a) Obtain a map that shows routes that they may take from San Francisco to New Orleans. Write directions for them from San Francisco to New Orleans via the shortest distance. Use major highways whenever possible.

b) Use the scale on the map to estimate the one-way distance to New Orleans.

c) If the Williams estimate that they will average 50 mph (including comfort stops), estimate the travel time, in hours, to New Orleans.

d) If the Williams want to travel about 400 miles per day, locate a town in the vicinity of where they will stop each evening.

e) If they begin each segment of the trip each day at 9 A.M., at about what time will they look for a hotel each evening?

f) Use the information provided in parts (a) through (e) to estimate the time of day they will arrive in New Orleans.

g) Estimate the mileage of a typical midsized car and the cost per gallon of a gallon of regular unleaded gasoline. Then estimate the cost of gasoline for the Williams' trip.

h) Estimate the cost of a typical breakfast, a typical lunch, and a typical dinner for two adults, and the cost of a typical motel room. Then estimate the total cost, including meals, gas, and lodging, for the Williams' trip from San Francisco to New Orleans (one way).

Problem Solving

3. Four acrobats who bill themselves as the "Tumbling Tumbleweeds" finish up their act with the amazing "Human Pillar," in which the acrobats form a tower, each one standing on the shoulders of the one below. Each acrobat (Ernie, Jed, Tex, and Zeke Tumbleweed) wears a different distinctive item of western garb (chaps, holster, Stetson hat, or leather vest) in the act. Can you identify the members of the "Human Pillar," from top to bottom, by name and apparel?

   a) Jed Tumbleweed is not on top, but he is somewhere above the man in the Stetson.

   b) Zeke Tumbleweed does not wear the holster.

   c) The man in the vest is not on top.

   d) The man in the chaps is somewhere above Tex but somewhere below Zeke.

ORDER	NAME	APPAREL

SECTION 2.1 Set Concepts

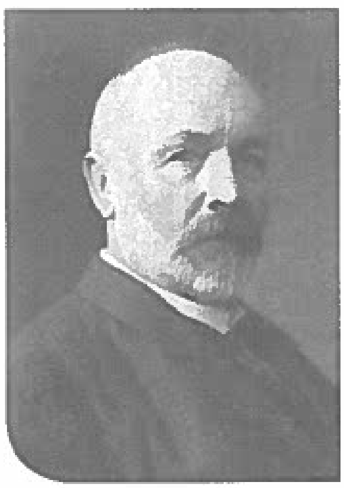
▲ *Many different restaurant categories include McDonald's*

Can you think of a few different categories or groups to which the restaurant McDonald's belongs? One way you could categorize McDonald's is as a fast food restaurant. Another way is as a restaurant selling hamburgers. A third way is as a restaurant selling breakfast. In this section, we will discuss ways to sort or classify objects. We will also discuss different methods that can be used to indicate collections of objects.

Why This Is Important Set classifications are important in a range of applications from placing students in courses to classifying stars in the universe.

Profile in Mathematics

Georg Cantor



Georg Cantor (1845–1918), born in St. Petersburg, Russia, is recognized as the founder of set theory. Cantor's creative work in mathematics was nearly lost when his father insisted that he become an engineer rather than a mathematician. His two major books on set theory, *Foundations of General Theory of Aggregates* and *Contributions to the Founding of the Theory of Transfinite Numbers,* were published in 1883 and 1895, respectively. See the Profile in Mathematics on page 85 for more information on Cantor and Leopold Kronecker.

We encounter sets in many different ways every day of our lives. A ***set*** is a collection of objects, which are called ***elements*** or ***members*** of the set. For example, the United States is a collection, or set, of 50 states plus the District of Columbia. The 50 individual states plus the District of Columbia are the members or elements of the set that is called the United States.

A set is ***well defined*** if its contents can be clearly determined. The set of U.S. presidents is a well-defined set because its contents, the presidents, can be named. The set of the three best movies is not a well-defined set because the word *best* is interpreted differently by different people. In this text, we use only well-defined sets.

Three methods are commonly used to indicate a set: (1) description, (2) roster form, and (3) set-builder notation.

The method of indicating a set by ***description*** is illustrated in Example 1.

