to the solution \( y(x) \). Now for nonlinear equations such as (4), the factor \( f^{(n+1)}(\xi) \) in the Lagrange error formula may grow too rapidly with \( n \), and the convergence can be thwarted. But if the differential equation is linear and its coefficients and nonhomogeneous term enjoy a feature known as analyticity, our wish is granted; the error does indeed diminish to zero as the degree \( n \) goes to infinity, and the sequence of Taylor polynomials can be guaranteed to converge to the actual solution on a certain (known) interval. For instance, the exponential, sine, and cosine functions in Example 1 all satisfy linear differential equations with constant coefficients, and their Taylor series converge to the corresponding function values. (Indeed, all their derivatives are bounded on any interval of finite length, so the Lagrange formula for their approximation errors approaches zero as \( n \) increases, for each value of \( x \).) This topic is the theme for the early sections of this chapter.

In Sections 8.5 and 8.6 we’ll see that solutions to second-order linear equations can exhibit very wild behavior near points \( x_0 \) where the coefficient of \( y'' \) is zero; so wild, in fact, that no Euler or Runge–Kutta algorithm could hope to keep up with them. But a clever modification of the Taylor polynomial method, due to Frobenius, provides very accurate approximations to the solutions in such regions. It is this latter feature, perhaps, that underscores the value of the Taylor methodology in the current practice of applied mathematics.

### 8.1 Exercises

In Problems 1–8, determine the first three nonzero terms in the Taylor polynomial approximations for the given initial value problem.

1. \( y' = x^2 + y^2 \); \( y(0) = 1 \)
2. \( y' = y^2 \); \( y(0) = 2 \)
3. \( y' = \sin y + e^x \); \( y(0) = 0 \)
4. \( y'' = \sin(x + y) \); \( y(0) = 0 \)
5. \( x^3 + tx = 0 \); \( x(0) = 1 \), \( x'(0) = 0 \)
6. \( y'' + y = 0 \); \( y(0) = 0 \), \( y'(0) = 1 \)
7. \( y''(\theta) + y(\theta)^3 = \sin \theta \); \( y(0) = 0 \), \( y'(0) = 0 \)
8. \( y'' + \sin y = 0 \); \( y(0) = 1 \), \( y'(0) = 0 \)

9. (a) Construct the Taylor polynomial \( p_3(x) \) of degree 3 for the function \( f(x) = \ln x \) around \( x = 1 \).
(b) Using the error formula (6), show that
\[
|\ln(1.5) - p_3(1.5)| \leq \left(\frac{0.5}{4}\right)^3 = 0.015625
\]
(c) Compare the estimate in part (b) with the actual error by calculating \( |\ln(1.5) - p_3(1.5)| \).
(d) Sketch the graphs of \( \ln x \) and \( p_3(x) \) (on the same axes) for \( 0 < x < 2 \).

10. (a) Construct the Taylor polynomial \( p_3(x) \) of degree 3 for the function \( f(x) = 1/(2 - x) \) around \( x = 0 \).

(b) Using the error formula (6), show that
\[
\left| f\left(\frac{1}{2}\right) - p_3\left(\frac{1}{2}\right) \right| = \left| 2 - p_3\left(\frac{1}{2}\right) \right| \leq \frac{2}{3^6}
\]
(c) Compare the estimate in part (b) with the actual error
\[
\left| 2 - p_3\left(\frac{1}{2}\right) \right|
\]
(d) Sketch the graphs of \( 1/(2 - x) \) and \( p_3(x) \) (on the same axes) for \(-2 < x < 2\).

11. Argue that if \( y = \phi(x) \) is a solution to the differential equation \( y'' + p(x)y' + q(x)y = g(x) \) on the interval \((a, b)\), where \( p, q, \) and \( g \) are each twice-differentiable, then the fourth derivative of \( \phi(x) \) exists on \((a, b)\).

12. Argue that if \( y = \phi(x) \) is a solution to the differential equation \( y'' + p(x)y' + q(x)y = g(x) \) on the interval \((a, b)\), where \( p, q, \) and \( g \) possess derivatives of all orders, then \( \phi \) has derivatives of all orders on \((a, b)\).

13. **Duffing’s Equation.** In the study of a nonlinear spring with periodic forcing, the following equation arises:
\[
y'' + ky + ry^3 = A \cos \omega t
\]
Let \( k = r = A = 1 \) and \( \omega = 10 \). Find the first three nonzero terms in the Taylor polynomial approximations to the solution with initial values \( y(0) = 0 \), \( y'(0) = 1 \).
14. **Soft versus Hard Springs.** For Duffing’s equation given in Problem 13, the behavior of the solutions changes as \( r \) changes sign. When \( r > 0 \), the restoring force \( ky + ry^3 \) becomes stronger than for the linear spring \((r = 0)\). Such a spring is called hard. When \( r < 0 \), the restoring force becomes weaker than the linear spring and the spring is called soft. Pendulums act like soft springs.

(a) Redo Problem 13 with \( r = -1 \). Notice that for the initial conditions \( y(0) = 0, y'(0) = 1 \), the soft and hard springs appear to respond in the same way for \( t \) small.

(b) Keeping \( k = A = 1 \) and \( \omega = 10 \), change the initial conditions to \( y(0) = 1 \) and \( y'(0) = 0 \). Now redo Problem 13 with \( r = \pm 1 \).

(c) Based on the results of part (b), is there a difference between the behavior of soft and hard springs for \( t \) small? Describe.

15. The solution to the initial value problem

\[
xy''(x) + 2y'(x) + xy(x) = 0 ;
\]

\[
y(0) = 1, \quad y'(0) = 0
\]

has derivatives of all orders at \( x = 0 \) (although this is far from obvious). Use L’Hôpital’s rule to compute the Taylor polynomial of degree 2 approximating this solution.

16. van der Pol Equation. In the study of the vacuum tube, the following equation is encountered:

\[
y'' + (0.1)(y^2 - 1)y' + y = 0
\]

Find the Taylor polynomial of degree 4 approximating the solution with the initial values \( y(0) = 1 \), \( y'(0) = 0 \).

---

### 8.2 Power Series and Analytic Functions

The differential equations studied in earlier sections often possessed solutions \( y(x) \) that could be written in terms of elementary functions such as polynomials, exponentials, sines, and cosines. However, many important equations arise whose solutions cannot be so expressed. In the previous chapters, when we encountered such an equation we either settled for expressing the solution as an integral (see Exercises 2.2, Problem 27, page 43) or as a numerical approximation (Sections 3.6, 3.7, and 5.3). However, the Taylor polynomial approximation scheme of the preceding section suggests another possibility. Suppose the differential equation (and initial conditions) permit the computation of every derivative \( y^{(n)} \) at the expansion point \( x_0 \). Are there any conditions that would guarantee that the sequence of Taylor polynomials would converge to the solution \( y(x) \) as the degree of the polynomials tends to infinity:

\[
\lim_{n \to \infty} \sum_{j=0}^{n} \frac{y^{(j)}(x_0)}{j!}(x - x_0)^j = \sum_{j=0}^{\infty} \frac{y^{(j)}(x_0)}{j!}(x - x_0)^j = y(x) ?
\]

In other words, when can we be sure that a solution to a differential equation is represented by its Taylor series? As we’ll see, the answer is quite favorable, and it enables a powerful new technique for solving equations.

The most efficient way to begin an exploration of this issue is by investigating the algebraic and convergence properties of generic expressions that include Taylor series—“long polynomials” so to speak, or more conventionally, power series.

#### Power Series

A **power series** about the point \( x_0 \) is an expression of the form

\[
\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots,
\]
By the reasoning in the previous paragraph, the derivatives of \( f \) have convergent power series representations

\[
f'(x) = \sum_{n=0}^{\infty} na_n(x - x_0)^{n-1} = 0 + \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1},
\]

\[
f''(x) = \sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2} = 0 + \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2},
\]

\[
\vdots
\]

\[
f^{(j)}(x) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-[j-1])a_n(x - x_0)^{n-j},
\]

\[
= \sum_{n=j}^{\infty} n(n-1)\cdots(n-[j-1])a_n(x - x_0)^{n-j}.
\]

But if we evaluate these series at \( x = x_0 \), we learn that

\[
f(x_0) = a_0,
\]

\[
f'(x_0) = 1 \cdot a_1,
\]

\[
f''(x_0) = 2 \cdot 1 \cdot a_2,
\]

\[
\vdots
\]

\[
f^{(j)}(x_0) = j! \cdot a_j,
\]

\[
\vdots
\]

that is, \( a_j = f^{(j)}(x_0)/j! \) and the power series must coincide with the Taylor series†

\[
\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j
\]

about \( x_0 \). Any power series—regardless of how it is derived—that converges in some neighborhood of \( x_0 \) to a function has to be the Taylor series of that function. For example, the expansion for \( \arctan x \) given in (9) of Example 2 must be its Taylor expansion.

