Letters

Global Exponential Stability of Competitive Neural Networks With Different Time Scales

A. Meyer-Baese, S. S. Pilyugin, and Y. Chen

Abstract—**The dynamics of cortical cognitive maps developed by self-organization must include the aspects of long and short-term memory. The behavior of such a neural network is characterized by an equation of neural activity as a fast phenomenon and an equation of synaptic modification as a slow part of the neural system. We present a new method of analyzing the dynamics of a biological relevant system with different time scales based on the theory of flow invariance. We are able to show the conditions under which the solutions of such a system are bounded being less restrictive than with the K–monotone theory, singular perturbation theory, or those based on supervised synaptic learning. We prove the existence and the uniqueness of the equilibrium. A strict Lyapunov function for the flow of a competitive neural system with different time scales is given and based on it we are able to prove the global exponential stability of the equilibrium point.**

Index Terms—**Flow invariance, global exponential stability, multitime scale neural network.**

I. Introduction

Dynamic neural networks which contain both feedforward and feedback connections between the neural layers play an important role in visual processing, pattern recognition, neural computing, and control. Moreover, biological networks possess synapses whose synaptic weights vary in time. Thus, competitive neural networks with a combined activity and weight dynamics constitute an important class of neural networks. Their capability of storing desired patterns as stable equilibrium points requires stability criteria which include the mutual interference between neuron and learning dynamics.

This paper investigates the dynamics of cortical cognitive maps, modeled by a system of competitive differential equations, from a rigorous analytic standpoint. The networks under study model the dynamics of both the neural activity levels, the short-term memory (STM), and the dynamics of unsupervised synaptic modifications, the long-term memory (LTM). The actual network models under consideration may be considered extensions of Grossberg's shunting network [4] or Amari's model for primitive neuronal competition [2]. These earlier networks are considered pools of mutually inhibitory neurons with fixed synaptic connections. Our results extend the previous studies to systems where the synapses can be modified by external stimuli. Also, the learning algorithm is unsupervised.

Summarizing, we present a mathematical analysis of a revised version of the Willshaw–Malsburg model [11] of topographic formation, solving the equations of synaptic self-organization coupled with the field equation of neural excitations. In other words, we study the dynamics of cortical cognitive maps developed by self-organization which can be found in the nervous system.

Recently, several articles have discussed neural systems with time-varying weights. In [8] the dynamical behavior of discrete-time neural networks is studied using stable dynamic backpropagation algorithms. Two new stable learning concepts, the multiplier and the constrained learning rate methods, are employed. They describe supervised learning algorithms, and evaluate an error function. Generalized dynamic neural networks described in [9] are recurrent neural networks with time-dependent weights. The algorithm for learning continuous trajectories is based on a variational formulation of the Pontryagin maximum principle, and is also supervised. A robust local stability condition has been presented in [10] for multilayer recurrent neural networks with two hidden layers. The NL_q theory was proposed as a stability theory for multilayer recurrent neural with application to neural control. All these above mentioned papers employ a supervised learning dynamics.

In this paper, we apply the theory of flow invariance on large-scale neural networks, which have two types of state variables (LTM and STM) describing the slow unsupervised and the fast dynamics of the system. We will give the mathematical conditions for showing when the STM and LTM trajectories are bounded. Our design is more general than that given in [6] since it is not required to assume a high gain approximation and it does not treat the two dynamics separately. In addition, it does not require the excitatory region to comprise only one neuron. We also give a strict Lyapunov function for the neural multi-time scale system, show the existence and uniqueness of the equilibrium, and prove global exponential stability for the equilibrium. Also, our proof is based on a different approach than in [7] where the analysis of only global asymptotic stability was based on the theory of singular perturbations. Besides showing here the uniqueness and existence of the equilibrium, we are able to prove milder and fewer necessary conditions for stability.