Example 1 *Description of Sets*

Write a description of the set containing the elements Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday.

Solution The set is the days of the week. ■

Listing the elements of a set inside a pair of ***braces***, $\{\ \ \}$, is called ***roster form***. The braces are an essential part of the notation because they identify the contents as a set. For example, $\{1, 2, 3\}$ is notation for the set whose elements are 1, 2, and 3, but (1, 2, 3) and [1, 2, 3] are not sets because parentheses and brackets do not indicate a set. For a set written in roster form, commas separate the elements of the set. The order in which the elements are listed is not important.

Sets are generally named with capital letters. For example, the name commonly selected for the set of ***natural numbers*** or ***counting numbers*** is N.

Definition: Natural Numbers

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

The three dots after the 5, called an ***ellipsis***, indicate that the elements in the set continue in the same manner. An ellipsis followed by a last element indicates that the elements continue in the same manner up to and including the last element. This notation is illustrated in Example 2(b).

Solution

a) Because set B consists of the natural numbers less than 6, we write

$$B = \{x \mid x \in N \text{ and } x < 6\}$$

Another acceptable answer is $B = \{x \mid x \in N \text{ and } x \leq 5\}$.

b) Set B is the set of all elements x such that x is a natural number and x is less than 6. ■

Example 5 *Roster Form to Set-Builder Notation*

a) Write set $C = \{$North America, South America, Europe, Asia, Australia, Africa, Antarctica$\}$ in set-builder notation.

b) Write in words how you would read set C in set-builder notation.

Solution

a) $C = \{x \mid x \text{ is a continent}\}$.

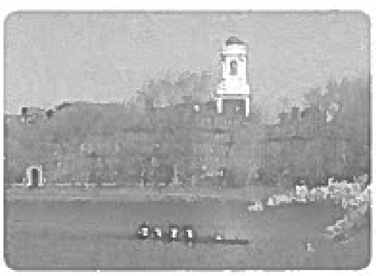
b) Set C is the set of all elements x such that x is a continent. ■

Example 6 *Set-Builder Notation to Roster Form*

Write set $A = \{x \mid x \in N \text{ and } 2 \leq x < 8\}$ in roster form.

Solution $A = \{2, 3, 4, 5, 6, 7\}$ ■

Example 7 *Oldest Colleges in the United States*



▲ *Harvard University*

The table shows the 10 oldest colleges in the United States. Let set C be the set of colleges that are located in Virginia that are among the 10 oldest colleges in the United States. Write set C in roster form.

Ten Oldest Colleges in the United States	State	Year Chartered
Harvard University	Massachusetts	1636
College of William and Mary	Virginia	1692
Yale University	Connecticut	1701
University of Pennsylvania	Pennsylvania	1740
Moravian College	Pennsylvania	1742
Princeton University	New Jersey	1746
Washington and Lee University	Virginia	1749
Columbia University	New York	1754
Brown University	Rhode Island	1764
Rutgers	New Jersey	1766

Source: National Center for Education Statistics

Solution By examining the table, we find that two colleges located in Virginia appear in the table. They are College of William and Mary and Washington and Lee University. Thus, set $C = \{$College of William and Mary, Washington and Lee University$\}$. ■

A set is said to be ***finite*** if it either contains no elements or the number of elements in the set is a natural number. The set $B = \{2, 4, 6, 8, 10\}$ is a finite set because the number of elements in the set is 5, and 5 is a natural number. A set that

Null or Empty Set

Some sets do not contain any elements, such as the set of zebras that live in your house.

Definition: Empty Set

The set that contains no elements is called the **empty set** or **null set** and is symbolized by $\{\ \}$ or $\varnothing$.

Note that $\{\varnothing\}$ is not the empty set. This set contains the element $\varnothing$ and has a cardinality of 1. The set $\{0\}$ is also not the empty set because it contains the element 0. It also has a cardinality of 1.

Example 8 *Natural Number Solutions*

Indicate the set of natural numbers that satisfies the equation $x + 2 = 0$.

Solution The values that satisfy the equation are those natural numbers that make the equation a true statement. Only the number -2 satisfies this equation. Because -2 is not a natural number, the solution set of this equation is $\{\ \}$ or $\varnothing$. ■

Universal Set

Another important set is a *universal set.*

Definition: Universal Set

A **universal set**, symbolized by U, is a set that contains all the elements for any specific discussion.

