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GLOBAL ASYMPTOTIC STABILITY OF A CLASS OF DYNAMICAL NEURAL NETWORKS

ANKE MEYER-BÄSE
Department of Electrical and Computer Engineering, Florida State University, Tallahassee, FL 32310-6046

SERGEI S. PILYUGIN
Department of Mathematics, University of Florida, Gainesville, FL 32611

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The dynamics of cortical cognitive maps developed by self–organization must include the aspects of long and short–term memory. The behavior of the network is such characterized by an equation of neural activity as a fast phenomenon and an equation of synaptic modification as a slow part of the neural biologically relevant system.

We present new stability conditions for analyzing the dynamics of a biological relevant system with different time scales based on the theory of flow invariance. We prove the existence and uniqueness of the equilibrium, and give a quadratic–type Lyapunov function for the flow of a competitive neural system with fast and slow dynamic variables and thus prove the global stability of the equilibrium point.

Keywords: Recurrent network; flow invariance; global asymptotic stability; time-varying weights.

1. Introduction: The Class of Neural Networks with Different Time-Scales

Dynamic neural networks which contain both feedforward and feedback connections between the neural layers play an important role in visual processing, pattern recognition, neural computing and control. Moreover, biological networks possess synapses whose synaptic weights vary in time. Thus, competitive neural networks with a combined activity and weight dynamics constitute an important class of neural networks. Their capability of storing desired patterns as stable equilibrium points requires stability criteria which include the mutual interference between neuron and learning dynamics.

This paper investigates the dynamics of cortical cognitive maps, modeled by a system of competitive differential equations, from a rigorous analytic standpoint. The networks under study model the dynamics of both the neural activity levels, the short-term memory (STM), and the dynamics of unsupervised synaptic modifications, the long-term memory (LTM). Such networks may be considered extensions of Grossberg's shunting network[5] or Amari's model for primitive neuronal competition.[1] These earlier networks are modeled as a pool of mutually inhibitory neurons with fixed synaptic connections. The present work extends the previous studies to systems where synaptic weights can be modified by external stimuli. In addition, the learning algorithm is unsupervised.

Summarizing, we present a mathematical analysis of a revised version of the Willshaw-Malsburg model[13] of topographic formation, solving the equations of synaptic self-organization coupled with the field equation of neural excitations. Specifically, we study the dynamics of cortical cognitive maps developed by self-organization which can be found in the nervous system.

Recently, several articles have discussed neural systems with time–varying weights. In Ref. 8 the dynamical behavior of discrete-time neural networks is studied using stable dynamic backpropagation algorithms. Two new stable learning concepts, the multiplier and the constrained learning rate methods, are employed. They describe supervised learning algorithms, and evaluate an error function. Generalized dynamic neural networks described in Ref. 4 are recurrent neural networks with time-dependent weights. The algorithm for learning continuous trajectories is based on a variational formulation of the Pontryagin maximum principle, and is also supervised. A robust local stability condition has been presented in Ref. 12 for multilayer recurrent neural networks with two hidden layers. The NL_q theory was proposed as a stability theory for multilayer recurrent neural networks with application to neural control. These papers consider the supervised learning dynamics for the lateral connection matrices.

In this paper, we apply the theory of flow-invariance to large-scale neural networks, which have two types of state variables (LTM and STM) describing the slow unsupervised and the fast dynamics of the system. We present the analytic conditions that warrant bounded STM and LTM trajectories. The design of our model is more general than that given in Ref. 9 since it does not require the assumption of high gain approximation and it does not treat the two dynamics separately. In addition, it does not require the excitatory region to comprise only one neuron. We also give a quadratic-type Lyapunov function for the neural multi-time scale system, show the existence and uniqueness of the equilibrium and prove global asymptotic stability for the equilibrium.

We consider a laterally inhibited network[a] with a deterministic signal Hebbian learning law[7] that is similar to the spatiotemporal system of Amari.[2] The general neural network equations describing the temporal evolution of the STM and LTM states for the jth neuron of a N-neuron network are:

$$\text{STM:}\quad \varepsilon\dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j \sum_{i=1}^{p} m_{ij} y_i \tag{1}$$

$$\text{LTM:}\quad \dot{m}_{ij} = -m_{ij} + y_i f(x_j) \tag{2}$$

where x_j is the current activity level, a_j is the time constant of the neuron, B_j is the contribution of the external stimulus term, $f(x_i)$ is the neuron's output, y_i is the external stimulus, and m_{ij} is the synaptic efficiency. ε is the fast time-scale associated with the STM state. D_{ij} represents a synaptic connection parameter between the ith neuron and the jth neuron. We assume here, that the recurrent neural network consists of both feedforward and feedback connections between the layers and neurons forming complicated dynamics.