We consider a laterally inhibited network with a deterministic signal Hebbian learning law [5] that is similar to the spatiotemporal system of Amari [1]. The general neural network equations describing the temporal evolution of the STM and LTM states for the jth neuron of a N-neuron network are

$$\text{STM:}\quad \epsilon \dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j \sum_{i=1}^{p} m_{ij} y_i \qquad (1)$$

$$\text{LTM:}\quad \dot{m}_{ij} = -m_{ij} + y_i f(x_j) \qquad (2)$$

where x_j is the current activity level, a_j is the time constant of the neuron, B_j is the contribution of the external stimulus term, $f(x_i)$ is the neuron's output, y_i is the external stimulus, and m_{ij} is the synaptic efficiency. ϵ is the fast time-scale associated with the STM state. D_{ij} represents a synaptic connection parameter between the ith neuron and the jth neuron. We assume here, that the recurrent neural network consists of both feedforward and feedback connections between the layers and neurons forming complicated dynamics.

The neural network is modeled by a system of deterministic equations with a time-dependent input vector rather than a source emitting input signals with a prescribed probability distribution.[1] By introducing

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[1]Our interest is to store patterns as equilibrium points in the N-dimensional space. In fact, in [2] is demonstrated the formation of stable one-dimensional cortical maps under the aspect of topological correspondence and under the restriction of a constant probability of the input signal.

the dynamic variable $S_j = \mathbf{y}^T \mathbf{m}_j$, we get a state-space representation of the LTM and STM equations of the system

$$\epsilon \dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j S_j \tag{3}$$
$$\dot{S}_j = -S_j + |\mathbf{y}|^2 f(x_j). \tag{4}$$

The input stimuli are assumed to be normalized vectors of unit magnitude $|\mathbf{y}|^2 = 1$. This system is subject to our analysis considerations to show that the LTM and STM trajectories are bounded.

II. Equilibrium and Global Asymptotic Stability Analysis of Neuro-Synaptic Systems

In this section, we present a new condition for the uniqueness and global exponential stability for neuro-synaptic systems which improves the previous stability results. The existence and uniquenss of the equilibrium is given based on flow-invariance while the global exponential stability is shown by a strict Lyapunov function.

The theory of flow-invariance gives a qualitative interpretation of the dynamics of a system, taking into account the invariance of the flow of the system. In other words a trajectory gets trapped in an invariant set.

Before we state the stability results based on the concept of flow-invariance we will first give some useful definitions used in nonlinear analysis.

A. Definitions

Definition 1: Let $F : R^N \to R^N$ be a Lipschitz continuous map and let S be a subset of R^N. We say that S is *flow-invariant* with respect to the system of differential equations

$$x'(t) = F(x(t)) \tag{5}$$

if any solution $x(t)$ starting in S at $t = 0$ remains in S for all $t \geq 0$ as long as $x(t)$ is defined. In dynamical systems terminology, such sets are called positively invariant under the flow generated by (5).

Definition 2: We say that the system (S) is *dissipative* in R^N if there exists a precompact (bounded) set $U \subset R^N$ such that for any solution $x(t)$ of (S) there exists $T \geq 0$ such that $x(t) \in U$ for all $t \geq T$. In other words, all solutions of (S) enter this bounded set U in finite time.

If (S) is dissipative, then all solutions of (S) are defined for $t \geq 0$, and there exists a compact set $A \subset U$ which attracts all solutions of (S). The set A is invariant under the flow of (S) and it is called the *global attractor* of (S) in R^N.