With these facts in mind we are ready to turn to the study of the effectiveness of power series techniques for solving differential equations. In the next sections, you will find it helpful to keep in mind that if \( f \) and \( g \) are analytic at \( x_0 \), then so are \( f + g, cf, fg \), and \( f/g \) if \( g(x_0) \neq 0 \). These facts follow from the algebraic properties of power series discussed earlier.

### 8.2 Exercises

**In Problems 1–6, determine the convergence set of the given power series.**

1. \( \sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n \)
2. \( \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n \)
3. \( \sum_{n=0}^{\infty} \frac{n^2}{2^n}(x + 2)^n \)
4. \( \sum_{n=1}^{\infty} \frac{4}{n^2 + 2n}(x - 3)^n \)
5. \( \sum_{n=1}^{\infty} \frac{3}{n^3}(x - 2)^n \)
6. \( \sum_{n=0}^{\infty} \frac{(n + 2)!}{n!}(x + 2)^n \)

†When the expansion point \( x_0 \) is zero, the Taylor series is also known as the Maclaurin series.
In Problems 9 and 10, find the power series expansion for the product
\[ f(x) = \sum_{n=0}^{\infty} a_n x^n, \]
\[ g(x) = \sum_{n=0}^{\infty} b_n x^n. \]

10. Determine the convergence set of the given power series.
(a) \[ \sum_{n=0}^{\infty} \frac{1}{n+1} x^n \]
(b) \[ \sum_{n=0}^{\infty} \frac{1}{2^n} x^n \]
(c) \[ \sum_{n=0}^{\infty} \frac{1}{n!} x^n \]
(d) \[ \sum_{n=0}^{\infty} \frac{1}{n^2} x^n \]
(e) \[ \sum_{n=0}^{\infty} \frac{1}{n^3} x^n \]
(f) \[ \sum_{n=0}^{\infty} \frac{1}{n^4} x^n \]

In Problems 11–14, find the first three nonzero terms in the power series expansion for the product \( f(x)g(x) \).

11. \( f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \)
\[ g(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \]

12. \( f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \)
\[ g(x) = \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \]

13. \( f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \)
\[ g(x) = (1 + x)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \]

14. \( f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \)
\[ g(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \]

15. Find the first few terms of the power series for the quotient
\[ q(x) = \left( \sum_{n=0}^{\infty} \frac{1}{2^n} x^n \right) \left/ \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) \right. \]
by completing the following:
(a) Let \( q(x) = \sum_{n=0}^{\infty} a_n x^n \), where the coefficients \( a_n \) are to be determined. Argue that \( \sum_{n=0}^{\infty} x^n/2^n \) is the Cauchy product of \( q(x) \) and \( \sum_{n=0}^{\infty} x^n/n! \).
(b) Use formula (6) of the Cauchy product to deduce the equations
\[ \frac{1}{2^0} = a_0, \quad \frac{1}{2^1} = a_0 + a_1, \]
\[ \frac{1}{2^2} = a_0 + a_1 + a_2, \]
\[ \frac{1}{2^3} = a_0 + a_1 + a_2 + a_3, \ldots. \]
(c) Solve the equations in part (b) to determine the constants \( a_0, a_1, a_2, a_3 \).

16. To find the first few terms in the power series for the quotient \( q(x) \) in Problem 15, treat the power series in the numerator and denominator as “long polynomials” and carry out long division. That is, perform
\[ 1 + x + \frac{1}{2} x^2 + \ldots \quad \middle/ \quad 1 + \frac{1}{2} x + \frac{1}{4} x^2 + \ldots. \]

In Problems 17–20, find a power series expansion for \( f(x) \), given the expansion for \( f(x) \).

17. \( f(x) = (1 + x)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \)

18. \( f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \)

19. \( f(x) = \sum_{k=0}^{\infty} a_k x^{2k} \)

20. \( f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \)
In Problems 21 and 22, find a power series expansion for \( g(x) := \int_0^x f(t) \, dt \), given the expansion for \( f(x) \).

21. \( f(x) = (1 + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n \)
22. \( f(x) = \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k} \)

In Problems 23–26, express the given power series as a series with generic term \( x^n \).

23. \( \sum_{n=1}^{\infty} a_n x^{n-1} \)
24. \( \sum_{n=2}^{\infty} n(n - 1)a_n x^{n+2} \)
25. \( \sum_{n=0}^{\infty} a_n x^{n+1} \)
26. \( \sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3} \)

27. Show that
\[ x^2 \sum_{n=0}^{\infty} n(n + 1)a_n x^n = \sum_{n=2}^{\infty} (n - 2)(n - 1)a_{n-2} x^n . \]

28. Show that
\[ 2 \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} nb_n x^{n-1} = b_1 + \sum_{n=1}^{\infty} [2a_{n-1} + (n + 1)b_n x^n] . \]

In Problems 29–34, determine the Taylor series about the point \( x_0 \) for the given functions and values of \( x_0 \).

29. \( f(x) = \cos x \), \( x_0 = \pi \)
30. \( f(x) = x^{-1} \), \( x_0 = 1 \)
31. \( f(x) = \frac{1 + x}{1 - x} \), \( x_0 = 0 \)
32. \( f(x) = \ln(1 + x) \), \( x_0 = 0 \)
33. \( f(x) = x^3 + 3x - 4 \), \( x_0 = 1 \)
34. \( f(x) = \sqrt{x} \), \( x_0 = 1 \)
35. The Taylor series for \( f(x) = \ln x \) about \( x_0 = 1 \) given in equation (13) can also be obtained as follows:
\( a \) Starting with the expansion \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \) and observing that
\[ \frac{1}{x} = \frac{1}{1 + (x - 1)} , \]

obtain the Taylor series for \( 1/x \) about \( x_0 = 1 \).
\( b \) Since \( \ln x = \int_1^x 1/t \, dt \), use the result of part (a) and termwise integration to obtain the Taylor series for \( f(x) = \ln x \) about \( x_0 = 1 \).
36. Let \( f(x) \) and \( g(x) \) be analytic at \( x_0 \). Determine whether the following statements are always true or sometimes false:
\( a \) \( 3f(x) + g(x) \) is analytic at \( x_0 \)
\( b \) \( f(x)/g(x) \) is analytic at \( x_0 \)
\( c \) \( f(x) \) is analytic at \( x_0 \)
\( d \) \[ f(x) \] is analytic at \( x_0 \)
37. Let
\[ f(x) = \begin{cases} e^{-1/x^2} , & x \neq 0 , \\ 0 , & x = 0 . \end{cases} \]

Show that \( f^{(n)}(0) = 0 \) for \( n = 0, 1, 2, \ldots \) and hence that the Maclaurin series for \( f(x) \) is \( 0 + 0 + 0 + \cdots \), which converges for all \( x \) but is equal to \( f(x) \) only when \( x = 0 \). This is an example of a function possessing derivatives of all orders (at \( x_0 = 0 \)), whose Taylor series converges, but the Taylor series (about \( x_0 = 0 \)) does not converge to the original function! Consequently, this function is not analytic at \( x = 0 \).
38. Compute the Taylor series for \( f(x) = \ln(1 + x^2) \) about \( x_0 = 0 \). [Hint: Multiply the series for \( (1 + x^2)^{-1} \) by \( 2x \) and integrate.]

