le 10 \end{cases}$

23. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ t - 1, & 1 < t < 3, \\ t^2 - 4, & 3 < t \le 10 \end{cases}$

24. $f(t) = \dfrac{t^2 - 3t + 2}{t^2 - 4}$

25. $f(t) = \dfrac{t^2 - t - 20}{t^2 + 7t + 10}$

26. $f(t) = \dfrac{t}{t^2 - 1}$

27. $f(t) = \begin{cases} 1/t, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1 - t, & 2 < t \le 10 \end{cases}$

28. $f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \ne 0, \\ 1, & t = 0 \end{cases}$

29. Which of the following functions are of exponential order?

(a) $t^3\sin t$

(b) $100e^{49t}$

(c) e^{t^3}

(d) $t\ln t$

(e) $\cosh(t^2)$

(f) $\dfrac{1}{t^2 + 1}$

(g) $\sin(t^2) + t^4e^{6t}$

(h) $3 - e^{t^2} + \cos 4t$

(i) $\exp\{t^2/(t+1)\}$

(j) $\sin(e^{t^2}) + e^{\sin t}$

30. For the transforms $F(s)$ in Table 7.1, what can be said about $\lim_{s\to\infty} F(s)$?

31. Thanks to Euler's formula (page 168) and the algebraic properties of complex numbers, several of the entries of Table 7.1 can be derived from a single formula; namely,

$$(6)\quad \mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{s - a + ib}{(s-a)^2 + b^2}, \qquad s > a.$$

(a) By computing the integral in the definition of the Laplace transform on page 353 with $f(t) = e^{(a+ib)t}$, show that

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{1}{s - (a + ib)}, \qquad s > a.$$

(b) Deduce (6) from part (a) by showing that

$$\frac{1}{s - (a+ib)} = \frac{s - a + ib}{(s-a)^2 + b^2}.$$

(c) By equating the real and imaginary parts in formula (6), deduce the last two entries in Table 7.1.

32. Prove that for fixed $s > 0$, we have

$$\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0.$$

33. Prove that if f is piecewise continuous on $[a, b]$ and g is continuous on $[a, b]$, then the product fg is piecewise continuous on $[a, b]$.

For easy reference, Table 7.2 lists some of the basic properties of the Laplace transform derived so far.

TABLE 7.2 Properties of Laplace Transforms

$\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$.
$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant c .
$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$.
$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$.
$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$.
$\mathscr{L}\left\{f^{(n)}\right\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.
$\mathscr{L}\{t^nf(t)\}(s) = (-1)^n\frac{d^n}{ds^n}\left(\mathscr{L}\{f\}(s)\right)$.

7.3 EXERCISES

In Problems 1–20, determine the Laplace transform of the given function using Table 7.1 and the properties of the transform given in Table 7.2. [Hint: In Problems 12–20, use an appropriate trigonometric identity.]

1. $t^2 + e^t \sin 2t$

2. $3t^2 - e^{2t}$

3. $e^{-t}\cos 3t + e^{6t} - 1$

4. $3t^4 - 2t^2 + 1$

5. $2t^2e^{-t} - t + \cos 4t$

6. $e^{-2t}\sin 2t + e^{3t}t^2$

7. $(t-1)^4$

8. $(1+e^{-t})^2$

9. $e^{-t}t \sin 2t$

10. $te^{2t}\cos 5t$

11. $\cosh bt$

12. $\sin 3t \cos 3t$

13. $\sin^2 t$

14. $e^{7t}\sin^2 t$

15. $\cos^3 t$

16. $t \sin^2 t$

17. $\sin 2t \sin 5t$

18. $\cos nt \cos mt$, $m \neq n$

19. $\cos nt \sin mt$, $m \neq n$

20. $t \sin 2t \sin 5t$

21. Given that $\mathscr{L}\{\cos bt\}(s) = s/(s^2 + b^2)$, use the translation property to compute $\mathscr{L}\{e^{at}\cos bt\}$.

22. Starting with the transform $\mathscr{L}\{1\}(s) = 1/s$, use formula (6) for the derivatives of the Laplace transform to show that $\mathscr{L}\{t\}(s) = 1/s^2$, $\mathscr{L}\{t^2\}(s) = 2!/s^3$, and, by using induction, that $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 1, 2, \ldots$.

23. Use Theorem 4 to show how entry 32 follows from entry 31 in the Laplace transform table on the inside back cover of the text.

24. Show that $\mathscr{L}\{e^{at}t^n\}(s) = n!/(s-a)^{n+1}$ in two ways:

(a) Use the translation property for $F(s)$.

(b) Use formula (6) for the derivatives of the Laplace transform.

25. Use formula (6) to help determine

(a) $\mathscr{L}\{t\cos bt\}$. **(b)** $\mathscr{L}\{t^2\cos bt\}$.

26. Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order.

(a) Show that there exist constants K and α such that

$$|f(t)| \le Ke^{\alpha t} \quad \text{for all } t \ge 0 \ .$$

(b) By using the definition of the transform and estimating the integral with the help of part (a), prove that

$$\lim_{s\to\infty} \mathscr{L}\{f\}(s) = 0 \ .$$

27. Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and assume $\lim_{t\to 0^+}[f(t)/t]$ exists. Show that

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)\,du \ ,$$

where $F(s) = \mathcal{L}\{f\}(s)$. [*Hint:* First show that $\frac{d}{ds}\mathcal{L}\{f(t)/t\}(s) = -F(s)$ and then use the result of Problem 26.]

28. Verify the identity in Problem 27 for the following functions. (Use the table of Laplace transforms on the inside back cover.)

(a) $f(t) = t^5$ **(b)** $f(t) = t^{3/2}$

29. The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, when all initial conditions are zero. If a linear system is governed by the differential equation

$$y''(t) + 6y'(t) + 10y(t) = g(t) , \qquad t > 0 ,$$

use the linearity property of the Laplace transform and Theorem 5 on the Laplace transform of higher-order derivatives to determine the transfer function $H(s) = Y(s)/G(s)$ for this system.

30. Find the transfer function, as defined in Problem 29, for the linear system governed by

$$y''(t) + 5y'(t) + 6y(t) = g(t) , \qquad t > 0 .$$

31. Translation in *t*. Show that for $c > 0$, the translated function

$$g(t) = \begin{cases} 0 , & 0 < t < c , \\ f(t - c) , & c < t \end{cases}$$

has Laplace transform

$$\mathcal{L}\{g\}(s) = e^{-cs}\mathcal{L}\{f\}(s) .$$

In Problems 32–35, let $g(t)$ be the given function $f(t)$ translated to the right by c units. Sketch $f(t)$ and $g(t)$ and find $\mathcal{L}\{g(t)\}(s)$. (See Problem 31.)

32. $f(t) \equiv 1$, $c = 2$

33. $f(t) = t$, $c = 1$

34. $f(t) = \sin t$, $c = \pi$

35. $f(t) = \sin t$, $c = \pi/2$

36. Use equation (5) to provide another derivation of the formula $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$. [*Hint*: Start with $\mathcal{L}\{1\}(s) = 1/s$ and use induction.]

37. Initial Value Theorem. Apply the relation

$$\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st}f'(t)\,dt = s\mathcal{L}\{f\}(s) - f(0) \tag{7}$$

to argue that for any function $f(t)$ whose derivative is piecewise continuous and of exponential order on $[0, \infty)$,

$$f(0) = \lim_{s\to\infty} s\mathcal{L}\{f\}(s) .$$

38. Verify the initial value theorem (Problem 37) for the following functions. (Use the table of Laplace transforms on the inside back cover.)

(a) 1 **(b)** e^t **(c)** e^{-t} **(d)** $\cos t$

(e) $\sin t$ **(f)** t^2 **(g)** $t \cos t$

7.4 INVERSE LAPLACE TRANSFORM

In Section 7.2 we defined the Laplace transform as an integral operator that maps a function $f(t)$ into a function $F(s)$. In this section we consider the problem of finding the function $f(t)$ when we are given the transform $F(s)$. That is, we seek an **inverse mapping** for the Laplace transform.

