Problem 1. Let $a_n > 0$, $s_n = a_1 + \ldots + a_n$, and suppose that $\sum a_n$ diverges. Prove that of the two series
\[
\sum \frac{a_n}{1 + a_n}, \quad \text{and} \quad \sum \frac{a_n}{s_n^2},
\]
the former diverges, and the latter converges. What can be said about convergence/divergence of
\[
\sum \frac{a_n}{1 + n a_n}, \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?
\]
Hint: $\{a_n\}$ is not necessarily monotone, and it does not necessarily converge to 0.

Problem 2. A sequence $a_n$ is said to be of bounded variation if $\sum |a_{n+1} - a_n|$ converges. Prove that if $a_n$ is of bounded variation, then the sequence $a_n$ converges. Is the converse true?

Problem 3. Let $f : [0, +\infty)$ be a function which is Riemann integrable on any finite interval $[0, M]$ and suppose that a finite limit $A = \lim_{x \to \infty} f(x)$ exists. Prove that the limit of the average (as defined below) value of $f$ also exists and that
\[
\lim_{x \to \infty} \frac{1}{x} \int_0^x f(z) \, dz = A.
\]

Problem 4. Prove or disprove: if a sequence of integrable functions $f_n$ converges pointwise to an integrable function $f$, and
\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx,
\]
then the convergence is uniform.

Problem 5. Let $a_n$ be a nonnegative sequence monotonically converging to zero. Prove or disprove: the series $\sum a_n$ and $\sum 3^k a_{3^k}$ converge or diverge together.

Problem 6. (a) Suppose the series $\sum a_n$ converges absolutely. Prove that for any integer $p = 2, 3, 4, \ldots$ the series $\sum a_n^p$ converges absolutely. (b) Suppose the series $\sum a_n$ converges conditionally. For which integers $p = 2, 3, 4, \ldots$ is the series $\sum a_n^p$ guaranteed to converge?