The neural network is modeled by a system of deterministic equations with a time-dependent input vector rather than a source emitting input signals with a prescribed probability distribution.[b] By introducing the dynamic variable $S_j = \mathbf{y}^T\mathbf{m}_j$, we obtain a state space representation of the LTM and STM equations of the system:

$$\varepsilon\dot{x}_j = -a_j x_j + \sum_{i=1}^{N} D_{ij} f(x_i) + B_j S_j \tag{3}$$

$$\dot{S}_j = -S_j + |\mathbf{y}|^2 f(x_j) \tag{4}$$

The input stimuli are assumed to be normalized vectors of unit magnitude $|\mathbf{y}|^2 = 1$. In the next section, we analyze this system and show that the STM and LTM trajectories are bounded.

2. Flow Invariant Sets of Competitive Neural Networks with Different Time-Scales

In this section, we present the conditions for the existence and uniqueness of the equilibrium.

[a]The feedback connections in the output layer perform lateral inhibition, with each neuron tending to inhibit the neuron to which it is laterally connected.[6]

[b]Our interest is to store patterns as equilibrium points in the N-dimensional space. In fact, in Ref. 1 is demonstrated the formation of stable one-dimensional cortical maps under the aspect of topological correspondence and under the restriction of a constant probability of the input signal.

The theory of flow–invariance gives a qualitative interpretation of the dynamics of a system, taking into account the invariance of the flow of the system. Frankly speaking, a trajectory is "trapped" in an invariant set.

Before stating the stability results based on the concept of flow–invariance, we define several important notions of nonlinear analysis.

Definition 1

Let F: $R^N \to R^N$ be a Lipschitz continuous map and let S be a subset of R^N. We say that S is *flow-invariant* with respect to the system of differential equations

$$x'(t) = F(x(t)), \tag{5}$$

if any solution $x(t)$ starting in S at $t = 0$ remains in S for all $t \geq 0$ as long as $x(t)$ is defined. In dynamical systems terminology, such sets are called positively invariant under the flow generated by Eq. (5).

Definition 2

We say that the system (5) is *dissipative* in R^N if there exists a precompact (bounded) set $U \subset R^N$ such that for any solution $x(t)$ of (5) there exists $T \geq 0$ such that $x(t) \in U$ for all $t \geq T$. In other words, all solutions of (5) enter this bounded set U in finite time.

If Eq. (5) is dissipative then all solutions of (5) are defined for $t \geq 0$, and there exists a compact set $A \subset U$ which attracts all solutions of (5). The set A is invariant under the flow of (5) and it is called the *global attractor* of (5) in R^N.

After we have introduced the definitions, we are ready to state the stability results based on the concept of flow–invariance.

Theorem 1

Consider the system of differential equations

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{N} D_{ij} f(x_j(t)) + B_i S_i(t), \quad i = 1, \ldots, N, \tag{6}$$

$$S_i'(t) = -S_i(t) + f(x_i(t)), \quad i = 1, \ldots, N, \tag{7}$$

and suppose that $a_i > 0$ for all $i = 1, \ldots, N$. Also suppose that f is locally Lipschitz and bounded, that is, there exists a constant $M > 0$ such that $-M \leq f(x) \leq M$ for all $x \in R$. Let

$$l_i = \frac{M}{a_i}\left(\sum_{j=1}^{N} |D_{ij}| + |B_i|\right) > 0, \quad i = 1, \ldots, N. \tag{8}$$

Then for any $\varepsilon > 0$ and for any initial condition $\{x_i(0), S_i(0)\} \in R^{2N}$ there exists a $T \geq 0$ such that

$$S_i(t) \in [-M - \varepsilon, M + \varepsilon],$$

$$x_i(t) \in [-l_i - \varepsilon, l_i + \varepsilon]$$

for all $i = 1, \ldots, N$ and all $t \geq T$.

Corollary 1

The system (6–7) is dissipative in R^{2N} and therefore it has a compact global attractor

$$A \subseteq H = \prod_{i=1}^{N}[-l_i, l_i] \times \prod_{i=1}^{N}[-M, M].$$

Corollary 2

It follows from the proof of Theorem 1 that the set H is flow invariant under (6–7). In other words, H is a positively invariant set of (6–7), that is, any solution starting in H at $t = 0$ remains in H for all $t \geq 0$.