To see the usefulness of such an inverse, let's consider the simple initial value problem

$$y'' - y = -t ; \qquad y(0) = 0 , \qquad y'(0) = 1 . \tag{1}$$

If we take the transform of both sides of equation (1) and use the linearity property of the transform, we find

$$\mathcal{L}\{y''\}(s) - Y(s) = -\frac{1}{s^2} ,$$

and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) \ ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$\begin{aligned} 0 &= [A(-1) + 2B](1) + C(5) \ , \\ 0 &= -A + 8 - 5 \ , \\ A &= 3 \ . \end{aligned}$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} \ .$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t) \\ &= 3\mathcal{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t) \\ &\quad + 4\mathcal{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t) \\ &= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} \ . \ \blacklozenge \end{aligned}$$

In Section 7.7, we discuss a different method (involving convolutions) for computing inverse transforms that does not require partial fraction decompositions. Moreover, the convolution method is convenient in the case of a rational function with a repeated quadratic factor in the denominator. Other helpful tools are described in Problems 33–36 and 38–43.

7.4 EXERCISES

In Problems 1–10, determine the inverse Laplace transform of the given function.

1. $\dfrac{6}{(s - 1)^4}$
2. $\dfrac{2}{s^2 + 4}$
3. $\dfrac{s + 1}{s^2 + 2s + 10}$
4. $\dfrac{4}{s^2 + 9}$
5. $\dfrac{1}{s^2 + 4s + 8}$
6. $\dfrac{3}{(2s + 5)^3}$
7. $\dfrac{2s + 16}{s^2 + 4s + 13}$
8. $\dfrac{1}{s^5}$
9. $\dfrac{3s - 15}{2s^2 - 4s + 10}$
10. $\dfrac{s - 1}{2s^2 + s + 6}$

In Problems 11–20, determine the partial fraction expansions for the given rational function.

11. $\dfrac{s^2 - 26s - 47}{(s - 1)(s + 2)(s + 5)}$
12. $\dfrac{-s - 7}{(s + 1)(s - 2)}$
13. $\dfrac{-2s^2 - 3s - 2}{s(s + 1)^2}$
14. $\dfrac{-8s^2 - 5s + 9}{(s + 1)(s^2 - 3s + 2)}$
15. $\dfrac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)}$
16. $\dfrac{-5s - 36}{(s + 2)(s^2 + 9)}$
17. $\dfrac{3s + 5}{s(s^2 + s - 6)}$
18. $\dfrac{3s^2 + 5s + 3}{s^4 + s^3}$

19. $\dfrac{1}{(s-3)(s^2+2s+2)}$ **20.** $\dfrac{s}{(s-1)(s^2-1)}$

In Problems 21–30, determine $\mathscr{L}^{-1}\{F\}$.

21. $F(s) = \dfrac{6s^2-13s+2}{s(s-1)(s-6)}$

22. $F(s) = \dfrac{s+11}{(s-1)(s+3)}$

23. $F(s) = \dfrac{5s^2+34s+53}{(s+3)^2(s+1)}$

24. $F(s) = \dfrac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$

25. $F(s) = \dfrac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

26. $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

27. $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$

28. $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$

29. $sF(s) + 2F(s) = \dfrac{10s^2+12s+14}{s^2-2s+2}$

30. $sF(s) - F(s) = \dfrac{2s+5}{s^2+2s+1}$

31. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} 0\ , & t = 2\ , \\ t\ , & t \neq 2\ . \end{cases}$

(b) $f_2(t) = \begin{cases} 5\ , & t = 1\ , \\ 2\ , & t = 6\ , \\ t\ , & t \neq 1, 6\ . \end{cases}$

(c) $f_3(t) = t$.

Which of the preceding functions is the inverse Laplace transform of $1/s^2$?

32. Determine the Laplace transform of each of the following functions:

(a) $f_1(t) = \begin{cases} t\ , & t = 1, 2, 3, \ldots\ , \\ e^t\ , & t \neq 1, 2, 3, \ldots\ . \end{cases}$

(b) $f_2(t) = \begin{cases} e^t\ , & t \neq 5, 8\ , \\ 6\ , & t = 5\ , \\ 0\ , & t = 8\ . \end{cases}$

(c) $f_3(t) = e^t$.

Which of the preceding functions is the inverse Laplace transform of $1/(s-1)$?

Theorem 6 in Section 7.3 can be expressed in terms of the inverse Laplace transform as

$$\mathscr{L}^{-1}\left\{\frac{d^nF}{ds^n}\right\}(t) = (-t)^n f(t)\ ,$$

where $f = \mathscr{L}^{-1}\{F\}$. *Use this equation in Problems 33–36 to compute* $\mathscr{L}^{-1}\{F\}$.

33. $F(s) = \ln\left(\dfrac{s+2}{s-5}\right)$ **34.** $F(s) = \ln\left(\dfrac{s-4}{s-3}\right)$

35. $F(s) = \ln\left(\dfrac{s^2+9}{s^2+1}\right)$ **36.** $F(s) = \arctan(1/s)$

37. Prove Theorem 7 on the linearity of the inverse transform. [*Hint:* Show that the right-hand side of equation (3) is a continuous function on $[0, \infty)$ whose Laplace transform is $F_1(s) + F_2(s)$.]

38. Residue Computation. Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg Q and suppose $s - r$ is a nonrepeated linear factor of $Q(s)$. Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $s - r$ is

$$\frac{A}{s-r}\ ,$$

where A (called the **residue**) is given by the formula

$$A = \lim_{s \to r} \frac{(s-r)P(s)}{Q(s)}\ .$$

39. Use the residue computation formula derived in Problem 38 to determine quickly the partial fraction expansion for

$$F(s) = \frac{2s+1}{s(s-1)(s+2)}\ .$$

40. Heaviside's Expansion Formula.[†] Let $P(s)$ and $Q(s)$ be polynomials with the degree of $P(s)$ less than the degree of $Q(s)$. Let

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)\ ,$$

where the r_i's are distinct real numbers. Show that

$$\mathscr{L}^{-1}\left\{\frac{P}{Q}\right\}(t) = \sum_{i=1}^{n} \frac{P(r_i)}{Q'(r_i)} e^{r_i t}\ .$$

[†] *Historical Footnote:* This formula played an important role in the "operational solution" to ordinary differential equations developed by Oliver Heaviside in the 1890s.

41. Use Heaviside's expansion formula derived in Problem 40 to determine the inverse Laplace transform of

$$F(s) = \frac{3s^2 - 16s + 5}{(s+1)(s-3)(s-2)} \ .$$

42. Complex Residues. Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg Q and suppose $(s-\alpha)^2 + \beta^2$ is a nonrepeated quadratic factor of Q. (That is, $\alpha \pm i\beta$ are complex conjugate zeros of Q.) Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $(s-\alpha)^2 + \beta^2$ is

$$\frac{A(s-\alpha) + \beta B}{(s-\alpha)^2 + \beta^2} \ ,$$

where the **complex residue** $\beta B + i\beta A$ is given by the formula

$$\beta B + i\beta A = \lim_{s \to \alpha + i\beta} \frac{[(s-\alpha)^2 + \beta^2]P(s)}{Q(s)} \ .$$

(Thus we can determine B and A by taking the real and imaginary parts of the limit and dividing them by β.)

43. Use the residue formulas derived in Problems 38 and 42 to determine the partial fraction expansion for

$$F(s) = \frac{6s^2 + 28}{(s^2 - 2s + 5)(s+2)} \ .$$

7.5 SOLVING INITIAL VALUE PROBLEMS

Our goal is to show how Laplace transforms can be used to solve initial value problems for linear differential equations. Recall that we have already studied ways of solving such initial value problems in Chapter 4. These previous methods required that we first find a *general solution* of the differential equation and then use the initial conditions to determine the desired solution. As we will see, the method of Laplace transforms leads to the solution of the initial value problem *without* first finding a general solution.

Other advantages to the transform method are worth noting. For example, the technique can easily handle equations involving forcing functions having jump discontinuities, as illustrated in Section 7.1. Further, the method can be used for certain linear differential equations with variable coefficients, a special class of integral equations, systems of differential equations, and partial differential equations.

Method of Laplace Transforms

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.
(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
(c) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

In step (a) we are tacitly assuming the solution is piecewise continuous on $[0, \infty)$ and of exponential order. Once we have obtained the inverse Laplace transform in step (c), we can verify that these tacit assumptions are satisfied.