8.3 POWER SERIES SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

In this section we demonstrate a method for obtaining a power series solution to a linear differential equation with polynomial coefficients. This method is easier to use than the Taylor series method discussed in Section 8.1 and sometimes gives a nice expression for the general term in the power series expansion. Knowing the form of the general term also allows us to test for the radius of convergence of the power series.
8.3 EXERCISES

In Problems 1–10, determine all the singular points of the given differential equation.
1. \((x + 1)y'' - x^2y' + 3y = 0\)
2. \(x^2y'' + 3y' - xy = 0\)
3. \((\theta^2 - 2)y'' + 2y' + (\sin \theta)y = 0\)
4. \((x^2 + x)y'' + 3y' - 6xy = 0\)
5. \((i^2 - t - 2)x^2 + (t + 1)x' - (t - 2)x = 0\)
6. \((x^2 - 1)y'' + (1 - x)y' + (x^2 - 2x + 1)y = 0\)
7. \((\sin x)y'' + (\cos x)y = 0\)
8. \(e^y'' - (x^2 - 1)y' + 2xy = 0\)
9. \((\sin \theta)y'' - (\ln \theta)y = 0\)
10. \(\ln(x - 1)y'' + (\sin 2x)y' - e^y = 0\)

In Problems 11–18, find at least the first four nonzero terms in a power series expansion about \(x = 0\) for a general solution to the given differential equation.
11. \(y' + (x + 2)y = 0\)
12. \(y' - y = 0\)
13. \(z'' - x^2z = 0\)
14. \((x^2 + 1)y'' + y = 0\)
15. \(y'' + (x - 1)y' + y = 0\)
16. \(y'' - 2y' + y = 0\)
17. \(w'' - x^2w' + w = 0\)
18. \((2x - 3)y'' - xy' + y = 0\)

In Problems 19–24, find a power series expansion about \(x = 0\) for a general solution to the given differential equation. Your answer should include a general formula for the coefficients.
19. \(y' - 2xy = 0\)
20. \(y'' + y = 0\)
21. \(y'' - xy' + 4y = 0\)
22. \(y'' - xy = 0\)
23. \(z'' - x^2z' - xz = 0\)
24. \((x^2 + 1)y'' - xy' + y = 0\)

In Problems 25–28, find at least the first four nonzero terms in a power series expansion about \(x = 0\) for the solution to the given initial value problem.
25. \(w'' + 3xw' - w = 0\) ; 
   \(w(0) = 2\), \(w'(0) = 0\)
26. \((x^2 - x + 1)y'' - y' - y = 0\) ; 
   \(y(0) = 0\), \(y'(0) = 1\)
27. \((x + 1)y'' - y = 0\) ; 
   \(y(0) = 0\), \(y'(0) = 1\)
28. \(y'' + (x - 2)y' - y = 0\) ; 
   \(y(0) = -1\), \(y'(0) = 0\)

In Problems 29–31, use the first few terms of the power series expansion to find a cubic polynomial approximation for the solution to the given initial value problem. Graph the linear, quadratic, and cubic polynomial approximations for \(-5 \leq x \leq 5\).
29. \(y'' + y' - xy = 0\) ; 
   \(y(0) = 1\), \(y'(0) = -2\)
30. \(y'' - 4xy' + 5y = 0\) ; 
   \(y(0) = -1\), \(y'(0) = 1\)
31. \((x^2 + 2)y'' + 2xy' + 3y = 0\) ; 
   \(y(0) = 1\), \(y'(0) = 2\)

32. Consider the initial value problem
   \(y'' - 2xy' - 2y = 0\) ; 
   \(y(0) = a_0\), \(y'(0) = a_1\).
   where \(a_0\) and \(a_1\) are constants.
   (a) Show that if \(a_0 = 0\), then the solution will be an odd function [that is, \(y(-x) = -y(x)\) for all \(x\)].
   What happens when \(a_1 = 0\)?
   (b) Show that if \(a_0\) and \(a_1\) are positive, then the solution is increasing on \((0, \infty)\).
   (c) Show that if \(a_0\) is negative and \(a_1\) is positive, then the solution is increasing on \((\infty, 0)\).
   (d) What conditions on \(a_0\) and \(a_1\) would guarantee that the solution is increasing on \((0, \infty)\)?

33. Use the ratio test to show that the radius of convergence of the series in equation (13) is infinite. [Hint: See Problem 7, Exercises 8.2, page 435.]

34. Emden’s Equation. A classical nonlinear equation that occurs in the study of the thermal behavior of a spherical cloud is Emden’s equation

   \(y'' + \frac{2}{x}y' + y^n = 0\)

   with initial conditions \(y(0) = 1\), \(y'(0) = 0\). Even though \(x = 0\) is not an ordinary point for this equation (which is nonlinear for \(n \neq 1\)), it turns out that there does exist a solution analytic at \(x = 0\). Assuming that \(n\) is a positive integer, show that the first few terms in a power series solution are

   \(y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\).

   [Hint: Substitute \(y = 1 + c_s x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots\) into the equation and carefully compute the first few terms in the expansion for \(y^n\).]
35. Variable Resistor. In Section 5.7, we showed that the charge \( q \) on the capacitor in a simple RLC circuit is governed by the equation

\[
Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t),
\]

where \( L \) is the inductance, \( R \) the resistance, \( C \) the capacitance, and \( E \) the voltage source. Since the resistance of a resistor increases with temperature, let’s assume that the resistor is heated so that the resistance at time \( t \) is \( R(t) = 1 + t/10 \) ohms (see Figure 8.5). If \( L = 0.1 \) H, \( C = 2 \) F, \( E(t) = 0 \), \( q(0) = 10 \) C, and \( q'(0) = 0 \) A, find at least the first four nonzero terms in a power series expansion about \( t = 0 \) for the charge on the capacitor.

36. Variable Spring Constant. As a spring is heated, its spring “constant” decreases. Suppose the spring is heated so that the spring “constant” at time \( t \) is \( k(t) = 6 - t \) N/m (see Figure 8.6). If the unforced mass–spring system has mass \( m = 2 \) kg and a damping constant \( b = 1 \) N-sec/m with initial conditions \( x(0) = 3 \) m and \( x'(0) = 0 \) m/sec, then the displacement \( x(t) \) is governed by the initial value problem

\[
2x''(t) + x'(t) + (6 - t)x(t) = 0; \quad x(0) = 3, \quad x'(0) = 0.
\]

Find at least the first four nonzero terms in a power series expansion about \( t = 0 \) for the displacement.

### 8.4 EQUATIONS WITH ANALYTIC COEFFICIENTS

In Section 8.3 we introduced a method for obtaining a power series solution about an ordinary point. In this section we continue the discussion of this procedure. We begin by stating a basic existence theorem for the equation

\[
y''(x) + p(x)y'(x) + q(x)y(x) = 0,
\]

which justifies the power series method.

#### Existence of Analytic Solutions

**Theorem 5.** Suppose \( x_0 \) is an ordinary point for equation (1). Then (1) has two linearly independent analytic solutions of the form

\[
y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n.
\]

Moreover, the radius of convergence of any power series solution of the form given by (2) is at least as large as the distance from \( x_0 \) to the nearest singular point (real or complex-valued) of equation (1).
Thus, the solution to the initial value problem in (10) is
\[ y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{11}{720}x^6 + \cdots . \]

Thus far we have used the power series method only for homogeneous equations. But the same method applies, with obvious modifications, to nonhomogeneous equations of the form
\[ y''(x) + p(x)y'(x) + q(x)y(x) = g(x) , \]
provided the forcing term \( g(x) \) and the coefficient functions are analytic at \( x_0 \). For example, to find a power series about \( x = 0 \) for a general solution to
\[ y''(x) - xy'(x) - y(x) = \sin x , \]
we use the substitution \( y(x) = \sum a_n x^n \) to obtain a power series expansion for the left-hand side of (16). We then equate the coefficients of this series with the corresponding coefficients of the Maclaurin expansion for \( \sin x \):
\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} . \]
Carrying out the details (see Problem 20), we ultimately find that an expansion for a general solution to (16) is
\[ y(x) = a_0 y_1(x) + a_1 y_2(x) + y_p(x) , \]
where
\[ y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots , \]
\[ y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots \]
are the solutions to the homogeneous equation associated with equation (16) and
\[ y_p(x) = \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{19}{5040}x^7 + \cdots \]
is a particular solution to equation (16).