When a universal set is given, only the elements in the universal set may be considered when working the problem. If, for example, the universal set for a particular problem is defined as $U = \{1, 2, 3, 4, \ldots, 10\}$, then only the natural numbers 1 through 10 may be used in that problem.

SECTION 2.1 Exercises

Warm Up Exercises

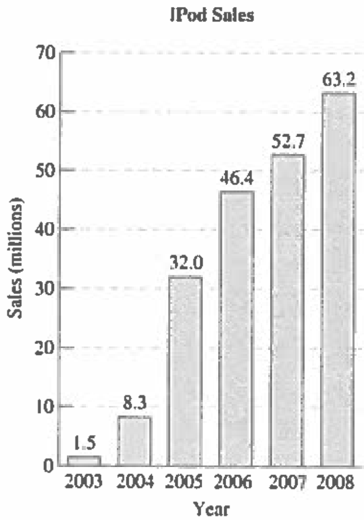
In Exercises 1–12, fill in the blank with an appropriate word, phrase, or symbol(s).

1. A collection of objects is called a(n) ________.
2. Three dots placed in a set to show that the set continues in the same manner is called a(n) ________.
3. The three ways a set can be written are ________, ________, and ________.
4. A set that contains no elements or the number of elements in the set is a natural number is called a(n)________ set.
5. A a set that is not finite is called a(n) ________. set.
6. Two sets that contain the same elements are called ________ sets.
7. Two sets that contain the same number of elements are called ________ sets.
8. The number of elements in a set is called the ________ number.
9. The set that contains no elements is called the ________ set.
10. The two ways to indicate an empty set are ________ and ________.
11. A set that contains all the elements for any specific discussion is called a(n) ________ set.
12. Two sets that have the same cardinal number can be placed in a(n) ________ correspondence.

40. The set of years in which iPod sales were less than 8 million

41. The set of years in which iPod sales were between 8 million and 60 million

42. The set of years in which iPod sales were more than 65 million



In Exercises 43–50, express each set in set-builder notation.

43. $B = \{7, 8, 9, 10, 11, 12, 13, 14\}$

44. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

45. $C = \{3, 6, 9, 12, \ldots\}$

46. $D = \{5, 10, 15, 20, \ldots\}$

47. E is the set of odd natural numbers.

48. A is the set of national holidays in the United States in July.

49. C is the set of months that contain less than 30 days.

50. $F = \{15, 16, 17, \ldots, 100\}$

In Exercises 51–58, write a description of each set.

51. $A = \{1, 2, 3, 4, 5, 6, 7\}$

52. $D = \{3, 6, 9, 12, 15, 18, \ldots\}$

53. $V = \{a, e, i, o, u\}$

54. $S = \{$Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy$\}$

▲ *Snow White and the Seven Dwarfs*

55. $T = \{$oak, maple, elm, pine, …$\}$

56. $E = \{x \mid x \in N \text{ and } 4 \leq x < 11\}$

57. $S = \{$spring, summer, fall, winter$\}$

58. $B = \{$John Lennon, Ringo Starr, Paul McCartney, George Harrison$\}$

▲ *The Beatles*

In Exercises 59–62, use the following list, which shows the 10 countries with the most cellular subscribers, in millions, as of 2008. Let the 10 countries in the list represent the universal set.

Country	Number of Subscribers
1. China	649.70
2. India	376.12
3. United States	260.00
4. Russia	172.00
5. Brazil	151.90
6. Indonesia	115.60
7. Japan	102.98
8. Germany	101.50
9. Pakistan	91.40
10. United Kingdom	70.00

Source: CIA

Use the list to determine each set in roster form.

59. $\{x \mid x$ is a country with at least 250 million cellular subscribers$\}$

60. $\{x \mid x$ is a country with fewer than 100 million cellular subscribers$\}$

61. $\{x \mid x$ is a country with between 100 million and 200 million cellular subscribers$\}$

62. $\{x \mid x$ is a country with between 250 million and 500 million cellular subscribers$\}$

In Exercises 87–90, determine whether the number used is a cardinal number or an ordinal number.