Example 1 Solve the initial value problem

$$y'' - 2y' + 5y = -8e^{-t} \ ; \qquad y(0) = 2 \ , \qquad y'(0) = 12 \ . \tag{1}$$

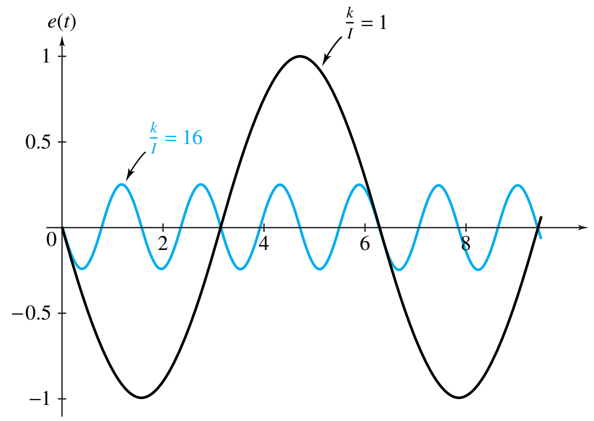

Figure 7.7 Error for automatic pilot when $k/I = 1$ and when $k/I = 16$

small by making k large relative to I, but then the term $\sqrt{k/I}$ becomes large, causing the error to oscillate more rapidly. (See Figure 7.7.) As with vibrations, the oscillations or oversteering can be controlled by introducing a damping torque proportional to $e'(t)$ but opposite in sign (see Problem 40).

7.5 EXERCISES

In Problems 1–14, solve the given initial value problem using the method of Laplace transforms.

1. $y'' - 2y' + 5y = 0$; $y(0) = 2$, $y'(0) = 4$

2. $y'' - y' - 2y = 0$; $y(0) = -2$, $y'(0) = 5$

3. $y'' + 6y' + 9y = 0$; $y(0) = -1$, $y'(0) = 6$

4. $y'' + 6y' + 5y = 12e^t$; $y(0) = -1$, $y'(0) = 7$

5. $w'' + w = t^2 + 2$; $w(0) = 1$, $w'(0) = -1$

6. $y'' - 4y' + 5y = 4e^{3t}$; $y(0) = 2$, $y'(0) = 7$

7. $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5$, $y'(0) = -4$

8. $y'' + 4y = 4t^2 - 4t + 10$; $y(0) = 0$, $y'(0) = 3$

9. $z'' + 5z' - 6z = 21e^{t-1}$; $z(1) = -1$, $z'(1) = 9$

10. $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0$, $y'(0) = 5$

11. $y'' - y = t - 2$; $y(2) = 3$, $y'(2) = 0$

12. $w'' - 2w' + w = 6t - 2$; $w(-1) = 3$; $w'(-1) = 7$

13. $y'' - y' - 2y = -8\cos t - 2\sin t$; $y(\pi/2) = 1$, $y'(\pi/2) = 0$

14. $y'' + y = t$; $y(\pi) = 0$, $y'(\pi) = 0$

In Problems 15–24, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.

15. $y'' - 3y' + 2y = \cos t$; $y(0) = 0$, $y'(0) = -1$

16. $y'' + 6y = t^2 - 1$; $y(0) = 0$, $y'(0) = -1$

17. $y'' + y' - y = t^3$; $y(0) = 1$, $y'(0) = 0$

18. $y'' - 2y' - y = e^{2t} - e^t$; $y(0) = 1$, $y'(0) = 3$

19. $y'' + 5y' - y = e^t - 1$; $y(0) = 1$, $y'(0) = 1$

20. $y'' + 3y = t^3$; $y(0) = 0$, $y'(0) = 0$

21. $y'' - 2y' + y = \cos t - \sin t$; $y(0) = 1$, $y'(0) = 3$

22. $y'' - 6y' + 5y = te^t$;
$y(0) = 2$, $y'(0) = -1$

23. $y'' + 4y = g(t)$; $y(0) = -1$; $y'(0) = 0$, where

$$g(t) = \begin{cases} t , & t < 2 , \\ 5 , & t > 2 \end{cases}$$

24. $y'' - y = g(t)$; $y(0) = 1$, $y'(0) = 2$, where

$$g(t) = \begin{cases} 1 , & t < 3 , \\ t , & t > 3 \end{cases}$$

In Problems 25–28, solve the given third-order initial value problem for $y(t)$ using the method of Laplace transforms.

25. $y''' - y'' + y' - y = 0$;
$y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$

26. $y''' + 4y'' + y' - 6y = -12$;
$y(0) = 1$, $y'(0) = 4$, $y''(0) = -2$

27. $y''' + 3y'' + 3y' + y = 0$;
$y(0) = -4$, $y'(0) = 4$, $y''(0) = -2$

28. $y''' + y'' + 3y' - 5y = 16e^{-t}$;
$y(0) = 0$, $y'(0) = 2$, $y''(0) = -4$

In Problems 29–32, use the method of Laplace transforms to find a general solution to the given differential equation by assuming $y(0) = a$ and $y'(0) = b$, where a and b are arbitrary constants.

29. $y'' - 4y' + 3y = 0$

30. $y'' + 6y' + 5y = t$

31. $y'' + 2y' + 2y = 5$

32. $y'' - 5y' + 6y = -6te^{2t}$

33. Use Theorem 6 in Section 7.3 to show that

$$\mathscr{L}\{t^2y'(t)\}(s) = sY''(s) + 2Y'(s) ,$$

where $Y(s) = \mathscr{L}\{y\}(s)$.

34. Use Theorem 6 in Section 7.3 to show that

$$\mathscr{L}\{t^2y''(t)\}(s) = s^2Y''(s) + 4sY'(s) + 2Y(s) ,$$

where $Y(s) = \mathscr{L}\{y\}(s)$.

In Problems 35–38, find solutions to the given initial value problem.

35. $y'' + 3ty' - 6y = 1$;
$y(0) = 0$, $y'(0) = 0$

36. $ty'' - ty' + y = 2$;
$y(0) = 2$, $y'(0) = -1$

37. $ty'' - 2y' + ty = 0$;
$y(0) = 1$, $y'(0) = 0$
[*Hint:* $\mathscr{L}^{-1}\{1/(s^2 + 1)^2\}(t) = (\sin t - t\cos t)/2$.]

38. $y'' + ty' - y = 0$;
$y(0) = 0$, $y'(0) = 3$

39. Determine the error $e(t)$ for the automatic pilot in Example 5 if the shaft is initially at rest in the zero direction and the desired direction is $g(t) = a$, where a is a constant.

40. In Example 5 assume that in order to control oscillations a component of torque proportional to $e'(t)$, but opposite in sign, is also fed back to the steering shaft. Show that equation (17) is now replaced by

$$Iy''(t) = -ke(t) - \mu e'(t) ,$$

where μ is a positive constant. Determine the error $e(t)$ for the automatic pilot with mild damping (i.e., $\mu < 2\sqrt{Ik}$) if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = a$, where a is a constant.

41. In Problem 40 determine the error $e(t)$ when the desired direction is given by $g(t) = at$, where a is a constant.

7.6 TRANSFORMS OF DISCONTINUOUS AND PERIODIC FUNCTIONS

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. Jump discontinuities occur naturally in physical problems such as electric circuits with on/off switches. To handle such behavior, Oliver Heaviside introduced the following step function.

When t is a positive integer, say $t = n$, then the recursive relation (19) can be repeatedly applied to obtain

$$\Gamma(n + 1) = n\Gamma(n) = n(n - 1)\Gamma(n - 1) = \cdots$$
$$= n(n - 1)(n - 2)\cdots 2\Gamma(1) .$$

It follows from the definition (18) that $\Gamma(1) = 1$, so we find

$$\Gamma(n + 1) = n! .$$

Thus, the gamma function extends the notion of factorial!

As an application of the gamma function, let's return to the problem of determining the Laplace transform of an arbitrary power of t. We will verify that the formula

(20) $$\mathcal{L}\{t^r\}(s) = \frac{\Gamma(r + 1)}{s^{r+1}}$$

holds for every constant $r > -1$.

By definition,

$$\mathcal{L}\{t^r\}(s) = \int_0^\infty e^{-st}t^r dt .$$

Let's make the substitution $u = st$. Then $du = s\,dt$, and we find

$$\mathcal{L}\{t^r\}(s) = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^r\left(\frac{1}{s}\right)du$$
$$= \frac{1}{s^{r+1}}\int_0^\infty e^{-u}u^r du = \frac{\Gamma(r + 1)}{s^{r+1}} .$$

Notice that when $r = n$ is a nonnegative integer, then $\Gamma(n + 1) = n!$, and so formula (20) reduces to the familiar formula for $\mathcal{L}\{t^n\}$.

