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Letters

Stability analysis of an unsupervised neural network with feedforward and feedback dynamics

A. Meyer-Baese[a,*], S. Pilyugin[b]

[a]Department of Electrical and Computer Engineering, Florida State University, Tallahassee, FL 32310-6046, USA
[b]Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract

We present a new method of analyzing the dynamics of self-organizing neural networks with different time scales based on the theory of flow invariance. We prove the existence and the uniqueness of the equilibrium. A strict Lyapunov function for the flow of a competitive neural system with different time scales is given and based on it we are able to prove the global asymptotic stability of the equilibrium point.

Keywords: Global asymptotic stability; Hebbian and anti-Hebbian learning

1. Introduction

Cortical cognitive maps developed by self-organization represent an important class of recurrent neural networks.

This paper investigates the dynamical behavior of self-organizing neural networks, combining both the dynamics of neural activity, the short-term memory (STM), and the dynamics of unsupervised synaptic modifications, the long-term memory (LTM). An anti-Hebbian rule is used for the modification of the strengths of feedback synapses between output units while a normal Hebbian rule is used for the weights of the feedforward synapses from input units to output units.

Recently, several articles have discussed neural systems with time-varying weights, mostly employing a supervised learning law [3,2,5]. In a prior work [4], the authors analyzed the global asymptotic stability of a competitive neural network with a combined LTM and STM-dynamics. However, the LTM dynamics is restricted to only the Hebbian learning law to adapt the feedforward synapses. The present work extends the previous neural model by allowing a more detailed dynamical behavior. We compare the newly obtained stability results with those in [4], and also assess conservatism of these stability conditions with respect to those in [4].

In the following, we analyze mathematically a self-organizing neural network with Hebbian and anti-Hebbian learning rules. The general neural network equations describing the temporal evolution of the STM and LTM states for the jth neuron of a N-neuron network are

$$\text{STM:}\quad \varepsilon\dot{x}_j = -a_j x_j + C_j \sum_{i=1}^{N} D_{ij} f(x_i) + B_j \sum_{i=1}^{N} m_{ij} y_i, \tag{1}$$

$$\text{LTM 1:}\quad \dot{m}_{ij} = -m_{ij} + y_i f(x_j), \tag{2}$$

$$\text{LTM 2:}\quad \dot{D}_{ij} = -D_{ij} - f(x_i) f(x_j), \tag{3}$$

where x_j is the current activity level, a_j the time constant of the neuron, C_j the contribution of the lateral stimulus term, B_j the contribution of the external stimulus term, $f(x_i)$ the neuron's output, y_i the external stimulus, and m_{ij} the synaptic efficiency. ε is the fast time-scale associated with the STM state. D_{ij} represents a synaptic connection parameter between the ith neuron and the jth neuron. We assume here, that the recurrent neural network consists of

*Corresponding author.
E-mail address: amb@eng.fsu.edu (A. Meyer-Baese).

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both feedforward and feedback connections between the layers and neurons forming complicated dynamics.

2. Equilibrium and global asymptotic stability analysis of neuro-synaptic systems

In this section, we present a new condition for the uniqueness and global asymptotic stability for neuro-synaptic systems which improves the previous stability results. The existence and uniqueness of the equilibrium is given based on flow-invariance while the global asymptotic stability is shown by a strict Lyapunov function.

The theory of flow-invariance gives a qualitative interpretation of the dynamics of a system, taking into account the invariance of the flow of the system. In other words a trajectory gets trapped in an invariant set.

Before we state the stability results based on the concept of flow-invariance, we will first give some useful definitions used in nonlinear analysis.

2.1. Definitions

Definition 1. Let $F: R^N \to R^N$ be a Lipschitz continuous map and let S be a subset of R^N. We say that S is *flow-invariant* with respect to the system of differential equation

$$x'(t) = F(x(t)), \tag{S}$$

if any solution $x(t)$ starting in S at $t = 0$ remains in S for all $t \geqslant 0$ as long as $x(t)$ is defined. In dynamical systems terminology, such sets are called positively invariant under the flow generated by (S).