### 8.4 Exercises

**In Problems 1–6, find a minimum value for the radius of convergence of a power series solution about \( x_0 \).**

1. \((x + 1)y'' - 3xy' + 2y = 0 \quad x_0 = 1\)
2. \(y'' - xy' - 3y = 0 \quad x_0 = 2\)
3. \((1 + x + x^2)y'' - 3y = 0 \quad x_0 = 1\)
4. \((x^2 - 5x + 6)y'' - 3xy' - y = 0 \quad x_0 = 0\)
5. \(y'' - (\tan x)y' + y = 0 \quad x_0 = 0\)
6. \((1 + x^3)y'' - xy' + 3x^2y = 0 \quad x_0 = 1\)

**In Problems 7–12, find at least the first four nonzero terms in a power series expansion about \( x_0 \) for a general solution to the given differential equation with the given value for \( x_0 \).**

7. \(y'' + 2(x - 1)y = 0 \quad x_0 = 1\)
8. \(y'' - 2xy = 0 \quad x_0 = -1\)
9. \((x^2 - 2x)y'' + 2y = 0 \quad x_0 = 1\)
10. \(x^2y'' - xy' + 2y = 0 \quad x_0 = 2\)
11. \(x^2y'' - y' + y = 0 \quad x_0 = 2\)
12. \(y'' + (3x - 1)y' - y = 0 \quad x_0 = -1\)

**In Problems 13–19, find at least the first four nonzero terms in a power series expansion of the solution to the given initial value problem.**

13. \(x' + (\sin t)x = 0 \quad x(0) = 1\)
14. \(y' - e^y = 0 \quad y(0) = 1\)
15. \((x^2 + 1)y'' - e^x y' + y = 0\)
   \(y(0) = 1\), \(y'(0) = 1\)

16. \(y'' + ty' + e^t y = 0\)
   \(y(0) = 0\), \(y'(0) = -1\)

17. \(y'' - (\sin x) y = 0\)
   \(y(\pi) = 1\), \(y'(\pi) = 0\)

18. \(y'' - (\cos x) y' - y = 0\)
   \(y(\pi/2) = 1\), \(y'(\pi/2) = 1\)

19. \(y'' - e^{2x} y' + (\cos x) y = 0\)
   \(y(0) = -1\), \(y'(0) = 1\)

20. To derive the general solution given by equations (17)–(20) for the nonhomogeneous equation (16), complete the following steps:
   (a) Substitute \(y(x) = \sum_{n=0}^{\infty} a_n x^n\) and the Maclaurin series for \(\sin x\) into equation (16) to obtain
   \[
   (2a_2 - a_0) + \sum_{k=1}^{\infty} \left[ (k + 2)(k + 1)a_{k+2} - (k + 1)a_k \right] x^k
   = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n + 1}.
   \]
   (b) Equate the coefficients of like powers of \(x\) on both sides of the equation in part (a) and thereby deduce the equations
   \[
   a_2 = \frac{a_0}{2}, \quad a_3 = \frac{1}{6} + \frac{a_1}{3}, \quad a_4 = \frac{a_0}{8},
   \quad a_5 = \frac{1}{40} + \frac{a_1}{15}, \quad a_6 = \frac{a_0}{48},
   \quad a_7 = \frac{19}{5040} + \frac{a_1}{105}.
   \]
   (c) Show that the relations in part (b) yield the general solution to (16) given in equations (17)–(20).

In Problems 21–28, use the procedure illustrated in Problem 20 to find at least the first four nonzero terms in a power series expansion about \(x = 0\) of a general solution to the given differential equation.

21. \(y' - xy = \sin x\)

22. \(w' + xw = e^x\)

23. \(z'' + xz' + z = x^2 + 2x + 1\)

24. \(y'' - 2xy' + 3y = x^2\)

25. \((1 + x^2)y'' - xy' + y = e^{-x}\)

26. \(y'' - xy' + 2y = \cos x\)

27. \((1 - x^2)y'' - y' + y = \tan x\)

28. \(y'' - (\sin x)y = \cos x\)

29. The equation
   \[
   (1 - x^2)y'' - 2xy' + n(n + 1)y = 0,
   \]
   where \(n\) is an unspecified parameter, is called Legendre’s equation. This equation occurs in applications of differential equations to engineering systems in spherical coordinates.
   (a) Find a power series expansion about \(x = 0\) for a solution to Legendre’s equation.
   (b) Show that for \(n\) a nonnegative integer, there exists an \(n\)th-degree polynomial that is a solution to Legendre’s equation. These polynomials, up to a constant multiple, are called Legendre polynomials.
   (c) Determine the first three Legendre polynomials (up to a constant multiple).

30. Aging Spring. As a spring ages, its spring “constant” decreases in value. One such model for a mass–spring system with an aging spring is
   \[
   mx''(t) + bx'(t) + ke^{-\eta t}x(t) = 0,
   \]
   where \(m\) is the mass, \(b\) the damping constant, \(k\) and \(\eta\) positive constants, and \(x(t)\) the displacement of the spring from its equilibrium position. Let \(m = 1\) kg, \(b = 2\) N·sec/m, \(k = 1\) N/m, and \(\eta = 1\) (sec)^{-1}. The system is set in motion by displacing the mass 1 m from its equilibrium position and then releasing it (\(x(0) = 1\), \(x'(0) = 0\)). Find at least the first four nonzero terms in a power series expansion about \(t = 0\) for the displacement.

31. Aging Spring without Damping. In the mass–spring system for an aging spring discussed in Problem 30, assume that there is no damping (i.e., \(b = 0\)), \(m = 1\), and \(k = 1\). To see the effect of aging, consider \(\eta\) as a positive parameter.
   (a) Redo Problem 30 with \(b = 0\) and \(\eta\) arbitrary but fixed.
   (b) Set \(\eta = 0\) in the expansion obtained in part (a). Does this expansion agree with the expansion for the solution to the problem with \(\eta = 0\)? [Hint: When \(\eta = 0\), the solution is \(x(t) = \cos t\).]
In Problems 1–10, use the substitution \( y = x^r \) to find a general solution to the given equation for \( x > 0 \).

1. \( x^2y''(x) + 6xy'(x) + 6y(x) = 0 \)
2. \( 2x^2y''(x) + 13xy'(x) + 15y(x) = 0 \)
3. \( x^2y''(x) - xy'(x) + 17y(x) = 0 \)
4. \( x^2y''(x) + 2xy'(x) - 3y(x) = 0 \)
5. \( \frac{d^2y}{dx^2} = \frac{5}{x} \frac{dy}{dx} - \frac{13}{x^2} y \)
6. \( \frac{d^2y}{dx^2} = \frac{1}{x} \frac{dy}{dx} - \frac{4}{x^2} y \)
7. \( x^3y'''(x) + 4x^2y''(x) + 10xy'(x) - 10y(x) = 0 \)
8. \( x^3y'''(x) + 4x^2y''(x) + xy'(x) = 0 \)
9. \( x^3y'''(x) + 3x^2y''(x) + 5xy'(x) - 5y(x) = 0 \)
10. \( x^3y'''(x) + 9x^2y''(x) + 19xy'(x) + 8y(x) = 0 \)

In Problems 11 and 12, use a substitution of the form \( y = (x - c)^r \) to find a general solution to the given equation for \( x > c \).

11. \( 2(x - 3)^2y''(x) + 5(x - 3)y'(x) - 2y(x) = 0 \)
12. \( 4(x + 2)^2y''(x) + 5y(x) = 0 \)
In Problems 13 and 14, use variation of parameters to find a general solution to the given equation for $x > 0$.

13. $x^2 y''(x) + 2 x y'(x) + 2 y(x) = x^{-1/2}$
14. $x^2 y''(x) + 2 x y'(x) - 2 y(x) = 6 x^{-2} + 3x$

In Problems 15–17, solve the given initial value problem.

15. $r^2 x^r(t) - 12x(t) = 0$ ;
   $x(1) = 3$, $x'(1) = 5$
16. $x^2 y''(x) + 5 x y'(x) + 4 y(x) = 0$ ;
   $y(1) = 3$, $y'(1) = 7$
17. $x^2 y'''(x) + 6 x^2 y''(x) + 29 x y'(x) - 29 y(x) = 0$ ;
   $y(1) = 2$, $y'(1) = -3$, $y''(1) = 19$

18. Suppose $r_0$ is a repeated root of the auxiliary equation $ar^2 + br + c = 0$. Then, as we well know, $y_1(t) = e^{r_0 t}$ is a solution to the equation $ay'' + by' + cy = 0$, where $a$, $b$, and $c$ are constants. Use a derivation similar to the one given in this section for the case when the indicial equation has a repeated root to show that a second linearly independent solution is $y_2(t) = te^{r_0 t}$.