7.6 EXERCISES

In Problems 1–4, sketch the graph of the given function and determine its Laplace transform.

1. $(t - 1)^2u(t - 1)$

2. $u(t - 1) - u(t - 4)$

3. $t^2u(t - 2)$

4. $tu(t - 1)$

In Problems 5–10, express the given function using window and step functions and compute its Laplace transform.

5. $g(t) = \begin{cases} 0 , & 0 < t < 1 , \\ 2 , & 1 < t < 2 , \\ 1 , & 2 < t < 3 , \\ 3 , & 3 < t \end{cases}$

6. $g(t) = \begin{cases} 0 , & 0 < t < 2 , \\ t + 1 , & 2 < t \end{cases}$

7.

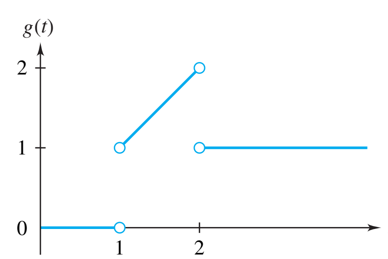

Figure 7.16 Function in Problem 7

8.

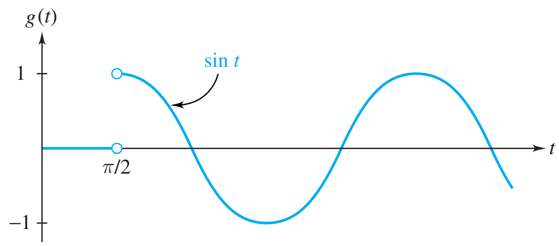

Figure 7.17 Function in Problem 8

9.

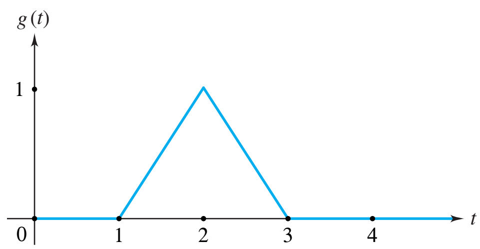

Figure 7.18 Function in Problem 9

10.

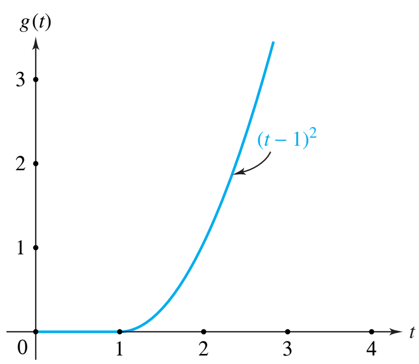

Figure 7.19 Function in Problem 10

In Problems 11–18, determine an inverse Laplace transform of the given function.

11. $\dfrac{e^{-2s}}{s-1}$ **12.** $\dfrac{e^{-3s}}{s^2}$

13. $\dfrac{e^{-2s}-3e^{-4s}}{s+2}$ **14.** $\dfrac{e^{-3s}}{s^2+9}$

15. $\dfrac{se^{-3s}}{s^2+4s+5}$ **16.** $\dfrac{e^{-s}}{s^2+4}$

17. $\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}$ **18.** $\dfrac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}$

19. The current $I(t)$ in an *RLC* series circuit is governed by the initial value problem

$$I''(t) + 2I'(t) + 2I(t) = g(t) ;$$
$$I(0) = 10 , \qquad I'(0) = 0 ,$$

where

$$g(t) := \begin{cases} 20 , & 0 < t < 3\pi , \\ 0 , & 3\pi < t < 4\pi , \\ 20 , & 4\pi < t . \end{cases}$$

Determine the current as a function of time t. Sketch $I(t)$ for $0 < t < 8\pi$.

20. The current $I(t)$ in an *LC* series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t) ;$$
$$I(0) = 1 , \qquad I'(0) = 3 ,$$

where

$$g(t) := \begin{cases} 3 \sin t , & 0 \le t \le 2\pi , \\ 0 , & 2\pi < t . \end{cases}$$

Determine the current as a function of time t.

In Problems 21–24, determine $\mathcal{L}\{f\}$, where $f(t)$ is periodic with the given period. Also graph $f(t)$.

21. $f(t) = t , \quad 0 < t < 2 ,$ and $f(t)$ has period 2.

22. $f(t) = e^t , \quad 0 < t < 1 ,$ and $f(t)$ has period 1.

23. $f(t) = \begin{cases} e^{-t} , & 0 < t < 1 , \\ 1 , & 1 < t < 2 , \end{cases}$

and $f(t)$ has period 2.

24. $f(t) = \begin{cases} t , & 0 < t < 1 , \\ 1 - t , & 1 < t < 2 , \end{cases}$

and $f(t)$ has period 2.

In Problems 25–28, determine $\mathcal{L}\{f\}$, where the periodic function is described by its graph.

25.

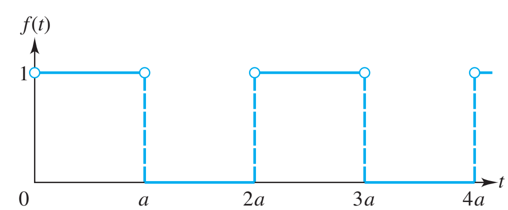

Figure 7.20 Square wave

26.

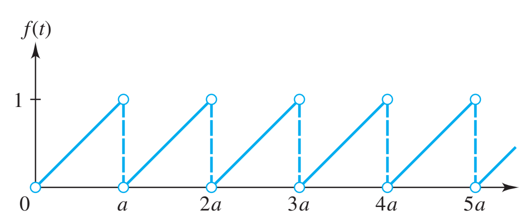

Figure 7.21 Sawtooth wave

27.

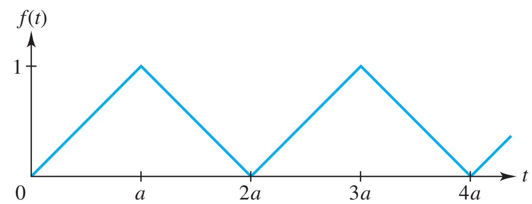

Figure 7.22 Triangular wave

28.

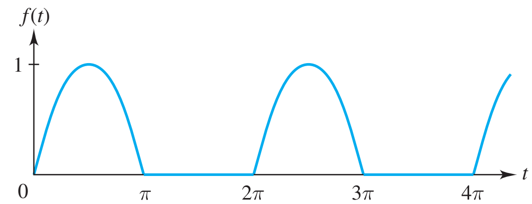

Figure 7.23 Half-rectified sine wave

In Problems 29–32, solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.

29. $y'' + y = u(t-3)$;
$y(0) = 0$, $y'(0) = 1$

30. $w'' + w = u(t-2) - u(t-4)$;
$w(0) = 1$, $w'(0) = 0$

31. $y'' + y = t - (t-4)u(t-2)$;
$y(0) = 0$, $y'(0) = 1$

32. $y'' + y = 3\sin 2t - 3(\sin 2t)u(t-2\pi)$;
$y(0) = 1$, $y'(0) = -2$

In Problems 33–40, solve the given initial value problem using the method of Laplace transforms.

33. $y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi)$;
$y(0) = 1$, $y'(0) = 1$

34. $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)$;
$y(0) = 0$, $y'(0) = 0$

35. $z'' + 3z' + 2z = e^{-3t}u(t-2)$;
$z(0) = 2$, $z'(0) = -3$

36. $y'' + 5y' + 6y = tu(t-2)$;
$y(0) = 0$, $y'(0) = 1$

37. $y'' + 4y = g(t)$; $y(0) = 1$, $y'(0) = 3$,

where $g(t) = \begin{cases} \sin t , & 0 \le t \le 2\pi , \\ 0 , & 2\pi < t \end{cases}$

38. $y'' + 2y' + 10y = g(t)$;
$y(0) = -1$, $y'(0) = 0$,

where $g(t) = \begin{cases} 10 , & 0 \le t \le 10 , \\ 20 , & 10 < t < 20 , \\ 0 , & 20 < t \end{cases}$

39. $y'' + 5y' + 6y = g(t)$;
$y(0) = 0$, $y'(0) = 2$,

where $g(t) = \begin{cases} 0 , & 0 \le t < 1 , \\ t , & 1 < t < 5 , \\ 1 , & 5 < t \end{cases}$

40. $y'' + 3y' + 2y = g(t)$;
$y(0) = 2$, $y'(0) = -1$,

where $g(t) = \begin{cases} e^{-t} , & 0 \le t < 3 , \\ 1 , & 3 < t \end{cases}$

41. Show that if $\mathcal{L}\{g\}(s) = [(s+\alpha)(1-e^{-Ts})]^{-1}$, where $T > 0$ is fixed, then

$$g(t) = e^{-\alpha t} + e^{-\alpha(t-T)}u(t-T) + e^{-\alpha(t-2T)}u(t-2T) + e^{-\alpha(t-3T)}u(t-3T) + \cdots . \tag{21}$$

[*Hint:* Use the fact that $1 + x + x^2 + \cdots = 1/(1-x)$.]