Definition 2. We say that $F: R^N \to R^N$ is a Lipschitz continuous map, if and only if there is a constant $K > 0$ such that $|F(x) - F(y)| \leqslant K|x - y|$ with $x, y \in R^N$.

Definition 3. We say that the system (S) is *dissipative* in R^N if there exists a precompact (bounded) set $U \subset R^N$ such that for any solution $x(t)$ of (S) there exists $T \geqslant 0$ such that $x(t) \in U$ for all $t \geqslant T$ [1].In other words, all solutions of (S) enter this bounded set U in finite time.

If (S) is dissipative then all solutions of (S) are defined for $t \geqslant 0$, and there exists a compact set $A \subset U$ which attracts all solutions of (S). The set A is invariant under the flow of (S) and it is called the *global attractor* of (S) in R^N.

2.1.1. Results

Theorem 1. *Suppose that f is locally Lipschitz and bounded, that is,* $|\mathbf{f}(x)| \leqslant M$. *Also suppose that* $a_i > 0$ *and* $|y_i| \leqslant 1$ *for all* $i = 1, \ldots, N$. *Then all solutions of* (1)–(3) *are defined for all* $t \in (-\infty, +\infty)$. *Furthermore, the system* (1)–(3) *is dissipative in* R^{3N}.

Proof. Since f is locally Lipschitz, then so is the vector field $F(\mathbf{x}, \mathbf{m}, \mathbf{D})$ of (1)–(3). Therefore, the solutions are defined locally. We introduce positive constants

$$K_1 = \max_j a_j, \quad K_2 = \max_j |C_j|, \quad K_3 = \max_j |B_j|,$$

$$K_4 = M, \quad K_5 = M^2$$

and observe that

$$|\dot{x}_j| \leqslant K_1|x_j| + K_2 M|D_{ij}| + K_3|m_{ij}|,$$

$$|\dot{m}_{ij}| \leqslant K_4 + |m_{ij}|,$$

$$|\dot{D}_{ij}| \leqslant K_5 + |D_{ij}|.$$

It follows that any local solution of (1)–(3) can be extended to a global solution defined for all $t \in (-\infty, +\infty)$. Let $h > 0$ and

$$L_j = \frac{1}{a_j}\sum_{i=1}^{N}(|B_j|M + |C_j|M^3).$$

There exist constants $0 < \delta_j < h$ such that

$$\sum_{i=1}^{N}\delta_j(|C_j| + |B_j|M) < \frac{a_j h}{2}, \quad j = 1, \ldots, N.$$

Consider a solution $(\mathbf{x}(t), \mathbf{m}(t), \mathbf{D}(t))$ of (1)–(3) with $t \geqslant 0$. If $m_{ij}(t) \leqslant -M - \delta_j$, then

$$\dot{m}_{ij}(t) \geqslant -(-M - \delta_j) - |f(x_j(t))||y_i| \geqslant \delta_j > 0.$$

Similarly, if $m_{ij}(t) \geqslant M + \delta_j$, then

$$\dot{m}_{ij} \leqslant -(M + \delta_j) + |f(x_j(t))||y_i| \leqslant -\delta_j < 0.$$

Therefore, for any $j = 1, \ldots, N$, there exists a $T_j^S \geqslant 0$ such that

$$m_{ij}(t) \in [-(M + \delta_j), M + \delta_j] \subseteq [-(M + h), M + h]$$

for all $t \geqslant T_j^S$. A similar argument shows that for any $j = 1, \ldots, N$, there exists a $T_j^t \geqslant 0$ such that

$$D_{ij}(t) \in [-(M^2 + \delta_j), M^2 + \delta_j] \subseteq [-(M^2 + h), M^2 + h]$$