Corollary 3

Since the set H can be contracted to a point and it is flow-invariant with respect to (6–7), the Brower fixed point theorem implies that there exists a point $e \in H$ which is fixed under the flow of (6–7). Consequently, $e \in H$ is an equilibrium of (6–7).

Theorem 2

Suppose that $f(x)$ is C^1 with $|f'(x)| < k$ for all x and

$$a_i > k\left(\sum_{j=1}^{N} |D_{ij}| + |B_i|\right), \quad i = 1, \ldots N, \tag{9}$$

then the equilibrium e is unique.

3. Global Asymptotic Stability of Competitive Neural Networks with Different Time-Scales

After proving the existence and uniqueness of the equilibrium point in the last section, we propose in this section a Lyapunov function for the multi-time scale neural network and prove global asymptotic stability of the equilibrium point.

Consider a competitive neural system which is described by the following system of nonlinear differential equations:

$$\varepsilon\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{S}, \varepsilon) \tag{10}$$

$$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{x}, \mathbf{S}) \tag{11}$$

where $\mathbf{f}$: $R^N \times R^N \to R^N$, $\mathbf{g}$: $R^N \times R^N \to R^N$ are continously differentiable and satisfy $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Equation (10) models the fast system and Eq. (11) the slow system. Both equations are a generalized representation of Eqs. (3) and (4).

This time-scale approach is asymptotic, that is exact, in the limit as the ratio ε of the speeds of the slow versus the fast dynamics tends to zero. When ε is small, approximations are obtained from reduced-order models in separate time-scales.

A reduced system is defined by setting $\varepsilon = 0$ in Eqs. (10) and (11) to obtain

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{S}, 0) \tag{12}$$

$$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{x}, \mathbf{S}) \tag{13}$$

Assuming that Eq. (12) has a unique root $\mathbf{x} = \mathbf{h}(\mathbf{S})$, the reduced system is rewritten as

$$\dot{\mathbf{S}} = \mathbf{f}(\mathbf{h}(\mathbf{S}), \mathbf{S}) = \mathbf{f_r}(\mathbf{S}) \tag{14}$$

A boundary–layer system is defined as

$$\frac{\partial \mathbf{x}}{\partial \tau} = \mathbf{g}(\mathbf{x}, \mathbf{S}(\tau), 0) \tag{15}$$

where $\tau = t/\varepsilon$ is a stretching time scale and the vector $\mathbf{S} \in \mathbf{R^N}$ is treated as a fixed unknown parameter.

We are now able to to draw conclusions about the behavior of the original system (10) and (11) based upon a study of a simplified system $\dot{\mathbf{S}} = \mathbf{f_r}(\mathbf{S})$ obtained from (11) setting $\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{S}, 0)$. In Ref. 11, it is shown that a quadratic-type Lyapunov function establishing asymptotic stability for a singularly perturbed system can be obtained as a weighted sum of the lower-ordered reduced and boundary–layer systems, assuming that the perturbation factor is sufficiently small. Theorem 3[11] states this formally:

Theorem 3

Suppose that there exist Lyapunov functions for the reduced and the boundary layer system and that $\mathbf{f}$ and $\mathbf{g}$ satisfy certain interaction conditions as shown in Ref. 11. Then the origin ($\mathbf{x} = \mathbf{S} = \mathbf{0}$) is an asymptotically stable equilibrium point of the singularly perturbed system (10) and (11) for all $\varepsilon < \varepsilon^*(d)$.

Moreover, for every $d \in (0, 1)$

$$v(\mathbf{x}, \mathbf{S}) = (1 - d)V(\mathbf{S}) + dW(\mathbf{x}, \mathbf{S}) \tag{16}$$

is a Lyapunov function for (10) and (11) for all $\varepsilon < \varepsilon^*(d)$, where V is the Lyapunov function for the reduced order system and W of the boundary layer system.

Appendix 3 gives the conditions on the interaction of the fast and slow states, and also the upper bound $\varepsilon^*(d)$.

If the equilibrium under study is not the origin, one can always translate the coordinates on $\mathbf{R}^{2N}$ so that the equilibrium of interest becomes the new origin. To apply Theorem 3 we must determine for our multi-time scale neural network (3) and (4) two Lyapunov functions: one for the boundary–layer system and one for the reduced-order system.