87. J. K. Rowling has written 7 Harry Potter books.

▲ *J. K. Rowling*

88. Study the chart on page 25 in the book.

89. Lincoln was the sixteenth president of the United States.

90. Emily paid $35 for her new blouse.

91. Describe three sets of which you are a member.

92. Describe three sets that have no members.

93. Write a short paragraph explaining why the universal set and the empty set are necessary in the study of sets.

Challenge Problem/Group Activity

94. a) In a given exercise, a universal set is not specified, but we know that actor Orlando Bloom is a member of the universal set. Describe five different possible universal sets of which Orlando Bloom is a member.

b) Write a description of one set that includes all the universal sets in part (a).

Internet/Research Activity

95. Georg Cantor is recognized as the founder and a leader in the development of set theory. Do research and write a paper on his life and his contributions to set theory and to the field of mathematics. References include history of mathematics books, encyclopedias, and the Internet.

SECTION 2.2 Subsets

▲ *The set of intercollegiate sports includes basketball.*

Consider the following sets. Set $A = \{\text{baseball, basketball, hockey}\}$. Set $B = \{\text{baseball, football, basketball, hockey, softball}\}$. Note that each element of set A is also an element of set B. In this section, we will discuss how to illustrate the relationship between two sets, A and B, when each element of set A is also an element of set B.

Why ***This Is Important*** The relationship between sets is important throughout life. For example, to gain a promotion at work, you may need to fulfill different sets of criteria.

In our complex world, we often break larger sets into smaller, more manageable sets, called *subsets*. For example, consider the set of people in your class. Suppose we categorize the set of people in your class according to the first letter of their last name (the A's, B's, C's, etc.). When we do so, each of these sets may be considered a subset of the original set. Each of these subsets can be separated further. For example, the set of people whose last name begins with the letter A can be categorized as either male or female or by their age. Each of these collections of people is also a subset. A given set may have many different subsets.

Definition: Subset

Set A is a **subset** of set B, symbolized by $A \subseteq B$, if and only if all the elements of set A are also elements of set B.

The symbol $A \subseteq B$ indicates that "set A is a subset of set B." The symbol $\not\subseteq$ is used to indicate "is not a subset." Thus, $A \not\subseteq B$ indicates that set A is not a subset of set B. *To show that set A is not a subset of set B, we must find at least one element of set A that is not an element of set B.*

MATHEMATICS TODAY

The Ladder of Life

In biology, the science of classifying all living things is called *taxonomy*. More than 2000 years ago, Aristotle formalized animal classification with his "ladder of life": higher animals, lower animals, higher plants, lower plants. Today, living organisms are classified into six kingdoms (or sets) called animalia, plantae, archaea, eubacteria, fungi, and protista. Even more general groupings of living things are made according to shared characteristics. The groupings, from most general to most specific, are kingdom, phylum, class, order, family, genus, and species. For example, a zebra, *Equus burchelli*, is a member of the genus *Equus*, as is the horse, *Equus caballus*. Both the zebra and the horse are members of the universal set called the kingdom of animals and the same family, Equidae; they are members of different species (*E. burchelli* and *E. caballus*), however.

Why This is Important Scientists use sets to classify and catorgize animals, plants, and all forms of life.

Let $A = \{\ \}$ and $B = \{1, 2, 3, 4\}$. Is $A \subseteq B$? To show $A \not\subseteq B$, you must find at least one element of set A that is not an element of set B. Because this cannot be done, $A \subseteq B$ must be true. Using the same reasoning, we can show that *the empty set is a subset of every set, including itself.*

Example 3 *Element or Subset?*

Determine whether the following are true or false.

a) $3 \in \{3, 4, 5\}$
b) $\{3\} \in \{3, 4, 5\}$
c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$
d) $\{3\} \subseteq \{3, 4, 5\}$
e) $3 \subseteq \{3, 4, 5\}$
f) $\{\ \} \subseteq \{3, 4, 5\}$

