
Feedback-mediated oscillatory coexistence in the chemostat

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1 Introduction

We study the mathematical model of a chemostat:

$$\begin{aligned}\dot{S} &= D(S_0 - S) - \sum_{i=1}^n x_i f_i(S) / \gamma_i, \\ \dot{x}_i &= x_i (f_i(S) - D), \quad i = 1, \dots, n, \quad n \geq 2\end{aligned}$$

where S is the nutrient concentration and x_i are the concentrations of the competing species. The f_i are the growth functions of the species and they are assumed to be monotonically increasing. The yield constants, denoted by γ_i , reflect that only a fraction of the nutrient of what the different species consume, leads to new biomass.

The two natural control parameters are the input nutrient concentration S_0 and the dilution rate D . If these are constant, then at most one species survives as dictated by the *principle of competitive exclusion* [11, 5, 17, 16]. But in practice, it often happens that S_0 and/or D change over time, and a well-established literature exists in case these variations are periodic. For results in the case S_0 is periodic, see [15, 10, 14], and for periodic variations in D see [4, 16]. In both cases coexistence is possible under certain conditions, contrasting the competitive exclusion principle. The case when S_0 is a general time-dependent function has been studied in [8].

More recently a program of feedback control in the chemostat has been initiated in [6], where the dilution rate is treated as a feedback variable and made dependent on the concentrations of the competitors in the following simple way:

$$D(x) = \epsilon + \sum_{i=1}^n k_i x_i,$$

where ϵ and the k_i are non-negative design parameters. The feedback approach is perhaps most natural in the lab setting. For instance, optical sensors can be

used to measure turbidity, giving a rough estimate of the concentrations of the species. An alternative way to measure concentrations is to use GFP's (Green Fluorescence Proteins), especially since nowadays GFP's can emit light of different color, thus allowing to distinguish between different species. These concentration estimates can be processed by a computer to (online) calculate the dilution rate. The result then determines the speed of the pump -the device that is being actuated- which supplies the reactor with fresh medium.

The introduction of feedback controls can lead to significant changes in the asymptotic behavior of solutions when compared to the case in which the control parameters are fixed. These changes have been well documented for the case $n = 2$. For example, it was shown in [6] that the system may be rendered coexistent for suitable choices of the feedback parameters. This result was generalized in [9] to include chemostat models with non-monotone growth functions. In both cases, coexistence takes its simplest form: a globally asymptotically stable positive steady state. For other choices of the feedback parameters – but still assuming $n = 2$ – it was shown in [7] that bistability can occur.

In [7] it was shown that if there are three competitors, no interior solution is persistent and at least one species disappears.

Therefore we propose to investigate a different type of feedback law, and see whether or not coexistence is possible when there are more than two competitors. We will show that this is indeed the case when $n = 3$, using the following feedback law:

$$D(x) = \epsilon - \sum_{n=1}^2 k_i x_i,$$

for certain values of the non-negative parameters ϵ and k_i .

Now, coexistence will take the form of an interior periodic solution, bifurcating from a periodic solution on the boundary through what is traditionally referred to as a *transcritical bifurcation* of the associated Poincaré map. In general, it is not difficult to establish the existence of the transcritical bifurcation. It is considerably more difficult to determine the stability of the bifurcating periodic solution because the precise location of the boundary periodic solution is typically unknown. Therefore, we will resort to numerical simulations, which suggest that asymptotically stable periodic solutions may exist.

We will show along the way that in a chemostat with $n = 2$ species and controlled by a feedback law of the new type, asymptotically stable periodic solutions are possible (these will be the boundary periodic solutions mentioned in the previous paragraph). Perhaps this result is of interest by itself because this type of dynamical behavior does not occur in chemostats with 2 species which are controlled by the former feedback law, see [7].

Due to space limitations the proofs of all subsequent results are dropped.

2 Oscillations in the chemostat

2.1 Two species

Consider the following chemostat model:

$$\begin{aligned}\dot{S} &= D(x, y)(1 - S) - xf(S) - yg(S), \\ \dot{x} &= x(f(S) - D(x, y)), \\ \dot{y} &= y(g(S) - D(x, y)),\end{aligned}\tag{1}$$

where

$$(S, x, y) \in T := \{(S, x, y) \in \mathbb{R}_+^3 \mid S + x + y \leq 1\}.$$

We assume that the f and g are nonnegative C^3 functions on \mathbb{R}_+ with $f(0) = g(0) = 0$ and $f', g' > 0$. As before we assume the existence of $\lambda \in (0, 1)$ and $D^* > 0$ such that $f(\lambda) = g(\lambda) = D^*$, $f'(\lambda) < g'(\lambda)$ and $f(S) > g(S)$ for $S \in (0, \lambda)$ and $f(S) < g(S)$ for $S > \lambda$. Moreover we assume that $f''(S), g''(S) < 0$ and $f'''(S), g'''(S) > 0$ for all $S \geq 0$.