42. The function $g(t)$ in (21) can be expressed in a more convenient form as follows:

(a) Show that for each $n = 0, 1, 2, \ldots$,

$$g(t) = e^{-\alpha t}\left[\frac{e^{(n+1)\alpha T} - 1}{e^{\alpha T} - 1}\right]$$

for $nT < t < (n+1)T$.

[*Hint:* Use the fact that $1 + x + x^2 + \cdots + x^n = (x^{n+1} - 1)/(x-1)$.]

(b) Let $v = t - (n+1)T$. Show that when $nT < t < (n+1)T$, then $-T < v < 0$ and

$$g(t) = \frac{e^{-\alpha v}}{e^{\alpha T} - 1} - \frac{e^{-\alpha t}}{e^{\alpha T} - 1} . \tag{22}$$

(c) Use the facts that the first term in (22) is periodic with period T and the second term is independent of n to sketch the graph of $g(t)$ in (22) for $\alpha = 1$ and $T = 2$.

43. Show that if $\mathcal{L}\{g\}(s) = \beta[(s^2 + \beta^2)(1 - e^{-Ts})]^{-1}$, then

$$g(t) = \sin \beta t + [\sin \beta(t-T)]u(t-T) + [\sin \beta(t-2T)]u(t-2T) + [\sin \beta(t-3T)]u(t-3T) + \cdots .$$

44. Use the result of Problem 43 to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(1-e^{-\pi s})}\right\}(t) = g(t) ,$$

where $g(t)$ is periodic with period 2π and

$$g(t) := \begin{cases} \sin t , & 0 \le t \le \pi , \\ 0 , & \pi \le t \le 2\pi . \end{cases}$$

In Problems 45 and 46, use the method of Laplace transforms and the results of Problems 41 and 42 to solve the initial value problem.

$$y'' + 3y' + 2y = f(t) \; ; \quad y(0) = 0 \; , \quad y'(0) = 0 \; ,$$

where $f(t)$ is the periodic function defined in the stated problem.

45. Problem 22 **46.** Problem 25 with $a = 1$

In Problems 47–50, find a Taylor series for $f(t)$ about $t = 0$. Assuming the Laplace transform of $f(t)$ can be computed term by term, find an expansion for $\mathscr{L}\{f\}(s)$ in powers of $1/s$. If possible, sum the series.

47. $f(t) = e^t$ **48.** $f(t) = \sin t$

49. $f(t) = \dfrac{1 - \cos t}{t}$ **50.** $f(t) = e^{-t^2}$

51. Using the recursive relation (19) and the fact that $\Gamma(1/2) = \sqrt{\pi}$, determine
(a) $\mathscr{L}\{t^{-1/2}\}$. **(b)** $\mathscr{L}\{t^{7/2}\}$.

52. Use the recursive relation (19) and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to show that

$$\mathscr{L}^{-1}\left\{s^{-(n+1/2)}\right\}(t) = \frac{2^n t^{n-1/2}}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}} \; ,$$

where n is a positive integer.

53. Verify (15) in Theorem 9 for the function $f(t) = \sin t$, taking the period as 2π. Repeat, taking the period as 4π.

54. By replacing s by $1/s$ in the Maclaurin series expansion for arctan s, show that

$$\arctan \frac{1}{s} = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots \; .$$

55. Find an expansion for $e^{-1/s}$ in powers of $1/s$. Use the expansion for $e^{-1/s}$ to obtain an expansion for $s^{-1/2}e^{-1/s}$ in terms of $1/s^{n+1/2}$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\{s^{-1/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi t}} \cos 2\sqrt{t} \; .$$

[*Hint:* Use the result of Problem 52.]

56. Use the procedure discussed in Problem 55 to show that

$$\mathscr{L}^{-1}\{s^{-3/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi}} \sin 2\sqrt{t} \; .$$

57. Find an expansion for $\ln[1 + (1/s^2)]$ in powers of $1/s$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\left\{\ln\left(1 + \frac{1}{s^2}\right)\right\}(t) = \frac{2}{t}(1 - \cos t) \; .$$

58. The **unit triangular pulse** $\Lambda(t)$ is defined by

$$\Lambda(t) := \begin{cases} 0 \; , & t < 0 \; , \\ 2t \; , & 0 < t < 1/2 \; , \\ 2 - 2t \; , & 1/2 < t < 1 \; , \\ 0 \; , & t > 1 \; . \end{cases}$$

(a) Sketch the graph of $\Lambda(t)$. Why is it so named? Why is its symbol appropriate?

(b) Show that $\Lambda(t) = \int_{-\infty}^{t} 2\{\Pi_{0,1/2}(\tau) - \Pi_{1/2,1}(\tau)\}d\tau$.

(c) Find the Laplace transform of $\Lambda(t)$.

59. The mixing tank in Figure 7.24 initially holds 500 L of a brine solution with a salt concentration of 0.2 kg/L. For the first 10 min of operation, valve A is open, adding 12 L/min of brine containing a 0.4 kg/L salt concentration. After 10 min, valve B is switched in, adding a 0.6 kg/L concentration at 12 L/min. The exit valve C removes 12 L/min, thereby keeping the volume constant. Find the concentration of salt in the tank as a function of time.

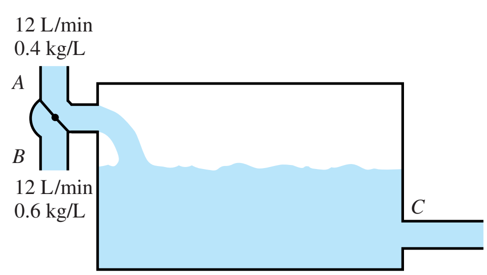

Figure 7.24 Mixing tank

60. Suppose in Problem 59 valve B is initially opened for 10 min and then valve A is switched in for 10 min. Finally, valve B is switched back in. Find the concentration of salt in the tank as a function of time.

61. Suppose valve C removes only 6 L/min in Problem 59. Can Laplace transforms be used to solve the problem? Discuss.

The inverse Laplace transform of $H(s)$ is the impulse response function

$$h(t) = \mathscr{L}^{-1}\{H\}(t) = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2 + 2^2}\right\}(t)$$
$$= \frac{1}{2}e^{-t}\sin 2t \ .$$

To solve the initial value problem, we need the solution to the corresponding homogeneous problem. The auxiliary equation for the homogeneous equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. Thus a general solution is $C_1e^{-t}\cos 2t + C_2e^{-t}\sin 2t$. Choosing C_1 and C_2 so that the initial conditions in (21) are satisfied, we obtain $y_k(t) = 2e^{-t}\cos 2t$.

Hence, a formula for the solution to the initial value problem (21) is

$$(h * g)(t) + y_k(t) = \frac{1}{2}\int_0^t e^{-(t-v)}\sin[2(t-v)]g(v)dv + 2e^{-t}\cos 2t \ . \ \blacklozenge$$

7.7 EXERCISES

In Problems 1–4, use the convolution theorem to obtain a formula for the solution to the given initial value problem, where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

1. $y'' - 2y' + y = g(t)$; $y(0) = -1$, $y'(0) = 1$

2. $y'' + 9y = g(t)$; $y(0) = 1$, $y'(0) = 0$

3. $y'' + 4y' + 5y = g(t)$; $y(0) = 1$, $y'(0) = 1$

4. $y'' + y = g(t)$; $y(0) = 0$, $y'(0) = 1$

In Problems 5–12, use the convolution theorem to find the inverse Laplace transform of the given function.