Solution

a) $3 \in \{3, 4, 5\}$ is a true statement because 3 is an element of the set $\{3, 4, 5\}$.
b) $\{3\} \in \{3, 4, 5\}$ is a false statement because $\{3\}$ is a set, and the set $\{3\}$ is not an element of the set $\{3, 4, 5\}$.
c) $\{3\} \in \{\{3\}, \{4\}, \{5\}\}$ is a true statement because $\{3\}$ is an element in the set. The elements of the set $\{\{3\}, \{4\}, \{5\}\}$ are themselves sets.
d) $\{3\} \subseteq \{3, 4, 5\}$ is a true statement because every element of the first set is an element of the second set.
e) $3 \subseteq \{3, 4, 5\}$ is a false statement because the 3 is not in braces, so it is not a set and thus cannot be a subset. The 3 is an element of the set as indicated in part (a).
f) $\{\ \} \subseteq \{3, 4, 5\}$ is a true statement because the empty set is a subset of every set. ■

Number of Subsets

How many distinct subsets can be made from a given set? The empty set has no elements and has exactly one subset, the empty set. A set with one element has two subsets. A set with two elements has four subsets. A set with three elements has eight subsets. This information is illustrated in Table 2.1.

Table 2.1 Number of Subsets

Set	Subsets	Number of Subsets
$\{\ \}$	$\{\ \}$	$1 = 2^0$
$\{a\}$	$\{a\}$ $\{\ \}$	$2 = 2^1$
$\{a, b\}$	$\{a, b\}$ $\{a\}, \{b\}$ $\{\ \}$	$4 = 2 \times 2 = 2^2$
$\{a, b, c\}$	$\{a, b, c\}$ $\{a, b\}, \{a, c\}, \{b, c\}$ $\{a\}, \{b\}, \{c\}$ $\{\ \}$	$8 = 2 \times 2 \times 2 = 2^3$

SECTION 2.2 Exercises

Warm Up Exercises

In Exercises 1–4, fill in the blank with an appropriate word, phrase, or symbol(s).

1. If all the elements of set A are also elements of set B, then set A is a(n) ________ of set B.

2. If all the elements of set A are also elements of set B, and set $A \neq$ set B, then set A is a (n) ________ subset of set B.

3. The expression for determining the number of distinct subsets for a set with n distinct elements is ________ .

4. The expression for determining the number of distinct proper subsets for a set with n distinct elements is ________ .

Practice the Skills

In Exercises 5–26, answer true or false. If false, give the reason.

5. { book } $\subseteq$ { magazine, newspaper, book }

6. { Italy } $\subseteq$ { Italy, Spain, France, Switzerland, Austria }

7. { McIntosh, Red Delicious } $\subseteq$ { Empire, Gala, Cortland, Red Delicious }

8. { pepper, salt } $\subseteq$ { salt, butter, mayonnaise }

9. { motorboat, kayak } $\subset$ { kayak, fishing boat, motorboat, sailboat }

10. { polar bear, tiger, lion } $\subset$ { tiger, lion, polar bear, penguin }

11. $\{4, 2, 7\} \subset \{4, 7, 2\}$

12. $\{c, a, r, t\} \subset \{t, r, a, c\}$

13. Xbox 360 $\in$ { PSIII, Wii, Xbox 360 }

14. LaGuardia $\in$ { JFK, LaGuardia, Newark }

15. { swimming } $\in$ { sailing, water skiing, swimming }

16. $\{\ \} \in \{1, 3, 5, 7\}$

17. $5 \notin \{2, 4, 6\}$

18. { } $\subseteq$ { table, chair, sofa }

19. { red } $\subset$ { red, blue, green }

20. $\{3, 5, 9\} \not\subset \{3, 9, 5\}$

21. $\{\ \} = \{\varnothing\}$

22. $\varnothing = \{\ \}$

23. $\{0\} = \varnothing$

24. $\{\ \} \subseteq \{\ \}$

25. $0 = \{\ \}$

26. $\{1\} \in \{\{1\}, \{2\}, \{3\}\}$

In Exercises 27–34, determine whether $A = B$, $A \subseteq B$, $B \subseteq A$, $A \subset B$, $B \subset A$, or if none of these applies. (There may be more than one answer.)