for all $t \geqslant T_j^t$. Let $T^{s,t} = \max_j\{T^s, T_j^t\}$ and consider $x_j(t)$ with $t \geqslant T^{s,t}$. If $x_j(t) \geqslant L_j + h$ and $t \geqslant T^{s,t}$ then

$$\dot{x}_j(t) \leqslant -a_j(L_j + h) + |B_j|\sum_{i=1}^{N}(M + \delta_j) + |C_j|\sum_{i=1}^{N}(M^3 + \delta_j),$$

therefore

$$\dot{x}_j(t) \leqslant -a_j h + \delta_j\sum_{i=1}^{N}(|C_j| + |B_j|) < -\frac{a_j h}{2} < 0.$$

Similarly, if $x_j(t) \leqslant -(L_j + h)$ and $t \geqslant T^{s,t}$ then

$$\dot{x}_j(t) \geqslant \frac{a_j h}{2} > 0.$$

Consequently, there exists $T_j^x \geqslant T^{s,t}$ such that

$$x_j(t) \in [-(L_j + h), L_j + h]$$

for all $t \geqslant T_j^x$. Let $T = \max_j T_j^x$, then

$$(\mathbf{x}(t), \mathbf{m}(t), \mathbf{D}(t)) \in H = \prod_j I_j^x \times \prod_j I_j^s \times \prod_j I_j^t$$

for all $t \geqslant T$. Here, $I_j^x = [-(L_j + h), L_j + h]$, $I_j^s = [-(M+h), M+h]$, and $I_j^x = [-(M^2 + h), M^2 + h]$. We conclude that (1)–(3) is dissipative in R^{3N}. □

Corollary 1. *The system* (1)–(3) *admits a compact global attractor* $A \subset H$.

Corollary 2. *Since* H *is a direct product of intervals, it is a contractible set. In addition to being contractible,* H *is forward invariant under the flow* ϕ_t *of* (1)–(3), *that is,* $\phi_t: H \to H$ *for any* $t \geqslant 0$. *Using the Brower fixed point theorem* [6] *and compactness of* H, *we conclude that there exists an equilibrium* $e \in H$ *of* (1)–(3).

In the following, we will assume that $C_j = B_j = 1$.

Theorem 2. *Let* $e = (\mathbf{x^0}, \mathbf{m^0}, \mathbf{D^0})$ *be an equilibrium of* (1)–(3). *Let* k *be Lipschitz constant for* f *and* $M = \max_i\{|f_i(\phi_i)|, |f(\phi_i + x^0)|\}$. *If* $1 > k$, $2 > M(1 + Mk)$ *and* $a_j > N \cdot (M + 1 + 3M^2k + k)/2$ *then* e *is an asymptotically globally attracting equilibrium of* (1)–(3).

Proof. Let $\phi_j = x_j - x_j^0$, $\psi_{ij} = m_{ij} - m_{ij}^0$, and $\xi_{ij} = D_{ij} - D_{ij}^0$. Substituting ϕ_j, ψ_{ij} and ξ_{ij} into (1)–(3), we obtain

$$\dot{\phi}_j = -a_j\phi_j + \sum_{i=1}^{N} \xi_{ij} f(\phi_i + x_i^0) - \sum_{i=1}^{N} f(x_i^0) f(x_j^0) f_i(\phi_i) + \sum_{i=1}^{N} \psi_{ij} y_i, \tag{4}$$

$$\dot{\psi}_{ij} = -\psi_j + f_j(\phi_j) y_i, \tag{5}$$

$$\dot{\xi}_{ij} = -\xi_{ij} - f(\phi_j + x_j^0) f_i(\phi_i) f(x_j^0), \tag{6}$$

where $f_j(\phi_j) = f(x_j^0 + \phi_j) - f(x_j^0)$. Let

$$V = \frac{1}{2}\sum_{j=1}^{N} \phi_j^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \psi_{ij}^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_{ij}^2. \tag{7}$$