In Ref. 3 is mentioned a global Lyapunov function for a competitive neural network (1) with only an activation dynamics:

$$L(x) = -\sum_{i=1}^{n} \int_0^{x_i} B_i(\zeta_i) f_i'(\zeta_i) d\zeta_i + \frac{1}{2} \sum_{j,k=1}^{n} m_{jk} f_j(x_j) f_k(x_k) \tag{17}$$

under the constraints: $m_{ij} = m_{ji}$, $a_i(x_i) \geq 0$ and $f_j(x_j) \geq 0$. This Lyapunov–function can be adapted to the boundary–layer system, if the LTM contribution S_i is treated as a fixed unknown parameter, yielding the Lyapunov–function:

$$W(\mathbf{x},\mathbf{S}) = \sum_{j=1}^{N}\int_0^{x_j} a_j(\zeta_j)f_j'(\zeta_j)d\zeta_j - \sum_{j=1}^{N} B_j S_j \int_0^{x_j} f_j'(\zeta_j)d\zeta_j - \frac{1}{2}\sum_{j=1}^{N} D_{ij} f_j(x_j) f_k(x_k) \qquad (18)$$

For the reduced-order system we can take the Lyapunov–function:

$$V(\mathbf{S}) = \frac{1}{2}\mathbf{S}^T\mathbf{S} = \sum_{i=1}^{N} S_i^2 \qquad (19)$$

As stated in Theorem 3, the Lyapunov–function for the STM and LTM dynamics is the superposition of the two previous Lyapunov–functions:

$$v(\mathbf{x},\mathbf{S}) = (1-d)V(\mathbf{S}) + dW(\mathbf{x},\mathbf{S}) \qquad (20)$$

4. Comparisons

In this section, we compare various stability theorems on competitive neural networks with different time scales.

In Ref. 9 the convergence to point attractors is proved based on the condition of high gain approximation which means that the output nonlinearity is approximated by a step function. It is also assumed that the synaptic connection parameter D_{ij} is given by

$$D_{ij} = \begin{cases} \alpha, & i = j\,, \\ -\beta, & i \neq j, \end{cases}$$

In this paper, we also employ the concept of flow invariance but are able to prove the uniqueness and existence of the equilibrium by only imposing that the output nonlinearity is bounded in R. Furthermore, we present a quadratic-type Lyapunov function and use it to prove global asymptotic stability.

In this work, we prove the existence of an invariant set being a compact global attractor, while in Ref. 10 local stability based on the theory of singular perturbation is shown. This approach treats both fast and slow dynamics separately, and requires certain growth conditions to be satisfied. Four distinct inequalities, which are too difficult to test, have to be fullfilled at the same time to ensure that the system matrices of the slow and fast system are Hurwitz. Our approach requires only a simple inequality to hold, and requires that the first derivative of the output nonlinearity to be bounded.

A direct comparison between the results obtained in this paper with those in Refs. 4, 8 and 12 cannot be made. The main difference lies in the learning mechanism. We use an unsupervised Hebbian learning law for the feedforward synapses, while the others employ a supervised learning law of the lateral connection matrices such as dynamic backpropagation algorithm[8,12] or an optimal control problem.[4] The cited papers assume that the target output signal (trajectory) is known, while we do not use or require this knowledge. Besides the supervised learning algorithms, which does not pertain to cortical cognitive maps, there are also minor dissimilarities in the neural network architecture: in the mentioned papers the nonlinearity is applied to the product between the current activity level vector and the feedback matrix, while we apply the nonlinearity directly to the current activity level vector and then multiply by the feedback matrix.

5. Conclusions

In this paper we presented new stability conditions for analyzing the dynamics of solutions of competitive neural networks with fast and slow dynamics. Based on the flow invariance technique we can show that the LTM and STM trajectories are bounded, being at the same time less restrictive than with K–monotone theory, or for systems with supervised LTM trajectories. Our method provides a lower bound for the neural time-constant, proves that flow invariance requires a bounded nonlinearity, and guarantees bounded solutions on a closed set. Besides showing the existence and uniqueness of the equilibrium, we also propose a quadratic-type Lyapunov function and based on it we prove global asymptotic stability of the equilibrium point.

Appendix 1: Proof of Theorem 1

Proof

Since f is locally Lipschitz, system (6–7) enjoys local existence and uniqueness of solutions. Moreover, since f is uniformly bounded, there exist constants $K_1, \ldots, K_5 > 0$ such that

$$|x_i'(t)| \leq K_1 + K_2|x_i(t)| + K_3|S_i(t)|,$$
$$|S_i'(t)| \leq K_4 + K_5|S_i(t)|,$$

thus all solutions are defined globally (for all $t \geq 0$).