5. $\dfrac{1}{s(s^2+1)}$

6. $\dfrac{1}{(s+1)(s+2)}$

7. $\dfrac{14}{(s+2)(s-5)}$

8. $\dfrac{1}{(s^2+4)^2}$

9. $\dfrac{s}{(s^2+1)^2}$

10. $\dfrac{1}{s^3(s^2+1)}$

11. $\dfrac{s}{(s-1)(s+2)}$ $\left[\textit{Hint: } \dfrac{s}{s-1} = 1 + \dfrac{1}{s-1} \ .\right]$

12. $\dfrac{s+1}{(s^2+1)^2}$

13. Find the Laplace transform of

$$f(t) := \int_0^t (t-v)e^{3v}dv \ .$$

14. Find the Laplace transform of

$$f(t) := \int_0^t e^v \sin(t-v)\, dv \ .$$

In Problems 15–22, solve the given integral equation or integro-differential equation for $y(t)$.

15. $y(t) + 3\displaystyle\int_0^t y(v)\sin(t-v)dv = t$

16. $y(t) + \displaystyle\int_0^t e^{t-v}y(v)dv = \sin t$

17. $y(t) + \displaystyle\int_0^t (t-v)y(v)dv = 1$

18. $y(t) + \displaystyle\int_0^t (t-v)y(v)dv = t^2$

19. $y(t) + \displaystyle\int_0^t (t-v)^2y(v)dv = t^3 + 3$

20. $y'(t) + \displaystyle\int_0^t (t-v)y(v)dv = t$, $y(0) = 0$

21. $y'(t) + y(t) - \displaystyle\int_0^t y(v)\sin(t-v)dv = -\sin t$, $y(0) = 1$

22. $y'(t) - 2\displaystyle\int_0^t e^{t-v}y(v)dv = t$, $y(0) = 2$

In Problems 23–28, a linear system is governed by the given initial value problem. Find the transfer function

$H(s)$ for the system and the impulse response function $h(t)$ and give a formula for the solution to the initial value problem.

23. $y'' + 9y = g(t)$;
$y(0) = 2$, $y'(0) = -3$

24. $y'' - 9y = g(t)$; $y(0) = 2$, $y'(0) = 0$

25. $y'' - y' - 6y = g(t)$;
$y(0) = 1$, $y'(0) = 8$

26. $y'' + 2y' - 15y = g(t)$;
$y(0) = 0$, $y'(0) = 8$

27. $y'' - 2y' + 5y = g(t)$;
$y(0) = 0$, $y'(0) = 2$

28. $y'' - 4y' + 5y = g(t)$;
$y(0) = 0$, $y'(0) = 1$

In Problems 29 and 30, the current $I(t)$ in an RLC circuit with voltage source $E(t)$ is governed by the initial value problem

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = e(t) ,$$

$$I(0) = a , \qquad I'(0) = b ,$$

where $e(t) = E'(t)$ (see Figure 7.25). For the given constants R, L, C, a, and b, find a formula for the solution $I(t)$ in terms of $e(t)$.

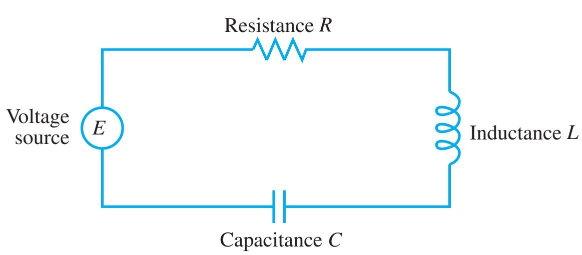

Figure 7.25 Schematic representation of an *RLC* series circuit

29. $R = 20\ \Omega$, $L = 5$ H, $C = 0.005$ F, $a = -1$ A, $b = 8$ A/sec.

30. $R = 80\ \Omega$, $L = 10$ H, $C = 1/410$ F, $a = 2$ A, $b = -8$ A/sec.

31. Use the convolution theorem and Laplace transforms to compute $1 * 1 * 1$.

32. Use the convolution theorem and Laplace transforms to compute $1 * t * t^2$.

33. Prove property (5) in Theorem 10.

34. Prove property (6) in Theorem 10.

35. Use the convolution theorem to show that

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(v)\,dv ,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

36. Using Theorem 5 in Section 7.3 and the convolution theorem, show that

$$\int_0^t \int_0^v f(z)\,dz\,dv = \mathscr{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}(t)$$

$$= t\int_0^t f(v)\,dv - \int_0^t vf(v)\,dv ,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

37. Prove directly that if $h(t)$ is the impulse response function characterized by equation (16), then for any continuous $g(t)$, we have $(h * g)(0) = (h * g)'(0) = 0$. [*Hint:* Use Leibniz's rule, described in Group Project E of Chapter 4.]

7.8 IMPULSES AND THE DIRAC DELTA FUNCTION

In mechanical systems, electrical circuits, bending of beams, and other applications, one encounters functions that have a very large value over a very short interval. For example, the strike of a hammer exerts a relatively large force over a relatively short time, and a heavy weight concentrated at a spot on a suspended beam exerts a large force over a very small section of the beam. To deal with violent forces of short duration, physicists and engineers use the delta function introduced by Paul A. M. Dirac. Relaxing our standards of rigor for the moment, we present the following somewhat informal definition.

where $Y(s)$ is the Laplace transform of the solution to (12) with zero initial conditions and $G(s)$ is the Laplace transform of $g(t)$. It is important to note that $H(s)$, and hence $h(t)$, does not depend on the choice of the function $g(t)$ in (12) [see equation (15) in Section 7.7]. However, it is useful to think of the impulse response function as the solution of the symbolic initial value problem

$$ay'' + by' + cy = \delta(t) \; ; \qquad y(0) = 0 \; , \qquad y'(0) = 0 \; . \tag{13}$$

Indeed, with $g(t) = \delta(t)$, we have $G(s) = 1$, and hence $H(s) = Y(s)$. Consequently $h(t) = y(t)$. So we see that the function $h(t)$ is the response to the impulse $\delta(t)$ for a mechanical system governed by the symbolic initial value problem (13).

7.8 EXERCISES

In Problems 1–6, evaluate the given integral.

1. $\int_{-\infty}^{\infty} (t^2 - 1)\delta(t)\,dt$
2. $\int_{-\infty}^{\infty} e^{3t}\delta(t)\,dt$
3. $\int_{-\infty}^{\infty} (\sin 3t)\delta\left(t - \frac{\pi}{2}\right)dt$
4. $\int_{-\infty}^{\infty} e^{-2t}\delta(t + 1)\,dt$
5. $\int_{0}^{\infty} e^{-2t}\delta(t - 1)\,dt$
6. $\int_{-1}^{1} (\cos 2t)\delta(t)\,dt$

In Problems 7–12, determine the Laplace transform of the given generalized function.

7. $\delta(t - 1) - \delta(t - 3)$
8. $3\delta(t - 1)$
9. $t\delta(t - 1)$
10. $t^3\delta(t - 3)$
11. $\delta(t - \pi)\sin t$
12. $e^t\delta(t - 3)$

In Problems 13–20, solve the given symbolic initial value problem.

13. $w'' + w = \delta(t - \pi)$; $w(0) = 0$, $w'(0) = 0$
14. $y'' + 2y' + 2y = \delta(t - \pi)$; $y(0) = 1$, $y'(0) = 1$
15. $y'' + 2y' - 3y = \delta(t - 1) - \delta(t - 2)$; $y(0) = 2$, $y'(0) = -2$
16. $y'' - 2y' - 3y = 2\delta(t - 1) - \delta(t - 3)$; $y(0) = 2$, $y'(0) = 2$
17. $y'' - y = 4\delta(t - 2) + t^2$; $y(0) = 0$, $y'(0) = 2$
18. $y'' - y' - 2y = 3\delta(t - 1) + e^t$; $y(0) = 0$, $y'(0) = 3$
19. $w'' + 6w' + 5w = e^t\delta(t - 1)$; $w(0) = 0$, $w'(0) = 4$
20. $y'' + 5y' + 6y = e^{-t}\delta(t - 2)$; $y(0) = 2$, $y'(0) = -5$

In Problems 21–24, solve the given symbolic initial value problem and sketch a graph of the solution.