Notice that all these assumptions hold if f and g are of Michaelis-Menten form $mS/(a + S)$.

The function $D(x, y)$ is a positive C^3 function to be determined later.

The model is well-posed since for $V := S + x + y$, we have that $\dot{V} = D(x, y)(1 - V)$, and so T is forward invariant. Moreover, an obvious reduction argument suggests the study of the following two-dimensional system:

$$\begin{aligned}\dot{x} &= x(f(1 - x - y) - D(x, y)) \\ \dot{y} &= y(g(1 - x - y) - D(x, y))\end{aligned}\tag{2}$$

where $(x, y) \in \Delta = \{(x, y) \in \mathbb{R}_+^2 \mid x + y \leq 1\}$.

We will assume that $D(x, y)$ is affine:

$$D(x, y) = -k_1x - k_2y + \epsilon,\tag{3}$$

where k_1, k_2 and ϵ are non-negative parameters to be determined later.

An interior steady state of system (2)-(3) is a solution $(x^*, y^*) \in \text{int}(\Delta)$ of:

$$\begin{aligned}x^* + y^* &= 1 - \lambda \\ k_1x^* + k_2y^* &= \epsilon - D^*.\end{aligned}\tag{4}$$

Assuming that an interior steady state (x^*, y^*) exists, we evaluate the trace and determinant of the variational matrix $J(x^*, y^*)$:

$$\begin{aligned}\text{tr}(J(x^*, y^*)) &= -x^*(f'(\lambda) - k_1) - y^*(g'(\lambda) - k_2), \\ \det(J(x^*, y^*)) &= x^*y^*(f'(\lambda) - g'(\lambda))(k_1 - k_2)\end{aligned}$$

Notice that if we set $k_1 = \bar{k}_1$ and $k_2 = \bar{k}_2$ where

$$f'(\lambda) =: \bar{k}_1 < \bar{k}_2 := g'(\lambda) \quad (5)$$

in the feedback (3), then $\text{tr}(J(x^*, y^*)) = 0$ and $\det(J(x^*, y^*)) = x^* y^* (\bar{k}_2 - \bar{k}_1)^2 > 0$, and hence the occurrence of a Hopf bifurcation becomes plausible.

This suggests treating (k_1, k_2) as a bifurcation parameter, while using $\epsilon > 0$ to guarantee that

1. $(x^*, y^*) \in \text{int}(\Delta)$.
2. $D(x, y) > 0$ in Δ .

Explicitly solving (4) for (x^*, y^*) shows that

$$(x^*, y^*) = \frac{1}{\bar{k}_2 - \bar{k}_1} (\bar{k}_2(1 - \lambda) - (\epsilon - D^*), (\epsilon - D^*) - \bar{k}_1(1 - \lambda)), \quad (6)$$

and so the first condition is satisfied if

$$\epsilon \in (\bar{k}_1(1 - \lambda) + D^*, \bar{k}_2(1 - \lambda) + D^*), \quad (7)$$

a nonempty interval since $\bar{k}_1 < \bar{k}_2$.

The affine function $D(x, y) = -\bar{k}_1 x - \bar{k}_2 y + \epsilon$ reaches its minimum in Δ at the point $(0, 1)$, and so the second condition holds if

$$\epsilon > \bar{k}_2. \quad (8)$$

Notice that both constraints (7) and (8) for ϵ are compatible only if

$$\bar{k}_2 < \bar{k}_2(1 - \lambda) + D^*,$$

or equivalently, $\bar{k}_2 < D^*/\lambda$. The latter inequality is satisfied: by the mean value theorem there is some $c \in (0, \lambda)$ such that $g'(c) = D^*/\lambda$ and since g' is decreasing as $g'' < 0$, this implies that $\bar{k}_2 \equiv g'(\lambda) < g'(c)$.