B. Results

Theorem 1: Consider the system of differential equations

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{N} D_{ij} f(x_j(t)) + B_i S_i(t),\ i{=}1,\ldots,N \tag{6}$$
$$S_i'(t) = -S_i(t) + f(x_i(t)),\ i = 1,\ldots,N \tag{7}$$

and suppose that $a_i > 0$ for all $i = 1,\ldots,N$. Also suppose that f is locally Lipschitz and bounded, that is, there exists a constant $M > 0$ such that $-M \leq f(x) \leq M$ for all $x \in R$. Let

$$l_i = \frac{M}{a_i}\left(\sum_{j=1}^{N} |D_{ij}| + |B_i|\right) > 0,\ i = 1,\ldots,N. \tag{8}$$

Then for any $\epsilon > 0$ and for any initial condition $\{x_i(0), S_i(0)\} \in R^{2N}$ there exists a $T \geq 0$ such that

$$S_i(t) \in [-M-\epsilon, M+\epsilon], \quad x_i(t) \in [-l_i-\epsilon, l_i+\epsilon]$$

for all $i = 1,\ldots,N$ and all $t \geq T$.

Proof: Since f is locally Lipschitz, system (6)–(7) enjoys local existence and uniqueness of solutions. Moreover, since f is uniformly bounded, there exist constants $K_1,\ldots,K_5 > 0$ such that

$$|x_i'(t)| \leq K_1 + K_2|x_i(t)| + K_3|S_i(t)|, |S_i'(t)| \leq K_4 + K_5|S_i(t)|$$

thus all solutions are defined globally (for all $t \geq 0$).

Given $\epsilon > 0$, we define

$$\delta_i = \begin{cases} \min(\frac{a_i\epsilon}{2|B_i|}, \epsilon), & B_i \neq 0 \\ \epsilon, & B_i = 0 \end{cases}$$

for $i = 1,\ldots,N$. It follows that $\delta_i > 0$ and $-|B_i|\delta_i + a_i\epsilon \geq a_i\epsilon/2$ for all $i = 1,\ldots,N$. Then for $t \geq 0$ and for $S_i(t) \leq -M - \delta_i$ the following inequality holds:

$$S_i'(t) \geq -(-M-\delta_i) + f(x_i(t)) = \delta_i + (f(x_i(t)) + M) \geq \delta_i > 0.$$

Similarly, for $t \geq 0$ and for $S_i(t) \geq M + \delta_i$ we have that

$$S_i'(t) \leq -(M+\delta_i) + f(x_i(t)) = -\delta_i + (f(x_i(t)) - M) \leq -\delta_i < 0.$$

Therefore, for any $i \in \{1,\ldots,N\}$ there exists a $T_i^s \geq 0$ such that

$$S_i(t) \in [-M-\delta_i, M+\delta_i] \subseteq [-M-\epsilon, M+\epsilon] \tag{9}$$

for all $t \geq T_i^s$. Let $T^s = \max_i T_i^s$, then (9) holds for all $i \in \{1,\ldots,N\}$ and for all $t \geq T^s$.

Now we consider $t \geq T^s$. For $x_i(t) \leq -l_i - \epsilon$, (6) and (9) imply that

$$x_i'(t) \geq a_i(l_i+\epsilon) + \sum_{j=1}^{N} D_{ij} f(x_j) + B_i(-M-\delta_i).$$

Using the definition of l_i given by (8), we find that for $t \geq T^s$ and $x_i(t) \leq -l_i - \epsilon$

$$x_i'(t) \geq a_i l_i + a_i\epsilon - M\left(\sum_{j=1}^{N}|D_{ij}| + |B_i|\right) - |B_i|\delta_i = -|B_i|\delta_i + a_i\epsilon \geq \frac{a_i\epsilon}{2} > 0.$$

Similarly, for $t \geq T^s$ and for $x_j(t) \geq l_i + \epsilon$, (6) and (9) imply that

$$x_i'(t) \leq -a_i(l_i+\epsilon) + \sum_{j=1}^{N} D_{ij} f(x_j) + B_i(M+\delta_i).$$

Using (8) again, we find that for $t \geq T^s$ and $x_i(t) \geq l_i + \epsilon$,

$$x_i'(t) \leq -a_i l_i - a_i\epsilon + M\left(\sum_{j=1}^{N}|D_{ij}| + |B_i|\right) + |B_i|\delta_i = -a_i\epsilon + |B_i|\delta_i \leq -\frac{a_i\epsilon}{2} < 0.$$