21. $y'' + y = \delta(t - 2\pi)$; $y(0) = 0$, $y'(0) = 1$
22. $y'' + y = \delta(t - \pi/2)$; $y(0) = 0$, $y'(0) = 1$
23. $y'' + y = -\delta(t - \pi) + \delta(t - 2\pi)$; $y(0) = 0$, $y'(0) = 1$
24. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi)$; $y(0) = 0$, $y'(0) = 1$

In Problems 25–28, find the impulse response function $h(t)$ by using the fact that $h(t)$ is the solution to the symbolic initial value problem with $g(t) = \delta(t)$ and zero initial conditions.

25. $y'' + 4y' + 8y = g(t)$
26. $y'' - 6y' + 13y = g(t)$
27. $y'' - 2y' + 5y = g(t)$
28. $y'' - y = g(t)$

29. A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass–spring system and begins to vibrate. After $\pi/2$ sec, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\frac{d^2x}{dt^2} + 9x = -3\delta\left(t - \frac{\pi}{2}\right)$$
$$x(0) = 1 \ , \quad \frac{dx}{dt}(0) = 0 \ ,$$

where $x(t)$ denotes the displacement from equilibrium at time t. What happens to the mass after it is struck?

30. You have probably heard that soldiers are told not to march in cadence when crossing a bridge. By solving the symbolic initial value problem

$$y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi) \ ;$$
$$y(0) = 0 \ , \qquad y'(0) = 0 \ ,$$

explain why soldiers are so instructed. [*Hint:* See Section 4.10.]

31. A linear system is said to be **stable** if its impulse response function $h(t)$ remains bounded as $t \to +\infty$. If the linear system is governed by

$$ay'' + by' + cy = g(t) \ ,$$

where b and c are not both zero, show that the system is stable if and only if the real parts of the roots to

$$ar^2 + br + c = 0$$

are less than or equal to zero.

32. A linear system is said to be **asymptotically stable** if its impulse response function satisfies $h(t) \to 0$ as $t \to +\infty$. If the linear system is governed by

$$ay'' + by' + cy = g(t) \ ,$$

show that the system is asymptotically stable if and only if the real parts of the roots to

$$ar^2 + br + c = 0$$

are strictly less than zero.

33. The Dirac delta function may also be characterized by the properties

$$\delta(t) = \begin{cases} 0 \ , & t \neq 0 \ , \\ \text{“infinite,”} & t = 0 \ , \end{cases}$$

and
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \ .$$

Formally using the mean value theorem for definite integrals, verify that if $f(t)$ is continuous, then the above properties imply

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0) \ .$$

34. Formally using integration by parts, show that

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt = -f'(0) \ .$$

Also show that, in general,

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)\,dt = (-1)^n f^{(n)}(0) \ .$$

35. Figure 7.29 shows a beam of length 2λ that is imbedded in a support on the left side and free on the right. The vertical deflection of the beam a distance x from the support is denoted by $y(x)$. If the beam has a concentrated load L acting on it in the center of the beam, then the deflection must satisfy the symbolic boundary value problem

$$EIy^{(4)}(x) = L\delta(x - \lambda) \ ;$$
$$y(0) = y'(0) = y''(2\lambda) = y'''(2\lambda) = 0 \ ,$$

where E, the modulus of elasticity, and I, the moment of inertia, are constants. Find a formula for the displacement $y(x)$ in terms of the constants λ, L, E, and I. [*Hint:* Let $y''(0) = A$ and $y'''(0) = B$. First solve the fourth-order symbolic initial value problem and then use the conditions $y''(2\lambda) = y'''(2\lambda) = 0$ to determine A and B.]

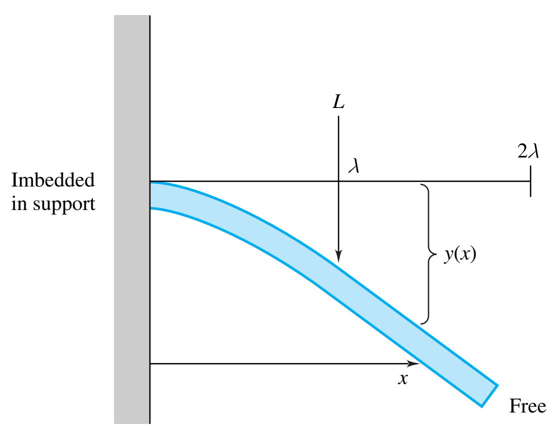

Figure 7.29 Beam imbedded in a support under a concentrated load at $x = \lambda$

To compute the inverse transform, we first write $X(s)$ in the partial fraction form

$$X(s) = \frac{3}{s+4} + \frac{1}{s-2} .$$

Hence, from the Laplace transform table on the inside back cover, we find that

$$x(t) = 3e^{-4t} + e^{2t} . \tag{4}$$

To determine $y(t)$, we could solve system (3) for $Y(s)$ and then compute its inverse Laplace transform. However, it is easier just to solve the first equation in system (1) for $y(t)$ in terms of $x(t)$. Thus,

$$y(t) = \frac{1}{2}x'(t) - 2t .$$

Substituting $x(t)$ from equation (4), we find that

$$y(t) = -6e^{-4t} + e^{2t} - 2t . \tag{5}$$

The solution to the initial value problem (1) consists of the pair of functions $x(t)$, $y(t)$ given by equations (4) and (5). ◆

7.9 EXERCISES

In Problems 1–19, use the method of Laplace transforms to solve the given initial value problem. Here x', y', etc., denotes differentiation with respect to t; so does the symbol D.

1. $x' = 3x - 2y$; $x(0) = 1$,
$y' = 3y - 2x$; $y(0) = 1$

2. $x' = x - y$; $x(0) = -1$,
$y' = 2x + 4y$; $y(0) = 0$

3. $z' + w' = z - w$; $z(0) = 1$,
$z' - w' = z - w$; $w(0) = 0$

4. $x' - 3x + 2y = \sin t$; $x(0) = 0$,
$4x - y' - y = \cos t$; $y(0) = 0$

5. $x' = y + \sin t$; $x(0) = 2$,
$y' = x + 2\cos t$; $y(0) = 0$

6. $x' - x - y = 1$; $x(0) = 0$,
$-x + y' - y = 0$; $y(0) = -5/2$

7. $(D - 4)[x] + 6y = 9e^{-3t}$; $x(0) = -9$,
$x - (D - 1)[y] = 5e^{-3t}$; $y(0) = 4$

8. $D[x] + y = 0$; $x(0) = 7/4$,
$4x + D[y] = 3$; $y(0) = 4$

9. $x'' + 2y' = -x$; $x(0) = 2$, $x'(0) = -7$,
$-3x'' + 2y'' = 3x - 4y$; $y(0) = 4$, $y'(0) = -9$

10. $x'' + y = 1$; $x(0) = 1$, $x'(0) = 1$,
$x + y'' = -1$; $y(0) = 1$, $y'(0) = -1$

11. $x' + y = 1 - u(t - 2)$; $x(0) = 0$,
$x + y' = 0$; $y(0) = 0$

12. $x' + y = x$; $x(0) = 0$, $y(0) = 1$,
$2x' + y'' = u(t - 3)$; $y'(0) = -1$

13. $x' - y' = (\sin t)u(t - \pi)$; $x(0) = 1$,
$x + y' = 0$; $y(0) = 1$

14. $x'' = y + u(t - 1)$; $x(0) = 1$, $x'(0) = 0$,
$y'' = x + 1 - u(t - 1)$; $y(0) = 0$,
$y'(0) = 0$

15. $x' - 2y = 2$; $x(1) = 1$,
$x' + x - y' = t^2 + 2t - 1$; $y(1) = 0$

16. $x' - 2x + y' = -(\cos t + 4\sin t)$; $x(\pi) = 0$,
$2x + y' + y = \sin t + 3\cos t$; $y(\pi) = 3$

17. $x' + x - y' = 2(t - 2)e^{t-2}$; $x(2) = 0$,
$x'' - x' - 2y = -e^{t-2}$; $x'(2) = 1$, $y(2) = 1$

18. $x' - 2y = 0$; $x(0) = 0$,
$x' - z' = 0$; $y(0) = 0$,
$x + y' - z = 3$; $z(0) = -2$

19. $x' = 3x + y - 2z$; $x(0) = -6$,
$y' = -x + 2y + z$; $y(0) = 2$,
$z' = 4x + y - 3z$; $z(0) = -12$

20. Use the method of Laplace transforms to solve
$x'' + y' = 2$; $x(0) = 3$, $x'(0) = 0$,
$4x + y' = 6$; $y(1) = 4$.
[*Hint:* Let $y(0) = c$ and then solve for c.]