Therefore, if we fix an ϵ in the nonempty interval

$$I := (\max(\bar{k}_1(1 - \lambda) + D^*, \bar{k}_2), \bar{k}_2(1 - \lambda) + D^*), \quad (9)$$

then there is some open neighborhood \mathcal{N} of (\bar{k}_1, \bar{k}_2) , such that for all $(k_1, k_2) \in \mathcal{N}$, $D(x, y) > 0$ in Δ , and system (2)-(3) has an interior steady state (x^*, y^*) (this is because both $D(x, y)$ and the point (x^*, y^*) depend continuously on (k_1, k_2)). Moreover, the interior steady state (x^*, y^*) undergoes a Hopf bifurcation at the bifurcation value (\bar{k}_1, \bar{k}_2) . To be more precise, a Hopf bifurcation occurs along any smooth parametric path $(k_1, k_2) = (k_1(\sigma), k_2(\sigma))$ such that $(\bar{k}_1, \bar{k}_2) = (k_1(0), k_2(0))$ and $x^* k_1'(0) + y^* k_2'(0) \neq 0$. The first condition ensures that the eigenvalues of $J(x^*, y^*)$ are purely imaginary at $\sigma = 0$. The second condition implies that this pair of complex conjugate eigenvalues of $J(x^*, y^*)$ crosses the imaginary axis transversally. Hence, a Hopf bifurcation occurs at $\sigma = 0$. In what follows, we will make use of one particular parametric path $(k_1, k_2) = (\bar{k}_1 + \sigma, \bar{k}_2)$ where the value of k_2 is fixed, and k_1 alone is treated as the bifurcation parameter. Such a path clearly satisfies the above conditions for the Hopf bifurcation. To determine the nature of the bifurcation (super- or subcritical), we followed the procedure outlined in [13].

This leads to the following result.

Theorem 1. Fix $\epsilon \in I$, and let \mathcal{N} be a (sufficiently small) open neighborhood of (\bar{k}_1, \bar{k}_2) . Assume that either $g''(\lambda) - f''(\lambda) \geq 0$ holds, or that $f(s)$ and $g(s)$ are of Michaelis-Menten type. Fixing $k_2 = \bar{k}_2$, a supercritical Hopf bifurcation occurs at the interior steady state of system (2)-(3) when k_1 passes through \bar{k}_1 . There exists a $\delta > 0$ such that for all $k_1 \in (\bar{k}_1, \bar{k}_1 + \delta)$, (2)-(3) has an asymptotically stable periodic solution having a Floquet multiplier in $(0, 1)$.

Since the set $\{(S, x, y) \in \mathbb{R}_+^3 \mid S + x + y = 1\}$ is forward invariant for (1)-(3), we immediately have:

Corollary 1. Under the conditions of Theorem 1, system (1)-(3) has an interior steady state in \mathcal{T} which undergoes a supercritical Hopf bifurcation when k_1 passes through \bar{k}_1 . There exists a $\delta > 0$ such that for all $k_1 \in (\bar{k}_1, \bar{k}_1 + \delta)$, (1)-(3) has an asymptotically stable periodic solution having two Floquet multipliers inside the unit circle.

2.2 Three species

Consider the following chemostat model:

$$\begin{aligned}\dot{S} &= D(x, y)(1 - S) - xf(S) - yg(S) - zh(S), \\ \dot{x} &= x(f(S) - D(x, y)), \\ \dot{y} &= y(g(S) - D(x, y)), \\ \dot{z} &= z(h(S) - D(x, y)),\end{aligned}\tag{10}$$

where

$$(S, x, y, z) \in \mathcal{T} := \{(S, x, y, z) \in \mathbb{R}_+^4 \mid S + x + y + z \leq 1\},$$

the functions f, g satisfy the assumptions made in the previous subsection and $D(x, y)$ is given by (3). We will choose the function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be of Michaelis-Menten form:

$$h(S) = \frac{mS}{a + S},\tag{11}$$

where the parameter $a > 0$ is fixed and m is treated as a bifurcation parameter. The set \mathcal{T} is clearly forward invariant under (10) with $D(x, y)$ given by (3) and $h(S)$ given by (11). Hence, the system is well-posed. In what follows, we use the symbol $\langle F \rangle := 1/T \int_0^T F(t) dt$ to denote the time average for any T -periodic function $F(t)$.