Consequently, for any $i \in \{1,\ldots,N\}$ there exists a $T_i^x \geq T^s \geq 0$ such that

$$x_i(t) \in [l_i - \epsilon, l_i + \epsilon] \tag{10}$$

for all $t \geq T_i^x$. Let $T = \max_i T_i^x$, then both (9) and (10) hold for all $i \in \{1,\ldots,N\}$ and all $t \geq T$. ∎

Corollary 1: The system (6)–(7) is dissipative in R^{2N} and, therefore, it has a compact global attractor

$$A \subseteq D = \prod_{i=1}^{N}[-l_i, l_i] \times \prod_{i=1}^{N}[-M, M].$$

Corollary 2: It follows from the proof of Theorem 1 that the set D is flow invariant under (6)–(7). In other words, D is a positively invariant set of (6)–(7), that is, any solution starting in D at $t = 0$ remains in D for all $t \geq 0$.

Corollary 3: Since the set H can be contracted to a point and D is flow-invariant with respect to (6)–(7), the Brower fixed point theorem implies that there exists a point $e \in D$ which is fixed under the flow of (6)–(7). Consequently, $e \in D$ is an equilibrium of (6)–(7).

Theorem 2: Suppose that $f(x)$ is C^1 with $|f'(x)| < k$ for all x and

$$a_i > k\left(\sum_{j=1}^{N}|D_{ij}| + |B_i|\right),\ i = 1, \ldots N \tag{11}$$

then the equilibrium e is unique.

Proof: At the equilibrium, $S_i = f(x_i)$ from (7). Substituting these expressions into (6), we obtain the system

$$0 = -a_i x_i + \sum_{j=1}^{N} D_{ij} f(x_j) + B_i f(x_i),\ i = 1, \ldots N.$$

Since $a_i > 0$, we can express x_i as

$$x_i = \frac{1}{a_i}\left(\sum_{j=1}^{N} D_{ij} f(x_j) + B_i f(x_i)\right) = F_i(x_1, \ldots, x_N).$$

The inequality (11) implies that

$$|F(x') - F(x'')| < |x' - x''|$$

where $F = (F_1, \ldots, F_N)$ so that F is a contracting map in the sup norm in R^N. Consequently, there exists a unique fixed point of F. The x_i-coordinates of this fixed point uniquely determine the S_i-coordinates of the equilibrium e via $S_i = f(x_i)$. We conclude that the equilibrium e is unique. ■

We let $e = (x_1^0, S_1^0, \ldots, x_N^0, S_N^0)$ be the equilibrium of (6)–(7) and introduce the change of variables $\phi_i = x_i - x_i^0$, $\psi_i = S_i - S_i^0$ which shifts e to the origin. Specifically, if we denote $f_i(\phi_i) = f(\phi_i + x_i^0) - S_i^0$, then $f_i(0) = 0$ and (6)–(7) may be rewritten as

$$\phi_i' = -a_i\phi_i + \sum_{j=1}^{N} D_{ij} f_j(\phi_j) + B_i \psi_i \tag{12}$$

$$\psi_i' = -\psi_i + f_i(\phi_i). \tag{13}$$

Theorem 3: Suppose that $f(x)$ is C^1 with $|f'(x)| \leq k$ for all x and $a_i > 0$. Let

$$d_i = \frac{1}{2}\left(\frac{|B_i|}{a_i} + k\right), \quad c_{ij} = \frac{1}{2}k\left(\frac{|D_{ij}|}{a_i} + \frac{|D_{ji}|}{a_j}\right)$$

for $i, j = 1, \ldots, N$. If

$$\max_i\left(d_i + \sum_{j=1}^{N} c_{ij}\right) < 1 \tag{14}$$

then e is a global attractor for system (12)–(13). Moreover, all solutions of (12)–(13) converge to e exponentially fast as $t \to \infty$.