19. Let $L[y](x) := x^3 y'''(x) + xy'(x) - y(x)$.
   (a) Show that $L[x^r](x) = (r - 1)^3 x^r$.
   (b) Using an extension of the argument given in this section for the case when the indicial equation has a double root, show that $L[y] = 0$ has the general solution
   $$y(x) = C_1 x + C_2 x \ln x + C_3 (\ln x)^2.$$

### 8.6 Method of Frobenius

In the previous section we showed that a homogeneous Cauchy–Euler equation has a solution of the form $y(x) = x^r$, $x > 0$, where $r$ is a certain constant. Cauchy–Euler equations have, of course, a very special form with only one singular point (at $x = 0$). In this section we show how the theory for Cauchy–Euler equations generalizes to other equations that have a special type of singularity.

To motivate the procedure, let’s rewrite the Cauchy–Euler equation,

(1) \[ ax^2 y''(x) + bxy'(x) + cy(x) = 0, \quad x > 0, \]

in the standard form

(2) \[ y''(x) + p(x) y'(x) + q(x) y(x) = 0, \quad x > 0, \]

where

\[ p(x) = \frac{p_0}{x}, \quad q(x) = \frac{q_0}{x^2}, \]

and $p_0$, $q_0$ are the constants $b/a$ and $c/a$, respectively. When we substitute $w(r, x) = x^r$ for $y$ into equation (2), we get

\[ [r(r - 1) + p_0 r + q_0] x^{r-2} = 0, \]

which yields the indicial equation

(3) \[ r(r - 1) + p_0 r + q_0 = 0. \]

Thus, if $r_1$ is a root of (3), then $w(r_1, x) = x^{r_1}$ is a solution to equations (1) and (2).

Let’s now assume, more generally, that (2) is an equation for which $x p(x)$ and $x^2 q(x)$, instead of being constants, are analytic functions. That is, in some open interval about $x = 0$,

(4) \[ xp(x) = p_0 + p_1 x + p_2 x^2 + \cdots = \sum_{n=0}^{\infty} p_n x^n, \]
(5) \[ x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \cdots = \sum_{n=0}^{\infty} q_n x^n. \]
into (54) ultimately gives

\[
(57) \quad [r(r-1) + 3r]a_0x^{r-1} + [(r+1)r + 3(r+1)]a_1x^r + \sum_{k=1}^{\infty} [(k + r + 1)(k + r + 3)a_{k+1} - a_{k-1}]x^{k+r} = 0 .
\]

Setting the coefficients equal to zero, we have

\[
(58) \quad [r(r-1) + 3r]a_0 = 0 ,
\]

\[
(59) \quad [(r+1)r + 3(r+1)]a_1 = 0 ,
\]

and, for \( k \geq 1 \), the recurrence relation

\[
(60) \quad (k + r + 1)(k + r + 3)a_{k+1} - a_{k-1} = 0 .
\]

With \( r = r_1 = 0 \), these equations lead to the following formulas: \( a_{2k+1} = 0, k = 0, 1, \ldots \), and

\[
(61) \quad a_{2k} = \frac{1}{[2 \cdot 4 \cdots (2k)][4 \cdot 6 \cdots (2k + 2)]}a_0 = \frac{1}{2^{2k}k!(k + 1)!}a_0 , \quad k \geq 0.
\]

Hence equation (54) has the power series solution

\[
(62) \quad w(0, x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^{2k}k!(k + 1)!}x^{2k} , \quad x > 0 . \quad \bullet
\]

Unlike in Example 5, if we work with the second root \( r = r_2 = -2 \) in Example 6, then we do not obtain a second linearly independent solution (see Problem 46).

In the preceding examples we were able to use the method of Frobenius to find a series solution valid to the right (\( x > 0 \)) of the regular singular point \( x = 0 \). For \( x < 0 \), we can use the change of variables \( x = -t \) and then solve the resulting equation for \( t > 0 \).

The method of Frobenius also applies to higher-order linear equations (see Problems 35–38).

### 8.6 Exercises

**In Problems 1–10, classify each singular point (real or complex) of the given equation as regular or irregular.**

1. \((x^2 - 1)y'' + xy' + 3y = 0\)
2. \(x^2y'' + 8xy' - 3xy = 0\)
3. \((x^2 + 1)z'' + 7x^2z' - 3xz = 0\)
4. \(x^2y'' - 5xy' + 7y = 0\)
5. \((x^2 - 1)^2y'' - (x - 1)y' + 3y = 0\)
6. \((x^2 - 4)y'' + (x + 2)y' + 3y = 0\)
7. \((i^2 - t - 2)^2x'' + (i^2 - 4)x' - tx = 0\)
8. \((x^2 - x)y'' + xy' + 7y = 0\)
9. \((x^2 + 2x - 8)^2y'' + (3x + 12)y' - x^2y = 0\)
10. \(x^3(x - 1)y'' + (x^2 - 3x)\sin(x)y' - xy = 0\)

**In Problems 11–18, find the indicial equation and the exponents for the specified singularity of the given differential equations.**

11. \(x^2y'' - 2xy' - 10y = 0\), at \( x = 0 \)
12. \(x^2y'' + 4xy' + 2y = 0\), at \( x = 0 \)
13. \((x^2 - x - 2)^2z'' + (x^2 - 4)z' - 6xz = 0\), at \( x = 2 \)
14. \((x^2 - 4)y'' + (x + 2)y' + 3y = 0\), at \( x = -2 \)
15. \(\theta^3y'' + \theta(\sin \theta)y' - (\tan \theta)y = 0\), at \( \theta = 0 \)
16. \((x^2 - 1)y'' - (x - 1)y' - 3y = 0\), at \( x = 1 \)
17. \((x - 1)^2y'' + (x^2 - 1)y' - 12y = 0\), at \( x = 1 \)
18. \(4x(\sin x)y'' - 3y = 0\), at \( x = 0 \)

**In Problems 19–24, use the method of Frobenius to find at least the first four nonzero terms in the series expansion about \( x = 0 \) for a solution to the given equation for \( x > 0 \).**

19. \(9x^2y'' + 9x^2y' + 2y = 0\)
20. \(2x(x - 1)y'' + 3(x - 1)y' - y = 0\)
21. \(x^2y'' + xy' + x^2y = 0\)
22. \(xy'' + y' - 4y = 0\)
23. \(x^2z'' + (x^2 + x)z' - z = 0\)
24. \(3xy'' + (2 - x)y' - y = 0\)

In Problems 25–30, use the method of Frobenius to find a general formula for the coefficient \(a_n\) in a series expansion about \(x = 0\) for a solution to the given equation for \(x > 0\).
25. \(4x^2y'' + 2x^2y' - (x + 3)y = 0\)
26. \(x^2y'' + (x^2 - x)y' + y = 0\)
27. \(xw'' - w' + xw = 0\)
28. \(3x^2y'' + 8xy' + (x - 2)y = 0\)
29. \(xy'' + (x - 1)y' - 2y = 0\)
30. \(x(x + 1)y'' + (x + 5)y' - 4y = 0\)

In Problems 31–34, first determine a recurrence formula for the coefficients in the (Frobenius) series expansion of the solution about \(x = 0\). Use this recurrence formula to determine if there exists a solution to the differential equation that is decreasing for \(x > 0\).
31. \(xy'' + (1 - x)y' - y = 0\)
32. \(x^2y'' - x(1 + x)y' + y = 0\)
33. \(3xy'' + 2(1 - x)y' - 4y = 0\)
34. \(xy'' + (x + 2)y' - y = 0\)

In Problems 35–38, use the method of Frobenius to find at least the first four nonzero terms in the series expansion about \(x = 0\) for a solution to the given linear third-order equation for \(x > 0\).
35. \(6x^3y''' + 13x^2y'' + (x + x^2)y' + xy = 0\)
36. \(6x^3y''' + 11x^2y'' - 2xy' - (x - 2)y = 0\)
37. \(6x^3y''' + 13x^2y'' - (x^2 + 3x)y' - xy = 0\)
38. \(6x^3y''' + (13x^2 - x^3)y'' + xy' - xy = 0\)

In Problems 39 and 40, try to use the method of Frobenius to find a series expansion about the irregular singular point \(x = 0\) for a solution to the given differential equation. If the method works, give at least the first four nonzero terms in the expansion. If the method does not work, explain what went wrong.
39. \(x^2y'' + (3x - 1)y' + y = 0\)
40. \(x^2y'' + y' - 2y = 0\)

In certain applications, it is desirable to have an expansion about the point at infinity. To obtain such an expansion, we use the change of variables \(z = 1/x\) and expand about \(z = 0\). In Problems 41 and 42, show that infinity is a regular singular point of the given differential equation by showing that \(z = 0\) is a regular singular point for the transformed equation in \(z\). Also find at least the first four nonzero terms in the series expansion about infinity of a solution to the original equation in \(x\).
41. \(x^2y''' - x^3y' - y = 0\)
42. \(18(x - 4)^2(x - 6)y'' + 9x(x - 4)y' - 32y = 0\)

43. Show that if \(r_1\) and \(r_2\) are roots of the indicial equation (16), with \(r_1\) the larger root (Re \(r_1 \geq \text{Re } r_2\)), then the coefficient of \(a_1\) in equation (19) is not zero when \(r = r_1\).