Theorem 2. Let the conditions of Corollary 1 hold and denote the asymptotically stable periodic solution of system (1)-(3) by $(1 - p_1(t) - p_2(t), p_1(t), p_2(t))$. Then for any $a > 0$, there exists a unique positive number m^* given by

$$m^* = \frac{\langle D(p_1(t), p_2(t)) \rangle}{\left\langle \frac{1 - p_1(t) - p_2(t)}{a + 1 - p_1(t) - p_2(t)} \right\rangle},$$

such that system (10) with $D(x, y)$ given by (3) and $h(s)$ given by (11) has a periodic solution in $\text{int}(\mathbb{R}_+^4) \cap \mathcal{T}$, arbitrarily close to the set $\{(S, x, y, z) \in \mathcal{T} \mid z = 0\}$, for all values of m which are sufficiently close to, and on one side of (either left or right) m^* .

3 Numerical results

In this section, we illustrate the analytical conclusions presented above with a numerical example of stable oscillatory coexistence of three species. The species labeled x , y , and z have the specific growth rates of Michaelis-Menten type, where

$$f(S) = \frac{2S}{0.5 + S}, \quad g(S) = \frac{3S}{1 + S}, \quad h(S) = \frac{mS}{0.1 + S},$$

where the maximal growth rate m of species z is treated as a bifurcation parameter.

The growth rates of species x and y have been chosen so that $\lambda = 1/2$ with $D^* = f(\lambda) = g(\lambda) = 1$. Furthermore, we consider the dilution rates of the form $D(x, y) = \epsilon - k_1x - \bar{k}_2y$, where $\bar{k}_2 = g'(\lambda) = 4/3$ is fixed and k_1 is used as a bifurcation parameter to induce a Hopf bifurcation in the (x, y) plane. For our particular choice of functions f and g and the dilution rate $D(x, y)$, the Hopf bifurcation occurs at the critical value $k_1 = \bar{k}_1 = f'(\lambda) = 1$. We chose the values $x^* = y^* = 1/4$ as the (x, y) coordinates of the bifurcating equilibrium, and set

$$\epsilon = D^* + x^*\bar{k}_1 + y^*\bar{k}_2 = \frac{19}{12}.$$

A direct calculation shows that the quantity

$$x^*Q^{xx} + y^*Q^{yy} = -\frac{367}{108}$$

is negative and hence the Hopf bifurcation is supercritical, see [13]. Specifically, a stable limit cycle in the (x, y) plane exists for all $k_1 > 1$ sufficiently close to 1.

The function $D(x, y)$ is affine in x and y and thus its range in the triangle Δ is determined by the values at the vertices. For the chosen values of ϵ and \bar{k}_2 , we find that

$$D(0, 0) = \epsilon > 0, \quad D(0, 1) = \epsilon - \bar{k}_2 = \frac{1}{4} > 0, \quad D(1, 0) = \epsilon - k_1,$$

hence $D(x, y)$ is positive for all $(x, y) \in \Delta$ as long as $0 < k_1 < \epsilon = 19/12$.

To compute a stable limit cycle in the (x, y) plane, we chose the value $k_1 = 1.075$ which clearly satisfies $1 < k_1 < \epsilon$. We performed a forward numerical

integration using Mathematica and computed a numerical orbit starting at a point close to the interior equilibrium (x^*, y^*) . This orbit converged to a stable limit cycle with estimated period $T = 159.86$ time units. Along this limit cycle, we found the numerical values for the following averages:

$$\left\langle \frac{1 - x(t) - y(t)}{1.1 - x(t) - y(t)} \right\rangle = 0.810, \quad \langle D(x(t), y(t)) \rangle = 0.947,$$

and found the bifurcation value $m^* = 1.169$ at which the transcritical bifurcation of limit cycles (see Theorem 2) occurs. For values $m = 1.169 + 0.001i$, $i = 1, \dots, 8$, we used forward numerical integration with initial condition (x_0, y_0, z_0) where (x_0, y_0) were chosen on the limit cycle in the (x, y) plane and $z_0 = 0.001$ was small. The results of these forward integrations are shown in Figure 1. These results suggest the existence of a family of stable periodic solutions that exist for $m > m^*$.