27. A = { penny, nickel, dime, quarter }
 B = { penny, quarter }

28. $A = \{x \mid x \in N \text{ and } x < 6\}$
 $B = \{x \mid x \in N \text{ and } 1 \le x \le 5\}$

29. Set A is the set of states that border the Atlantic Ocean. Set B is the set of states east of the Mississippi River.

30 $A = \{1, 3, 5, 7, 9\}$
 $B = \{3, 9, 5, 7, 6\}$

31. A = { x | x is a brand of soft drink }
 B = { A & W, Coca-Cola, Dr Pepper, Mountain Dew }

32. A = { x | x is a sport that uses a ball }
 B = { basketball, soccer, tennis }

33. Set A is the set of natural numbers between 2 and 7. Set B is the set of natural numbers greater than 2 and less than 7.

34. Set A is the set of all cars manufactured by General Motors. Set B is the set of sports cars manufactured by General Motors.

In Exercises 35–38, list all the subsets of the sets given.

35. $D = \varnothing$

36. $A = \{\bigcirc\}$

37. B = { cow, horse }

SECTION 2.3 Venn Diagrams and Set Operations

▲ *Some Laptops have A 14-inch display, some laptops have 4 GB of memory, and some laptops have a 14-inch display and 4 GB of memory.*

Suppose you go to a store to purchase a new laptop and tell a computer salesperson that you wish to purchase a laptop with a 14-inch display *and* 4 GB of memory. The salesperson was a bit distracted and thought you said you wanted to purchase a laptop with a 14-inch display *or* 4 GB of memory. Which laptops are in the set of laptops with a 14-inch display *and* 4 GB of memory? Which laptops are in the set of laptops with a 14-inch display *or* 4 GB of memory? These two questions are quite different. The first involves laptops joined by the word *and*. The second involves laptops joined by the word *or*. In this section, you will learn how to illustrate these and other set relationships.

Why *This is Important* Words such as *and* and *or* have important meaning in a variety of everyday applications, such as ordering from a menu or understanding the meaning of a legal document.

A useful technique for illustrating set relationships is the Venn diagram, named for English mathematician John Venn (1834–1923). Venn invented the diagrams and used them to illustrate ideas in his text on symbolic logic, published in 1881.

In a Venn diagram, a rectangle usually represents the universal set, U. The items inside the rectangle may be divided into subsets of the universal set. The subsets are usually represented by circles. In Fig. 2.1, the circle labeled A represents set A, which is a subset of the universal set.

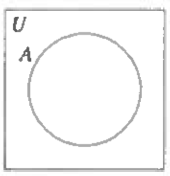


Figure 2.1

Two sets may be represented in a Venn diagram in any of four different ways, as shown in Fig. 2.2. Two sets A and B are ***disjoint*** when they have no elements in common. Two disjoint sets A and B are illustrated in Fig. 2.2(a). If set A is a proper subset of set B, $A \subset B$, the two sets may be illustrated as in Fig. 2.2(b). If set A contains exactly the same elements as set B, that is, $A = B$, the two sets may be illustrated as in Fig. 2.2(c). Two sets A and B with some elements in common are shown in Fig. 2.2(d), which is regarded as the most general form of a Venn diagram.

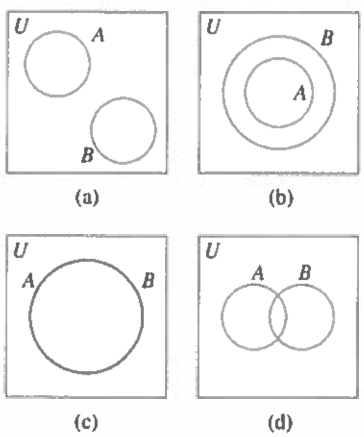


Figure 2.2

If we label the regions of the diagram in Fig. 2.2(d) using I, II, III, and IV, we can illustrate the four possible cases with this one diagram, Fig. 2.3.

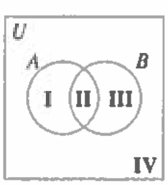


Figure 2.3

CASE 1: DISJOINT SETS When sets A and B are disjoint, they have no elements in common. Therefore, region II of Fig. 2.3 is empty.

CASE 2: SUBSETS When $A \subseteq B$, every element of set A is also an element of set B. Thus, there can be no elements in region I of Fig. 2.3. If $B \subseteq A$, however, then region III of Fig. 2.3 is empty.