44. To obtain a second linearly independent solution to equation (20):
   (a) Substitute \(w(r, x)\) given in (21) into (20) and conclude that the coefficients \(a_k\), \(k \geq 1\), must satisfy the recurrence relation
   \[ (k + r - 1)(2k + 2r - 1)a_k + [(k + r - 1)(k + r - 2) + 1]a_{k-1} = 0 \]
   (b) Use the recurrence relation with \(r = 1/2\) to derive the second series solution
   \[ w\left(\frac{1}{2}, x\right) = a_0\left(x^{1/2} - \frac{3}{4}x^{3/2} + \frac{7}{32}x^{5/2} - \frac{133}{1920}x^{7/2} + \ldots\right) \]
   (c) Use the recurrence relation with \(r = 1\) to obtain \(w(1, x)\) in (28).
45. In Example 5, show that if we choose \(r = r_2 = -3\), then we obtain two linearly independent solutions to equation (45). [Hint: \(a_0\) and \(a_3\) are arbitrary constants.]
46. In Example 6, show that if we choose \(r = r_2 = -2\), then we obtain a solution that is a constant multiple of the solution given in (62). [Hint: Show that \(a_0\) and \(a_1\) must be zero while \(a_2\) is arbitrary.]
47. In applying the method of Frobenius, the following recurrence relation arose: \(a_{k+1} = 15^2a_k/(k + 1)^9\), \(k = 0, 1, 2, \ldots\).
   (a) Show that the coefficients are given by the formula \(a_k = 15^5a_0/(k!)^9\), \(k = 0, 1, 2, \ldots\).
   (b) Use the formula obtained in part (a) with \(a_0 = 1\) to compute \(a_5, a_{10}, a_{15}, a_{20}\), and \(a_{25}\) on your computer or calculator. What goes wrong?
   (c) Now use the recurrence relation to compute \(a_k\) for \(k = 1, 2, 3, \ldots, 25\), assuming \(a_0 = 1\).
   (d) What advantage does the recurrence relation have over the formula?
Substituting these values for the $b_n$’s back into (39), we obtain the solution

\[
y_2(x) = C \left\{ y_1(x) \ln x + 2x^{-2} - \frac{3}{32} x^2 - \frac{7}{1152} x^4 + \cdots \right\} \\
+ b_2 \left\{ 1 + \frac{1}{8} x^2 + \frac{1}{192} x^4 + \cdots \right\},
\]

where $C$ and $b_2$ are arbitrary constants. Since the factor multiplying $b_2$ is the first solution $y_1(x)$, we can obtain a second linearly independent solution by choosing $C = 1$ and $b_2 = 0$:

\[
y_2(x) = y_1(x) \ln x + 2x^{-2} - \frac{3}{32} x^2 - \frac{7}{1152} x^4 + \cdots.
\]

See Figure 8.12.

In closing we note that if the roots $r_1$ and $r_2$ of the indicial equation associated with a differential equation are complex, then they are complex conjugates. Thus, the difference $r_1 - r_2$ is imaginary and, hence, not an integer, and we are in case (a) of Theorem 7. However, rather than employing the display (11)–(12) for the linearly independent solutions, one usually takes the real and imaginary parts of (11); Problem 26 provides an elaboration of this situation.

### 8.7 Exercises

In Problems 1–14, find at least the first three nonzero terms in the series expansion about $x = 0$ for a general solution to the given equation for $x > 0$. (These are the same equations as in Problems 19–32 of Exercises 8.6.)

1. $9x^2y'' + 9x^2y' + 2y = 0$
2. $2x(x - 1)y'' + 3(x - 1)y' - y = 0$
3. $x^2y'' + xy' + x^2y = 0$
4. $xy'' + y' - 4y = 0$
5. $x^2z'' + (x^2 + x)z' - z = 0$
6. $3xy'' + (2 - x)y' - y = 0$
7. $4x^2y'' + 2x^2y' - (x + 3)y = 0$
8. $x^2y'' + (x^2 - x)y' + y = 0$
9. $xw'' - w' - xw = 0$
10. $3x^2y'' + 8xy' + (x - 2)y = 0$
11. \( xy'' + (x - 1)y' - 2y = 0 \)
12. \( x(x + 1)y'' + (x + 5)y' - 4y = 0 \)
13. \( xy'' + (1 - x)y' - y = 0 \)
14. \( x^2y'' - x(1 + x)y' + y = 0 \)

In Problems 15 and 16, determine whether the given equation has a solution that is bounded near the origin, all solutions are bounded near the origin, or none of the solutions are bounded near the origin. (These are the same equations as in Problems 33 and 34 of Exercises 8.6.) Note that you need to analyze only the indicial equation in order to answer the question.

15. \( 3xy'' + 2(1 - x)y' - 4y = 0 \)
16. \( xy'' + (x + 2)y' - y = 0 \)

In Problems 17–20, find at least the first three nonzero terms in the series expansion about \( x = 0 \) for a general solution to the given linear third-order equation for \( x > 0 \). (These are the same equations as in Problems 35–38 in Exercises 8.6.)

17. \( 6x^3y''' + 13x^2y'' + (x + x^2)y' + xy = 0 \)
18. \( 6x^3y''' + 11x^2y'' - 2xy' - (x - 2)y = 0 \)
19. \( 6x^3y''' + 13x^2y'' - (x^2 + 3x)y' - xy = 0 \)
20. \( 6x^3y''' + (13x^2 - x^3)y'' + xy' - xy = 0 \)

21. **Buckling Columns.** In the study of the buckling of a column whose cross section varies, one encounters the equation

\[
(45) \quad x^n y''(x) + \alpha^2 y(x) = 0, \quad x > 0,
\]

where \( x \) is related to the height above the ground and \( y \) is the deflection away from the vertical. The positive constant \( \alpha \) depends on the rigidity of the column, its moment of inertia at the top, and the load. The positive integer \( n \) depends on the type of column. For example, when the column is a truncated cone [see Figure 8.13(a)], we have \( n = 4 \).

(a) Use the substitution \( x = t^{-1} \) to reduce (45) with \( n = 4 \) to the form

\[
\frac{d^2y}{dt^2} + \frac{2}{t} \frac{dy}{dt} + \alpha^2 y = 0, \quad t > 0.
\]

(b) Find at least the first six nonzero terms in the series expansion about \( t = 0 \) for a general solution to the equation obtained in part (a).

(c) Use the result of part (b) to give an expansion about \( x = \infty \) for a general solution to (45).

22. In Problem 21 consider a column with a rectangular cross section with two sides constant and the other two changing linearly [see Figure 8.13(b)]. In this case, \( n = 1 \). Find at least the first four nonzero terms in the series expansion about \( x = 0 \) for a general solution to equation (45) when \( n = 1 \).

23. Use the method of Frobenius and a reduction of order procedure (see page 198) to find at least the first three nonzero terms in the series expansion about the irregular singular point \( x = 0 \) for a general solution to the differential equation

\[
x^2y'' + y' - 2y = 0.
\]

24. The equation

\[
xy''(x) + (1 - x)y'(x) + ny(x) = 0,
\]

where \( n \) is a nonnegative integer, is called Laguerre’s differential equation. Show that for each \( n \), this equation has a polynomial solution of degree \( n \). These polynomials are denoted by \( L_n(x) \) and are called Laguerre polynomials. The first few Laguerre polynomials are

\[
L_0(x) = 1, \quad L_1(x) = -x + 1, \quad L_2(x) = x^2 - 4x + 2.
\]

25. Use the results of Problem 24 to obtain the first few terms in a series expansion about \( x = 0 \) for a general solution for \( x > 0 \) to Laguerre’s differential equation for \( n = 0 \) and 1.

26. To obtain two linearly independent solutions to

\[
(46) \quad x^2y'' + (x + x^2)y' + y = 0, \quad x > 0,
\]

complete the following steps.