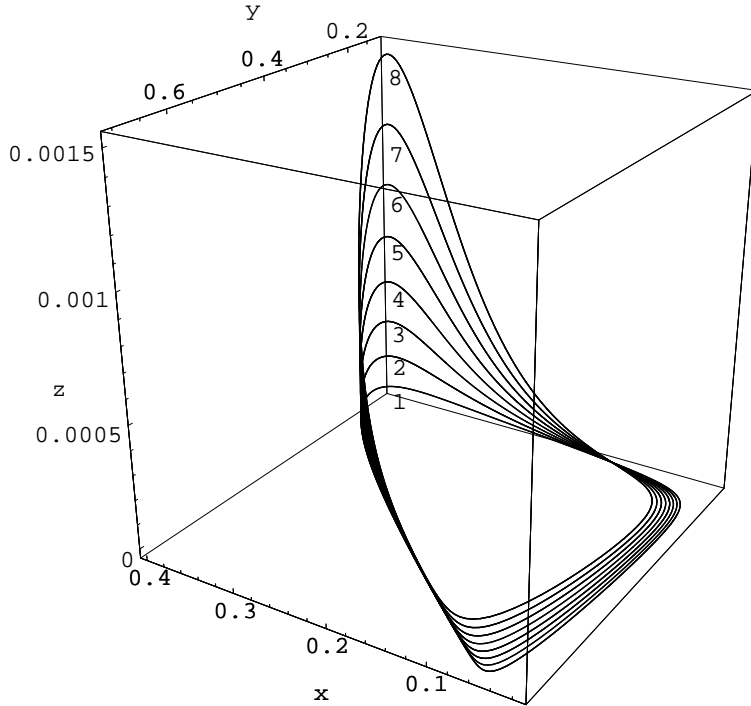


Fig. 1. Numerically computed stable periodic solutions with all three components (x, y, z) strictly positive. These periodic orbits are numbered according to the value of $i = 1, \dots, 8$ where $m = 1.169 + 0.001i$.

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References

1. J. ARINO, S. S. PILYUGIN, AND G.S.K. WOLKOWICZ, *Considerations on yield, nutrient uptake, cellular growth, and competition in chemostat models*, Can. Appl. Math. Quart. 11 (2003), pp. 107–142.
2. G. J. BUTLER AND P. WALTMAN, *Bifurcation from a limit cycle in a two-predator-one prey ecosystem modeled on a chemostat*, J. Math. Biol. 12 (1981), pp. 295–310.
3. G. J. BUTLER, S. B. HSU, AND P. WALTMAN, *Coexistence of competing predators in a chemostat*, J. Math. Biol. 17 (1983), pp. 133–151.
4. G. J. BUTLER, S. B. HSU, AND P. WALTMAN, *A mathematical model of the chemostat with periodic washout rate*, SIAM J. Appl. Math. 45 (1985), pp. 435–449.
5. G. J. BUTLER AND G.S.K. WOLKOWICZ, *A mathematical model of the chemostat with general class of functions describing nutrient uptake*, SIAM J. Appl. Math. 45 (1985), pp. 138–151.
6. P. DE LEENHEER AND H. L. SMITH, *Feedback control for chemostat models*, J. Math. Biol. 46 (2003), pp. 48–70.
7. P. DE LEENHEER, B. LI, AND H.L. SMITH, *Competition in the chemostat: some remarks*, Can. Appl. Math. Quart. 11 (2003), pp. 229–248.
8. S.F. ELLERMEYER, S.S. PILYUGIN, AND R. REDHEFFER, *Persistence criteria for a chemostat with variable nutrient input*, J. Differential Equations 171 (2001), pp. 132–147.
9. J.-L. GOUZÉ AND G. ROBLEDO, *Feedback control for nonmonotone competition models in the chemostat*, Nonlinear Analysis: Real World Applications 6 (2005), pp. 671–690.
10. J. K. HALE AND A.S. SOMOLINOS, *Competition for fluctuating nutrient*, J. Math. Biol. 18 (1983), pp. 255–280.
11. S. B. HSU, *Limiting behavior of competing species*, SIAM J. Appl. Math. 34 (1978), pp. 760–763.
12. S. S. PILYUGIN AND P. WALTMAN, *Multiple limit cycles in the chemostat with variable yield*, Math. Biosci. 182 (2003), pp. 151–166.
13. S. S. PILYUGIN AND P. WALTMAN, *Divergence criterion for generic planar systems*, SIAM J. Appl. Math. 64 (2003), pp. 81–93.
14. N.S. RAO AND E. O. ROXIN, *Controlled growth of competing species*, SIAM J. Appl. Math. 50 (1990), pp. 853–864.
15. H.L. SMITH, *Competitive coexistence in an oscillating chemostat*, SIAM J. Appl. Math. 40 (1981), pp. 498–522.
16. H.L. SMITH AND P. WALTMAN, *The Theory of the Chemostat*, Cambridge University Press, Cambridge, 1995.
17. G.S.K. WOLKOWICZ AND Z. LU, *Global dynamics of a mathematical model of competition in the chemostat: general response functions and differential death rates*, SIAM J. Appl. Math. 48 (1992), pp. 222–233.