CASE 3: EQUAL SETS When set A = set B, all the elements of set A are elements of set B and all the elements of set B are elements of set A. Thus, regions I and III of Fig. 2.3 are empty.

CASE 4: OVERLAPPING SETS When sets A and B have elements in common, those elements are in region II of Fig. 2.3. The elements that belong to set A but not to set B are in region I. The elements that belong to set B but not to set A are in region III.

In each of the four cases, any element belonging to the universal set but not belonging to set A or set B is placed in region IV.

Next we introduce set operations. Venn diagrams will be helpful in understanding set operations. The basic operations of arithmetic are $+$, $-$, $\times$, and $\div$. When we see these symbols, we know what procedure to follow to determine the answer. Some of the operations in set theory are $'$, $\cap$, $\cup$, $-$, and $\times$. They represent complement, intersection, union, difference, and Cartesian product, respectively.

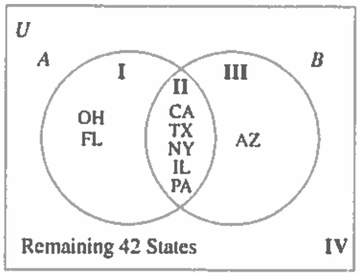


Figure 2.7

Solution First determine the intersection of sets A and B. The states common to both sets are California, Texas, New York, Illinois, and Pennsylvania. Therefore,

$$A \cap B = \{\text{California, Texas, New York, Illinois, Pennsylvania}\}$$

Place these elements in region II of Fig. 2.7. Complete region I by determining the elements in set A that have not been placed in region II. Therefore, Ohio and Florida are placed in region I. Complete region III by determining the elements in set B that have not been placed in region II. Thus, Arizona is placed in region III. Finally, place those elements in U that are not in either set within the rectangle but are outside both circles. This group includes the remaining 42 states, which are placed in region IV. ■

Example 3 *The Intersection of Sets*

Given

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 8\}$$
$$B = \{1, 3, 6, 7, 8\}$$
$$C = \{\ \}$$

find

a) $A \cap B$. b) $A \cap C$. c) $A' \cap B$. d) $(A \cap B)'$.

Solution

a) $A \cap B = \{1, 2, 3, 8\} \cap \{1, 3, 6, 7, 8\} = \{1, 3, 8\}$. The elements common to both set A and set B are 1, 3, and 8.

b) $A \cap C = \{1, 2, 3, 8\} \cap \{\ \} = \{\ \}$. There are no elements common to both set A and set C.

c) To determine $A' \cap B$, we must first determine A'.

$$A' = \{4, 5, 6, 7, 9, 10\}$$
$$A' \cap B = \{4, 5, 6, 7, 9, 10\} \cap \{1, 3, 6, 7, 8\} = \{6, 7\}$$

d) To find $(A \cap B)'$, first determine $A \cap B$.

$$A \cap B = \{1, 3, 8\} \text{ from part (a)}$$
$$(A \cap B)' = \{1, 3, 8\}' = \{2, 4, 5, 6, 7, 9, 10\}$$ ■

Union

The word *union* means to unite or join together, as in marriage, and that is exactly what is done when we perform the operation of union.

Definition: Union

The **union** of set A and set B, symbolized by $A \cup B$, is the set containing all the elements that are members of set A or of set B (or of both sets).

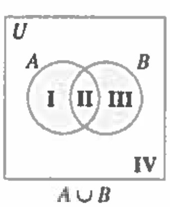


Figure 2.8

The three shaded regions of Fig. 2.8, regions I, II, and III, together represent the union of sets A and B. If an element is common to both sets, it is listed only once in the union of the sets.