(a) Verify that (46) has a regular singular point at \( x = 0 \) and that the associated indicial equation has complex roots \( \pm i \).
As discussed in Section 8.5, we can express
\[ x^{\alpha + i\beta} = x^\alpha x^{i\beta} \]
\[ = x^\alpha \cos(\beta \ln x) + ix^\alpha \sin(\beta \ln x) . \]
Deduce from this formula that \( \frac{d}{dx} x^{\alpha + i\beta} = (\alpha + i\beta)x^{\alpha - 1 + i\beta} \).

Set \( y(x) = \sum_{n=0}^{\infty} a_n x^{\alpha + i\beta} \), where the coefficients now are complex constants, and substitute this series into equation (46) using the result of part (b).

Setting the coefficients of like powers equal to zero, derive the recurrence relation
\[ a_n = -\frac{n - 1 + i}{(n + i)^2 + 1} a_{n-1} , \quad \text{for } n \geq 1 . \]

Taking \( a_0 = 1 \), compute the coefficients \( a_1 \) and \( a_2 \) and thereby obtain the first few terms of a complex solution to (46).

By computing the real and imaginary parts of the solution obtained in part (e), derive the following linearly independent real solutions to (46):
\[ y_1(x) = \left[ \cos(\ln x) \right] \left\{ 1 - \frac{2}{5} x + \frac{1}{10} x^2 + \cdots \right\} \]
\[ + \left[ \sin(\ln x) \right] \left\{ \frac{1}{5} x - \frac{1}{20} x^2 + \cdots \right\} , \]
\[ y_2(x) = \left[ \cos(\ln x) \right] \left\{ -\frac{1}{5} x + \frac{1}{20} x^2 + \cdots \right\} \]
\[ + \left[ \sin(\ln x) \right] \left\{ 1 - \frac{2}{5} x + \frac{1}{10} x^2 + \cdots \right\} . \]

8.8 SPECIAL FUNCTIONS

In advanced work in applied mathematics, engineering, and physics, a few special second-order equations arise very frequently. These equations have been extensively studied, and volumes have been written on the properties of their solutions, which are often referred to as special functions.

For reference purposes we include a brief survey of three of these equations: the hypergeometric equation, Bessel’s equation, and Legendre’s equation. Bessel’s equation governs the radial dependence of the solutions to the classical partial differential equations of physics in spherical coordinate systems (see Section 10.7, page 625); Legendre’s equation governs their latitudinal dependence. C. F. Gauss formulated the hypergeometric equation as a generic equation whose solutions include the special functions of Legendre, Chebyshev, Gegenbauer, and Jacobi. For a more detailed study of special functions, we refer you to Basic Hypergeometric Series, 2nd ed., by G. Gasper and M. Rahman (Cambridge University Press, Cambridge, 2004); Special Functions, by E. D. Rainville (Chelsea, New York, 1971); and Higher Transcendental Functions, by A. Erdelyi (ed.) (McGraw-Hill, New York, 1953), 3 volumes.

Hypergeometric Equation

The linear second-order differential equation
\[ x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha \beta y = 0 , \]
where \( \alpha, \beta, \) and \( \gamma \) are fixed parameters, is called the hypergeometric equation. This equation has singular points at \( x = 0 \) and 1, both of which are regular. Thus a series expansion about \( x = 0 \) for a solution to (1) obtained by the method of Frobenius will converge at least for \( 0 < x < 1 \) (see Theorem 6, page 461). To find this expansion, observe that the indicial equation associated with \( x = 0 \) is
\[ r(r-1) + \gamma r = r [r - (1 - \gamma)] = 0 , \]
which has roots 0 and 1 - \( \gamma \). Let us assume that \( \gamma \) is not an integer and use the root \( r = 0 \) to obtain a solution to (1) of the form
\[ y_1(x) = \sum_{n=0}^{\infty} a_n x^n . \]
In Problems 1–4, express a general solution to the given equation using Gaussian hypergeometric functions.

1. \( x(1 - x)y'' + \left( \frac{1}{2} - 4x \right)y' - 2y = 0 \)
2. \( 3x(1 - x)y'' + (1 - 27x)y' - 45y = 0 \)
3. \( 2x(1 - x)y'' + (1 - 6x)y' - 2y = 0 \)
4. \( 2x(1 - x)y'' + (3 - 10x)y' - 6y = 0 \)

In Problems 5–8, verify the following formulas by expanding each function in a power series about \( x = 0 \).

5. \( F(1, 1; 2; x) = -x^{-1} \ln(1 - x) \)
6. \( F(\alpha; \beta; 1; x) = (1 - x)^{-\alpha} \)
7. \( F\left(\frac{1}{2}, 1; \frac{3}{2}; x^2 \right) = \frac{1}{2} x^{-1} \ln(1 + x) \)
8. \( F\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2 \right) = x^{-1} \arctan x \)

In Problems 9 and 10, use the method in Section 8.7 to obtain two linearly independent solutions to the given hypergeometric equation.

9. \( x(1 - x)y'' + (1 - 3x)y' - y = 0 \)
10. \( x(1 - x)y'' + (2 - 2x)y' - \frac{1}{4}y = 0 \)

11. Show that the confluent hypergeometric equation
    \( xy'' + (\gamma - x) y' - \alpha y = 0 \),
    where \( \alpha \) and \( \gamma \) are fixed parameters and \( \gamma \) is not an integer,
    has two linearly independent solutions
    \[ y_1(x) = F_1(\alpha; \gamma; x) := 1 + \sum_{n=1}^{\infty} \frac{(-\alpha)_n}{n!(\gamma)_n} x^n \]
    and
    \[ y_2(x) = x^{1-\gamma} F_1(\alpha + 1 - \gamma; 2 - \gamma; x) \].
12. Use the property of the gamma function given in (19) to derive relation (20).

In Problems 13–18, express a general solution to the given equation using Bessel functions of either the first or second kind.

13. \( 4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \)
14. \( 9x^2y'' + 9xy' + (9x^2 - 16)y = 0 \)
15. \( x^2y'' + xy' + (x^2 - 1)y = 0 \)
16. \( x^2y'' + xy' + x^2y = 0 \)
17. \( 9x^2y'' + 9xy' + (9x^2 - 4)x = 0 \)
18. \( x^2z'' + xz' + (x^2 - 16)z = 0 \)

In Problems 19 and 20, a Bessel equation is given. For the appropriate choice of \( \nu \), the Bessel function \( J_\nu(x) \) is one solution. Use the method in Section 8.7 to obtain a second linearly independent solution.

19. \( x^2y'' + xy' + (x^2 - 1)y = 0 \)
20. \( x^2y'' + xy' + (x^2 - 4)y = 0 \)

21. Show that \( x^\nu J_\nu(x) \) satisfies the equation
    \( xy'' + (1 - 2\nu)y' + xy = 0 \), \( x > 0 \),
    and use this result to find a solution for the equation
    \( xy'' - 2y' + xy = 0 \), \( x > 0 \).

In Problems 22 through 24, derive the indicated recurrence formulas.

22. Formula (32)  
23. Formula (33)
24. Formula (34)

25. Show that
    \[ J_{1/2}(x) = (2/\pi x)^{1/2} \sin x \quad \text{and} \quad J_{-1/2}(x) = (2/\pi x)^{1/2} \cos x \)

26. The Bessel functions of order \( \nu = n + 1/2, n \) any integer, are related to the spherical Bessel functions. Use relation (33) and the results of Problem 25 to show that such Bessel functions can be represented in terms of \( \sin x \), \( \cos x \), and powers of \( x \).

27. Use Theorem 7 in Section 8.7 to determine a second linearly independent solution to Bessel’s equation of order 0 in terms of the Bessel function \( J_0(x) \).

28. Show that between two consecutive positive roots (zeros) of \( J_1(x) \), there is a root of \( J_0(x) \). This interlacing property of the roots of Bessel functions is illustrated in Figure 8.14. \([\text{Hint: Use relation (31) and Rolle’s theorem from calculus.}]\)

29. Use formula (43) to determine the first five Legendre polynomials.

30. Show that the Legendre polynomials of even degree are even functions of \( x \), while those of odd degree are odd functions.

31. (a) Show that the orthogonality condition (44) for Legendre polynomials implies that
    \[ \int_{-1}^{1} P_n(x) q(x) dx = 0 \]
    for any polynomial \( q(x) \) of degree at most \( n - 1 \). \([\text{Hint: The polynomials } P_0, P_1, \ldots, P_{n-1} \text{ are}]\)
linearly independent and hence span the space of all polynomials of degree at most \( n - 1 \). Thus, \( q(x) = a_0 P_0(x) + \cdots + a_{n-1} P_{n-1}(x) \) for suitable constants \( a_k \).