Given $\varepsilon > 0$, we define

$$\delta_i = \begin{cases} \min\left(\frac{a_i\varepsilon}{2|B_i|}, \varepsilon\right), & B_i \neq 0, \\ \varepsilon, & B_i = 0, \end{cases}$$

for $i = 1, \ldots, N$. It follows that $\delta_i > 0$ and $-|B_i|\delta_i + a_i\varepsilon \geq \frac{a_i\varepsilon}{2}$ for all $i = 1, \ldots, N$. Then for $t \geq 0$ and for $S_i(t) \leq -M - \delta_i$ the following inequality holds:

$$\begin{aligned} S_i'(t) &\geq -(-M - \delta_i) + f(x_i(t)) \\ &= \delta_i + (f(x_i(t)) + M) \geq \delta_i > 0. \end{aligned}$$

Similarly, for $t \geq 0$ and for $S_i(t) \geq M + \delta_i$ we have that

$$\begin{aligned} S_i'(t) &\leq -(M + \delta_i) + f(x_i(t)) \\ &= -\delta_i + (f(x_i(t)) - M) \leq -\delta_i < 0. \end{aligned}$$

Therefore, for any $i \in \{1, \ldots, N\}$ there exists a $T_i^s \geq 0$ such that

$$S_i(t) \in [-M - \delta_i, M + \delta_i] \subseteq [-M - \varepsilon, M + \varepsilon] \quad (21)$$

for all $t \geq T_i^s$. Let $T^s = \max_i T_i^s$, then Eq. (21) holds for all $i \in \{1, \ldots, N\}$ and for all $t \geq T^s$.

Now we consider $t \geq T^s$. For $x_i(t) \leq -l_i - \varepsilon$, Eqs. (6) and (21) imply that

$$x_i'(t) \geq a_i(l_i + \varepsilon) + \sum_{j=1}^N D_{ij} f(x_j) + B_i(-M - \delta_i).$$

Using the definition of l_i given by Eq. (8), we find that for $t \geq T^s$ and $x_i(t) \leq -l_i - \varepsilon$,

$$\begin{aligned} x_i'(t) &\geq a_i l_i + a_i\varepsilon - M\left(\sum_{j=1}^N |D_{ij}| + |B_i|\right) - |B_i|\delta_i \\ &= -|B_i|\delta_i + a_i\varepsilon \geq \frac{a_i\varepsilon}{2} > 0. \end{aligned}$$

Similarly, for $t \geq T^s$ and for $x_j(t) \geq l_i + \varepsilon$, Eqs. (6) and (21) imply that

$$x_i'(t) \leq -a_i(l_i + \varepsilon) + \sum_{j=1}^N D_{ij} f(x_j) + B_i(M + \delta_i).$$

Using Eq. (8) again, we find that for $t \geq T^s$ and $x_i(t) \geq l_i + \varepsilon$,

$$\begin{aligned} x_i'(t) &\leq -a_i l_i - a_i\varepsilon + M\left(\sum_{j=1}^N |D_{ij}| + |B_i|\right) + |B_i|\delta_i \\ &= -a_i\varepsilon + |B_i|\delta_i \leq -\frac{a_i\varepsilon}{2} < 0. \end{aligned}$$

Consequently, for any $i \in \{1, \ldots, N\}$ there exists a $T_i^x \geq T^s \geq 0$ such that

$$x_i(t) \in [l_i - \varepsilon, l_i + \varepsilon] \quad (22)$$

for all $t \geq T_i^x$. Let $T = \max_i T_i^x$, then both Eqs. (21) and (22) hold for all $i \in \{1, \ldots, N\}$ and all $t \geq T$. □

Appendix 2: Proof of Theorem 2

Proof

At the equilibrium, $S_i = f(x_i)$ from (7). Substituting these expressions into (6), we obtain the system

$$0 = -a_i x_i + \sum_{j=1}^N D_{ij} f(x_j) + B_i f(x_i), \quad i = 1, \ldots N.$$

Since $a_i > 0$, we can express x_i as

$$\begin{aligned} x_i &= \frac{1}{a_i}\left(\sum_{j=1}^N D_{ij} f(x_j) + B_i f(x_i)\right) \\ &= F_i(x_1, \ldots, x_N). \end{aligned}$$