21. For the interconnected tanks problem of Section 5.1, page 242, suppose that the input to tank A is now controlled by a valve which for the first 5 min delivers 6 L/min of pure water, but thereafter delivers 6 L/min of brine at a concentration of 2 kg/L. Assuming that all other data remain the same (see Figure 5.1, page 242), determine the mass of salt in each tank for $t > 0$ if $x_0 = 0$ and $y_0 = 4$.

22. Recompute the coupled mass–spring oscillator motion in Problem 1, Exercises 5.6 (page 289), using Laplace transforms.

In Problems 23 and 24, find a system of differential equations and initial conditions for the currents in the networks given by the schematic diagrams; the initial currents are all assumed to be zero. Solve for the currents in each branch of the network. (See Section 5.7 for a discussion of electrical networks.)

23.

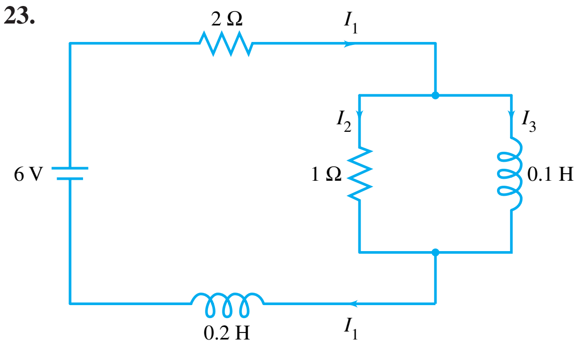

Figure 7.30 *RL* network for Problem 23

24.

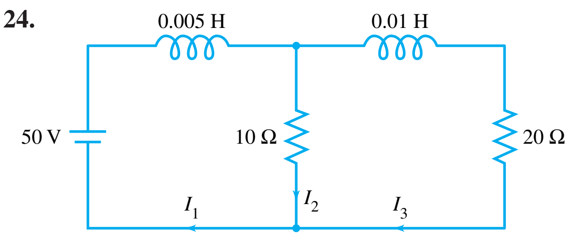

Figure 7.31 *RL* network for Problem 24

Chapter Summary

The use of the Laplace transform helps to simplify the process of solving initial value problems for certain differential and integral equations, especially when a forcing function with jump discontinuities is involved. The Laplace transform $\mathscr{L}\{f\}$ of a function $f(t)$ is defined by

$$\mathscr{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\,dt$$

for all values of s for which the improper integral exists. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α [that is, $|f(t)|$ grows no faster than a constant times $e^{\alpha t}$ as $t \to \infty$], then $\mathscr{L}\{f\}(s)$ exists for all $s > \alpha$.

The Laplace transform can be interpreted as an integral operator that maps a function $f(t)$ to a function $F(s)$. The transforms of commonly occurring functions appear in Table 7.1, page 359, method. For this purpose, the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$ is useful. The transform of a periodic forcing function $f(t)$ with period T is given by

$$\mathcal{L}\{f\}(s) = \frac{\int_0^T e^{-st}f(t)\,dt}{1 - e^{-sT}} .$$

The Dirac delta function $\delta(t)$ is useful in modeling a system that is excited by a large force applied over a short time interval. It is not a function in the usual sense but can be roughly interpreted as the derivative of a unit step function. The transform of $\delta(t-a)$ is

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as} , \qquad a \geq 0 .$$

REVIEW PROBLEMS

In Problems 1 and 2, use the definition of the Laplace transform to determine $\mathcal{L}\{f\}$.

1. $f(t) = \begin{cases} 3 , & 0 \leq t \leq 2 , \\ 6 - t , & 2 < t \end{cases}$

2. $f(t) = \begin{cases} e^{-t} , & 0 \leq t \leq 5 , \\ -1 , & 5 < t \end{cases}$

In Problems 3–10, determine the Laplace transform of the given function.

3. $t^2 e^{-9t}$

4. $e^{3t} \sin 4t$

5. $e^{2t} - t^3 + t^2 - \sin 5t$

6. $7e^{2t} \cos 3t - 2e^{7t} \sin 5t$

7. $t \cos 6t$

8. $(t+3)^2 - (e^t + 3)^2$

9. $t^2 u(t-4)$

10. $f(t) = \cos t, -\pi/2 \leq t \leq \pi/2$ and $f(t)$ has period π.

In Problems 11–17, determine the inverse Laplace transform of the given function.

11. $\dfrac{7}{(s+3)^3}$

12. $\dfrac{2s-1}{s^2 - 4s + 6}$

13. $\dfrac{4s^2 + 13s + 19}{(s-1)(s^2 + 4s + 13)}$

14. $\dfrac{s^2 + 16s + 9}{(s+1)(s+3)(s-2)}$

15. $\dfrac{2s^2 + 3s - 1}{(s+1)^2(s+2)}$

16. $\dfrac{1}{(s^2 + 9)^2}$

17. $\dfrac{e^{-2s}(4s + 2)}{(s-1)(s+2)}$

18. Find the Taylor series for $f(t) = e^{-t^2}$ about $t = 0$. Then, assuming that the Laplace transform of $f(t)$ can be computed term by term, find an expansion for $\mathcal{L}\{f\}(s)$ in powers of $1/s$.

In Problems 19–24, solve the given initial value problem for $y(t)$ using the method of Laplace transforms.

19. $y'' - 7y' + 10y = 0$;
$y(0) = 0 , \quad y'(0) = -3$

20. $y'' + 6y' + 9y = 0$;
$y(0) = -3 , \quad y'(0) = 10$

21. $y'' + 2y' + 2y = t^2 + 4t$;
$y(0) = 0 , \quad y'(0) = -1$

22. $y'' + 9y = 10e^{2t}$;
$y(0) = -1 , \quad y'(0) = 5$

23. $y'' + 3y' + 4y = u(t-1)$;
$y(0) = 0 , \quad y'(0) = 1$

24. $y'' - 4y' + 4y = t^2 e^t$;
$y(0) = 0 , \quad y'(0) = 0$

In Problems 25 and 26, find solutions to the given initial value problem.

25. $ty'' + 2(t-1)y' - 2y = 0$;
$y(0) = 0 , \quad y'(0) = 0$

26. $ty'' + 2(t-1)y' + (t-2)y = 0$;
$y(0) = 1 , \quad y'(0) = -1$

In Problems 27 and 28, solve the given equation for $y(t)$.

27. $y(t) + \int_0^t (t - v)y(v)\,dv = e^{-3t}$

28. $y'(t) - 2\int_0^t y(v)\sin(t - v)\,dv = 1$;

$y(0) = -1$

29. A linear system is governed by

$$y'' - 5y' + 6y = g(t) \ .$$

Find the transfer function and the impulse response function.

30. Solve the symbolic initial value problem

$$y'' + 4y = \delta\left(t - \frac{\pi}{2}\right) \ ;$$

$$y(0) = 0 \ , \qquad y'(0) = 1 \ .$$

In Problems 31 and 32, use Laplace transforms to solve the given system.

31. $x' + y = 0$; $\quad x(0) = 0$,
$x + y' = 1 - u(t - 2)$; $\quad y(0) = 0$

32. $x'' + 2y' = u(t - 3)$; $\quad x(0) = 1$, $\quad x'(0) = -1$,
$x + y' = y$; $\quad y(0) = 0$

TECHNICAL WRITING EXERCISES

1. Compare the use of Laplace transforms in solving linear differential equations with constant coefficients with the use of logarithms in solving algebraic equations of the form $x^r = a$.
2. Explain why the method of Laplace transforms works so well for linear differential equations with constant coefficients *and* integro-differential equations involving a convolution.
3. Discuss several examples of initial value problems in which the method of Laplace transforms cannot be applied.
4. A linear system is said to be **asymptotically stable** if its impulse response function $h(t) \to 0$ as $t \to +\infty$. Assume $H(s)$, the Laplace transform of $h(t)$, is a rational function in reduced form with the degree of its numerator less than the degree of its denominator. Explain in detail how the asymptotic stability of the linear system can be characterized in terms of the zeros of the denominator of $H(s)$. Give examples.
5. Compare and contrast the solution of initial value problems by Laplace transforms versus the methods of Chapter 4.