Proof: We prove global convergence by presenting a strict Lyapunov function for (12)–(13). Let

$$V = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\phi_i^2}{a_i} + \psi_i^2\right)$$

then

$$\frac{d}{dt}V = \sum_{i=1}^{N}\phi_i\left(-\phi_i + \sum_{j=1}^{N}\frac{D_{ij}}{a_i}f_j(\phi_j) + \frac{B_i}{a_i}\psi_i\right) + \sum_{i=1}^{N}\psi_i\left(-\psi_i + f_i(\phi_i)\right). \tag{15}$$

Since $f_i(0) = 0$ and $|f_i'(x)| = |f'(x + x_i^0)| < k$, we have that $|f_i(\phi_i)| < k|\phi_i|$. Consequently, equality in (15) can be replaced by the inequality

$$\frac{d}{dt}V < -\sum_{i=1}^{N}\left(\phi_i^2 + \psi_i^2\right) + \sum_{i,j=1}^{N}\frac{|D_{ij}|}{a_i}k|\phi_i||\phi_j| + \sum_{i=1}^{N}\left(\frac{|B_i|}{a_i} + k\right)|\phi_i||\psi_i|.$$

The right-hand side of this inequality is given by the quadratic form with the matrix $-Q$ where Q has the following block structure:

$$Q_{ij} = \begin{cases}\begin{pmatrix}1 - c_{ii} & -d_i \\ -d_i & 1\end{pmatrix}, & i = j, \\ \begin{pmatrix}-c_{ij} & 0 \\ 0 & 0\end{pmatrix}, & i \neq j,\end{cases} \quad i, j = 1, \ldots, N \tag{16}$$

where d_i and c_{ij} are as defined above. Inequality (14) together with Gerschgorin's Theorem imply that $-Q$ is positive definite, or equivalently that Q is negative definite. Let $\alpha > 0$ be the smallest eigenvalue of $-Q$. Then

$$\frac{d}{dt}V < -\alpha\sum_{i=1}^{N}\left(\phi_i^2 + \psi_i^2\right)$$

and consequently, V is a strict Lyapunov function for (12)–(13). Moreover, there exists $\beta > 0$ such that

$$\beta V \leq \sum_{i=1}^{N}\left(\phi_i^2 + \psi_i^2\right)$$

so that

$$\frac{d}{dt}V < -\alpha\beta V. \tag{17}$$

Equation (17) implies that V converges to zero exponentially fast, and thus solutions $(\phi(t), \psi(t))$ of (12)–(13) converge to the origin also exponentially fast. In terms of the original system (6)–(7), its solutions $(x(t), S(t))$ converge to e exponentially fast. ■

III. Comparisons

In this section, we compare various stability theorems on competitive neural networks with different time scales.

In [6], the convergence to point attractors is proved based on the condition of high-gain approximation which means that the output nonlinearity is approximated by a step function. It is also assumed that the synaptic connection parameter D_{ij} is given by

$$D_{ij} = \begin{cases}\alpha, & i = j, \\ -\beta, & i \neq j,\end{cases}.$$

In this paper, we also employ the concept of flow invariance but are able to prove the uniqueness and existence of the equilibrium by only imposing that the output nonlinearity is bounded in R.

While only the local stability is given in [6], the global asymptotic stability is proved in [7] based on the theory of singular perturbation. This approach treats both fast and slow dynamics separatedly, and requires certain growth conditions to be satisfied by both the slow and fast system in order to determine a coupled Lyapunov function. However, the imposed conditions are too difficult to test, and they give only global asymptotic stability. Our approach gives global exponential stability, requires only a simple inequality to hold, and requires that the first derivative of the output nonlinearity to be bounded.