The inequality (9) implies that

$$\|F(x') - F(x''\| < \|x' - x''\|,$$

where $F = (F_1, \ldots, F_N)$ and $x = (x_1, \ldots, x_N)$ so that F is a contracting map in the sup norm in R^N. Consequently, there exists a unique fixed point of F. The x_i-coordinates of this fixed point uniquely determine the S_i-coordinates of the equilibrium e via $S_i = f(x_i)$. We conclude that the equilibrium e is unique. □

Appendix 3: Conditions on the Interaction of the Fast and Slow States

The following assumptions have to be made:[11]

1. The reduced system (14) has a Lyapunov function $V : \mathbf{R^n} \to \mathbf{R}_+$ such that for all $\mathbf{S} \in \mathbf{B_S}$

$$(\nabla_S V(\mathbf{S}))^T \mathbf{f_r}(\mathbf{S}) \le -\alpha_1 \psi^2(\mathbf{S}), \ \alpha_1 > 0 \quad (23)$$

where $\psi(\mathbf{S})$ is a scalar-valued function of $\mathbf{S}$ that vanishes at $\mathbf{S} = \mathbf{0}$ and is different from zero for all other $\mathbf{S} \in \mathbf{B_S}$.
This condition guarantees that $\mathbf{S} = \mathbf{0}$ is an asymptotically stable equlibrium point of the reduced system (14).
2. The boundary–layer system (15) has a Lyapunov function $W(\mathbf{S}, \mathbf{x})$: $\mathbf{R^n} \times \mathbf{R^m} \to \mathbf{R}_+$ such that for all $\mathbf{S} \in \mathbf{B_S}$ and $\mathbf{x} \in \mathbf{B_x}$

$$(\nabla_x W(\mathbf{S}, \mathbf{x}))^T \mathbf{g}(\mathbf{S}, \mathbf{x}, 0) \le -\alpha_2 \phi^2(\mathbf{x} - \mathbf{h}(\mathbf{S})), \ \alpha_2 > 0 \quad (24)$$

where $\phi(\mathbf{x} - \mathbf{h}(\mathbf{S}))$ is a scalar-valued function $(\mathbf{S} - \mathbf{h}(\mathbf{S})) \in \mathbf{R^m}$ that vanishes at $\mathbf{x} = \mathbf{h}(\mathbf{S})$ and is different from zero for all other $\mathbf{x} \in \mathbf{B_S}$ and $\mathbf{x} \in \mathbf{B_x}$.
This condition guarantees that $\mathbf{x} = \mathbf{h}(\mathbf{S})$ is an asymptotically stable equlibrium point of the boundary–layer system (15).
3. The following three inequalities hold $\forall \mathbf{S} \in \mathbf{B_S}$ and $\forall \mathbf{x} \in \mathbf{B_x}$:

(a)

$$\nabla_S W(\mathbf{S}, \mathbf{x}))^T \mathbf{f}(\mathbf{S}, \mathbf{x}) \le c_1 \phi^2(\mathbf{x} - \mathbf{h}(\mathbf{S})) + c_2 \psi(\mathbf{S}) \phi(\mathbf{x} - \mathbf{h}(\mathbf{S})) \quad (25)$$

(b)

$$(\nabla_S V(\mathbf{S}))^T [\mathbf{f}(\mathbf{S}, \mathbf{x}) - \mathbf{f}(\mathbf{S}, \mathbf{h}(\mathbf{S}))] \le \beta_1 \psi(\mathbf{S}) \phi(\mathbf{x} - \mathbf{h}(\mathbf{S})) \quad (26)$$

(c)

$$(\nabla_x W(\mathbf{S}, \mathbf{x}))^T [\mathbf{g}(\mathbf{S}, \mathbf{x}, \varepsilon) - \mathbf{g}(\mathbf{S}, \mathbf{x}, 0)] \le \varepsilon K_1 \phi^2(\mathbf{x} - \mathbf{h}(\mathbf{S})) + \varepsilon K_2 \psi(\mathbf{S}) \phi(\mathbf{x} - \mathbf{h}(\mathbf{S})) \quad (27)$$

The constants c_1, c_2, β_1, K_1 and K_2 are nonnegative.

The inequalities above determine the permissible interaction between the slow and fast variables. They are basically smoothness requirements of f and g.

The upper bound $\varepsilon^*(d)$ is a positive number and is given by

$$\varepsilon^*(d) = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + [\beta_1(1-d) + \beta_2 d]^2 / 4d(1-d)} \quad (28)$$

where $\beta_2 = K_2 + c_2$, $\gamma = K_1 + c_1$.

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