Differentiating V with respect to the flow of (1)–(3) and considering the inequality $|ab| \leqslant \frac{1}{2}(a^2 + b^2), \forall a, b \in R$, we find that

$$\begin{aligned}
\dot{V} &= \sum_{j=1}^{N} \dot{\phi}_j\phi_j + \sum_{j=1}^{N}\sum_{i=1}^{N} \dot{\psi}_{ij}\psi_{ij} + \sum_{j=1}^{N}\sum_{i=1}^{N} \dot{\xi}_{ij}\xi_{ij} \\
&\leqslant -\sum_{j=1}^{N} a_j|\phi_j|^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} |\psi_{ij}|^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} |\xi_{ij}|^2 \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} M|\phi_j||\xi_{ij}| + \sum_{i=1}^{N}\sum_{j=1}^{N} |\psi_{ij}||\phi_j| \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} M^2k|\phi_i|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} k|\psi_{ij}||\phi_i| \\
&\quad + \sum_{j=1}^{N}\sum_{i=1}^{N} M^2k|\phi_i||\xi_{ij}| \\
&\leqslant -\sum_{j=1}^{N} a_j|\phi_j|^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} |\psi_{ij}|^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} |\xi_{ij}|^2 \\
&\quad + \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{M}{2}(|\xi_{ij}|^2 + |\phi_j|^2) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(|\psi_{ij}|^2 + |\phi_j|^2) \\
&\quad + M^2k\sum_{j=1}^{N}\sum_{i=1}^{N} |\phi_j|^2 + \frac{k}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}(|\psi_{ij}|^2 + |\phi_j|^2) \\
&\quad + \frac{M^2k}{2}\sum_{j=1}^{N}\sum_{i=1}^{N}(|\xi_{ij}|^2 + |\phi_j|^2) \\
&\leqslant -\sum_{j=1}^{N}\left(a_j - N\frac{M + 1 + 3M^2k + k}{2}\right)|\phi_j|^2 \\
&\quad - \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1-k}{2}|\psi_{ij}^2| \\
&\quad - \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{2 - M - M^2k}{2}|\xi_{ij}|^2 \leqslant 0.
\end{aligned} \tag{8}$$

We conclude that V is a strict Lyapunov function for (1)–(3) and that the origin is an asymptotically globally attracting equilibrium of (1)–(3). Therefore, e is an asymptotically globally attracting equilibrium of (1)–(3). □

3. Conclusions

In this paper we prove global asymptotic stability of self-organizing neural networks with Hebbian and anti-Hebbian learning rules. Based on the flow invariance technique we can show the conditions that the LTM and STM trajectories are bounded. We also presented a strict Lyapunov function and based on it we have shown global asymptotic stability of the equilibrium point. Besides proving the existence and uniqueness of the equilibrium, we are presenting milder and more general conditions than for a simpler neural system based on a Hebbian adaptation rule for the feedforward synapses.

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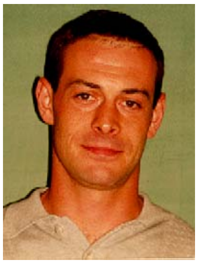
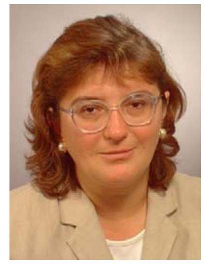


Anke Meyer-Baese is currently an associate professor in the department of Mathematics at the University of Florida. He received his Ph.D. in Mathematics from Emory University and was a post-doctoral fellow in mathematical biology at Emory University and Georgia Institute of Technology. His research interests include biological applications of differential equations and dynamical systems.



Sergei S. Pilyugin is currently an associate professor in the Department of Electrical and Computer Engineering at Florida State University in Tallahaseee. She received her Ph.D. in electrical engineering from the Darmstadt Institute of Technology in Germany.

Her research interests include biomedical applications of pattern recognition techniques and stability aspects of neural dynamical systems.