(b) Prove that if \( Q_n(x) \) is a polynomial of degree \( n \) such that

\[
\int_{-1}^{1} Q_n(x) P_k(x) dx = 0
\]

for \( k = 0, 1, \ldots, n-1 \), then \( Q_n(x) = c P_n(x) \) for some constant \( c \).

**Hint:** Select \( c \) so that the coefficient of \( x^n \) for \( Q_n(x) - c P_n(x) \) is zero. Then, since \( P_0, \ldots, P_{n-1} \) is a basis,

\[
Q_n(x) - c P_n(x) = a_0 P_0(x) + \cdots + a_{n-1} P_{n-1}(x).
\]

Multiply the last equation by \( P_k(x) \) (for \( 0 \leq k \leq n-1 \)) and integrate from \( x = -1 \) to \( x = 1 \) to show that each \( a_k \) is zero.

32. Deduce the recurrence formula (51) for Legendre polynomials by completing the following steps.

(a) Show that the function

\[
Q_{n-1}(x) := (n+1)P_{n+1}(x) - (2n+1)x P_n(x)
\]

is a polynomial of degree \( n - 1 \). **Hint:** Compute the coefficient of \( x^{n+1} \) term using the representation (42). The coefficient of \( x^n \) is also zero because \( P_{n+1}(x) \) and \( x P_n(x) \) are both odd or both even functions, a consequence of Problem 30.

(b) Using the result of Problem 31(a), show that

\[
\int_{-1}^{1} Q_{n-1}(x) P_k(x) dx = 0 \quad \text{for} \quad k = 0, 1, \ldots, n-2.
\]

(c) From Problem 31(b), conclude that \( Q_{n-1}(x) = c P_{n-1}(x) \) and, by taking \( x = 1 \), show that \( c = -n \). **Hint:** Recall that \( P_m(1) = 1 \) for all \( m \).

From the definition of \( Q_{n-1}(x) \) in part (a), the recurrence formula now follows.

33. To prove Rodrigues’s formula (52) for Legendre polynomials, complete the following steps.

(a) Let \( v_n := \left( \frac{d^n}{dx^n} \right) (x^2 - 1)^n \) and show that \( v_n(x) \) is a polynomial of degree \( n \) with the coefficient of \( x^n \) equal to \( (2n)!/n! \).

(b) Use integration by parts \( n \) times to show that, for any polynomial \( q(x) \) of degree less than \( n \),

\[
\int_{-1}^{1} v_n(x) q(x) dx = 0.
\]

**Hint:** For example, when \( n = 2 \),

\[
\int_{-1}^{1} \frac{d^2}{dx^2} \left( (x^2 - 1)^2 \right) q(x) dx = q(x) \frac{d}{dx} \left( (x^2 - 1)^2 \right) \bigg|_{-1}^{1} - \left. \left( q'(x)(x^2 - 1)^2 \right) \right|_{-1}^{1} + \int_{-1}^{1} q''(x)(x^2 - 1)^2 dx.
\]

Since \( n = 2 \), the degree of \( q(x) \) is at most 1, and so \( q''(x) \equiv 0 \). Thus

\[
\int_{-1}^{1} \frac{d^2}{dx^2} \left( (x^2 - 1)^2 \right) q(x) dx = 0.
\]

(c) Use the result of Problem 31(b) to conclude that \( P_n(x) = cv_n(x) \) and show that \( c = 1/2^n n! \) by comparing the coefficients of \( x^n \) in \( P_n(x) \) and \( v_n(x) \).

34. Use Rodrigues’s formula (52) to obtain the representation (43) for the Legendre polynomials \( P_n(x) \).

**Hint:** From the binomial formula,

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( (x^2 - 1)^n \right)
\]

\[
= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( \sum_{m=0}^{n} \frac{n!}{(n-m)!} (-1)^m x^{2n-2m} \right).
\]

35. The generating function in (53) for Legendre polynomials can be derived from the recurrence formula (51) as follows. Let \( x \) be fixed and set \( f(z) := \sum_{n=0}^{\infty} P_n(x) z^n \). The goal is to determine an explicit formula for \( f(z) \).

(a) Show that multiplying each term in the recurrence formula (51) by \( z^n \) and summing the terms from \( n = 1 \) to \( \infty \) leads to the differential equation

\[
\frac{df}{dz} = \frac{x - z}{1 - 2xz + z^2} f.
\]

**Hint:**

\[
\sum_{n=1}^{\infty} (n+1)P_{n+1}(x) z^n = \sum_{n=0}^{\infty} (n+1)P_{n+1}(x) z^n - P_1(x) = \frac{df}{dz} - x.
\]

(b) Solve the differential equation derived in part (a) and use the initial conditions \( f(0) = P_0(x) = 1 \) to obtain \( f(z) = \left( 1 - 2xz + z^2 \right)^{-1/2} \).
36. Find a general solution about \( x = 0 \) for the equation
\[
(1 - x^2)y'' - 2xy' + 2y = 0
\]
by first finding a polynomial solution and then using the reduction of order formula given in Exercises 6.1, Problem 31 to find a second (series) solution.

37. The **Hermite polynomials** \( H_n(x) \) are polynomial solutions to Hermite’s equation
\[
y'' - 2xy' + 2ny = 0.
\]
The Hermite polynomials are generated by
\[
e^{2n-1^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} r^n.
\]
Use this equation to determine the first four Hermite polynomials.

38. The **Chebyshev** (Tchebichef) polynomials \( T_n(x) \) are polynomial solutions to Chebyshev’s equation
\[
(1 - x^2)y'' - xy' + n^2y = 0.
\]
The Chebyshev polynomials satisfy the recurrence relation
\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),
\]
with \( T_0(x) = 1 \) and \( T_1(x) = x \). Use this recurrence relation to determine the next three Chebyshev polynomials.

39. The **Laguerre polynomials** \( L_n(x) \) are polynomial solutions to Laguerre’s equation
\[
xy'' + (1 - x)y' + ny = 0.
\]
The Laguerre polynomials satisfy Rodrigues’s formula,
\[
L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}).
\]
Use this formula to determine the first four Laguerre polynomials.

40. **Reduction to Bessel’s Equation.** The class of equations of the form
\[
y''(x) + c x^n y(x) = 0, \quad x > 0,
\]
where \( c \) and \( n \) are positive constants, can be solved by transforming the equation into Bessel’s equation.

\( \text{(a)} \) First, use the substitution \( y = x^{1/2} z \) to transform (54) into an equation involving \( x \) and \( z \).

\( \text{(b)} \) Second, use the substitution
\[
s = \frac{2\sqrt{c}}{n + 2} x^{n/2 + 1}
\]
to transform the equation obtained in part (a) into the Bessel equation
\[
s^2 \frac{d^2 z}{ds^2} + s \frac{dz}{ds} + \left(s^2 - \frac{1}{(n + 2)^2}\right) z = 0, \quad s > 0.
\]

\( \text{(c)} \) A general solution to the equation in part (b) can be given in terms of Bessel functions of the first and second kind. Substituting back in for \( s \) and \( z \), obtain a general solution for equation (54).

41. \( \text{(a)} \) Show that the substitution \( z(x) = \sqrt{x} y(x) \) renders Bessel’s equation (22) in the form
\[
z'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right) z = 0.
\]

\( \text{(b)} \) For \( x \gg 1 \), equation (55) would apparently be approximated by the equation
\[
z'' + z = 0.
\]
Write down the general solution to (56), reset \( y(x) = z(x)/\sqrt{x} \), and argue the plausibility of formula (36).

\( \text{(c)} \) For \( \nu = \pm 1/2 \), equation (55) reduces to equation (56) exactly. Relate this observation to Problem 25.

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**Chapter Summary**

Initial value problems that do not fall into the “solvable” categories that have been studied (such as constant-coefficient or equidimensional equations) can often be analyzed by interpreting the differential equation as a prescription for computing the higher derivatives of the unknown function. The **Taylor polynomial method** uses the equation to construct a polynomial approximation matching the initial values of a finite number of derivatives of the unknown. If the equation permits the extrapolation of this procedure to polynomials of arbitrarily high degree, **power series** representations of the solution can be constructed.