A comparison between the results obtained in this paper with those in [8]–[10] cannot be made. We employ in this paper a Hebbian learning law while the others employ a supervised learning law such as dynamic backpropagation algorithm [8], [10] or an optimal control problem [9]. Besides the supervised learning algorithms which does not pertain to cortical cognitive maps, there are also dissimilarities in the neural-network architecture.

IV. Conclusion

In this paper, we prove global exponential stability of competitive neural networks with fast and slow dynamics describing cognitive cortical maps developed by self-organization. Based on the flow invariance technique we can show the conditions that the LTM and STM trajectories are bounded, being at the same time less restrictive than with the K–monotone theory, or for systems with supervised LTM trajectories. We also presented a strict Lyapunov function and based on it we have shown global exponential stability of the equilibrium point. Besides proving the existence and uniqueness of the equilibrium, we are presenting milder and more general conditions than based on singular perturbation theory.

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Further Results on Adaptive Control for a Class of Nonlinear Systems Using Neural Networks

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***Abstract*—Zhang *et al.* presented an excellent neural-network (NN) controller for a class of nonlinear control designs. The singularity issue is completely avoided. Based on a modified Lyapunov function, their lemma illustrates the existence of an ideal control which is important in establishing the NN approximator. In this note, we provide a Lyapunov function to realize an alternative ideal control which is more direct and simpler. The major contributions of this note are divided into two parts. First, it proposes a control scheme which results in a smaller dimensionality of NN than that of Zhang *et al.* In this way, the proposed NN controller is easier to implement and more reliable for practical purposes. Second, by removing certain restrictions from the design reported by Zhang *et al.*, we further develop a new NN controller, which can be applied to a wider class of systems.**

***Index Terms*—Adaptive control, Lyapunov function, neural networks, nonlinear systems.**

I. Introduction

In [1], the nonlinear system under consideration is of the following form:

$$\begin{cases} \dot{x}_i = & x_{i+1}, \quad i = 1, 2, .., n-1 \\ \dot{x}_n = & a(x) + b(x)u \\ y = & x_1 \end{cases} \tag{1}$$

where $x = [x_1, x_2, \dots x_n]^T \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output, respectively. The objective is to enable the output y to follow a desired trajectory y_d.

Assumption 1: [1] The sign of $b(x)$ is known, and a known continuous function $\bar{a}(x) \geq 0$ and a constant $b_0 > 0$ exist such that $|a(x)| \leq \bar{a}(x)$ and $|b(x)| \geq b_0$, $\forall x \in R^n$.

Define vector x_d, e and a filtered tracking error s as

$$\begin{aligned} x_d &= [y_d, \dot{y}_d, \dots, y_d^{(n-1)}]^T \\ e &= x - x_d = [e_1, e_2, \dots e_n]^T \\ s &= \left(\frac{d}{dt} + \lambda\right)^{n-1} e_1 = [\Lambda^T \;\; 1]e, \quad \text{with} \quad \lambda > 0 \end{aligned} \tag{2}$$

where $\Lambda = [\lambda^{n-1}, (n-1)\lambda^{n-2}, \dots, (n-1)\lambda]^T$.

The time derivative of s can be written as

$$\dot{s} = a(x) + b(x)u + v \tag{3}$$

where $v = -y_d^{(n)} + [0 \;\; \Lambda^T]e$.

Assumption 2: [1] A desired trajectory vector x_d is continuous and available, and $x_d \in \Omega_d$ with Ω_d being a compact set.

Remark 1.1: In Assumption 1, $b(x)$ is required to satisfy $|b(x)| \geq b_0 > 0$. This assumption poses a controllable condition on the system (1) and it is made in many control schemes (Krstic *et al.* [5]; Sepulchre *et al.* [6]). Without losing generality, we will assume that $b(x) \geq b_0 > 0$ as in [1], [7, Ch. 8], and [8]. However, the following analysis may easily be modified for a system with $b(x) < 0$.

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