Example 4 *Determining Sets from a Venn Diagram*

Use the Venn diagram in Fig. 2.9 to determine the following sets.

a) U b) A c) B' d) $A \cap B$
e) $A \cup B$ f) $(A \cup B)'$ g) $n(A \cup B)$

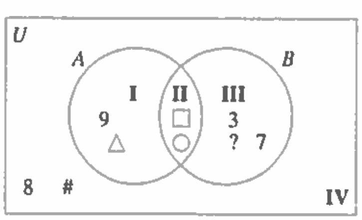


Figure 2.9

Solution

a) $(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\} \cap \{a, b, e, f, g\}$
$= \{a, b, e, g\}$

b) $(A \cup B) \cap C' = \{a, b, c, d, e, g\} \cap \{a, c, d, g\}$
$= \{a, c, d, g\}$

c) $A' \cap B' = \{c, d, f\} \cap \{b, f, g\}$
$= \{f\}$ ■

The Meaning of *and* and *or*

The words *and* and *or* are very important in many areas of mathematics. We use these words in several chapters in this book, including Ch.12, Probability. The word ***and*** is generally interpreted to mean ***intersection***, whereas ***or*** is generally interpreted to mean ***union***. Suppose $A = \{1, 2, 3, 5, 6, 8\}$ and $B = \{1, 3, 4, 7, 9, 10\}$. The elements that belong to set A *and* set B are 1 and 3. These are the elements in the intersection of the sets. The elements that belong to set A *or* set B are 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. These are the elements in the union of the sets.

The Relationship Between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$

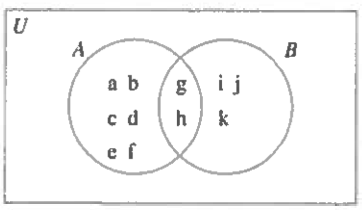


Figure 2.10

Having looked at unions and intersections, we can now determine a relationship between $n(A \cup B)$, $n(A)$, $n(B)$, and $n(A \cap B)$. Suppose set A has eight elements, set B has five elements, and $A \cap B$ has two elements. How many elements are in $A \cup B$? Let's make up some arbitrary sets that meet the criteria specified and draw a Venn diagram. If we let set $A = \{a, b, c, d, e, f, g, h\}$, then set B must contain five elements, two of which are also in set A. Let set $B = \{g, h, i, j, k\}$. We construct a Venn diagram by filling in the intersection first, as shown in Fig. 2.10. The number of elements in $A \cup B$ is 11. The elements g and h are in both sets, and if we add $n(A) + n(B)$, we are counting these elements twice.

To find the number of elements in the union of sets A and B, we can add the number of elements in sets A and B and then subtract the number of elements common to both sets.

The Number of Elements in $A \cup B$

For any finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example 7 *How Many Visitors Speak Spanish or French?*

The results of a survey of visitors at the Grand Canyon showed that 25 speak Spanish, 14 speak French, and 4 speak both Spanish and French. How many speak Spanish or French?

Solution If we let set A be the set of visitors who speak Spanish and let set B be the set of visitors who speak French, then we need to determine $n(A \cup B)$. We can use the above formula to find $n(A \cup B)$.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A \cup B) = 25 + 14 - 4$$
$$= 35$$

Thus, 35 of the visitors surveyed speak either Spanish or French. ■

d) To determine $A - C'$, we must first determine C'.

$$C' = \{a, c, d, f, h, i, j, k\}$$

$A - C'$ is the set of elements that are in set A but not set C'. The elements that are in set A but not set C' are b, e, and g. Therefore, $A - C' = \{b, e, g\}$. ■

Next we discuss the Cartesian product.

Cartesian Product

Definition: Cartesian Product

The **Cartesian product** of set A and set B, symbolized by $A \times B$ and read "A cross B," is the set of all possible *ordered pairs* of the form (a, b), where $a \in A$ and $b \in B$.

To determine the ordered pairs in a Cartesian product, select the first element of set A and form an ordered pair with each element of set B. Then select the second element of set A and form an ordered pair with each element of set B. Continue in this manner until you have used each element of set A.

Example 10 *The Cartesian Product of Two Sets*

Given $A = \{$orange, banana, apple$\}$ and $B = \{1, 2\}$, determine the following.

a) $A \times B$ b) $B \times A$ c) $A \times A$ d) $B \times B$

Solution

a) $A \times B = \{$(orange, 1), (orange, 2) (banana, 1), (banana, 2), (apple, 1), (apple, 2)$\}$

b) $B \times A = \{$(1, orange), (1, banana), (1, apple), (2, orange), (2, banana), (2, apple)$\}$

c) $A \times A = \{$(orange, orange), (orange, banana), (orange, apple), (banana, orange), (banana, banana), (banana, apple), (apple, orange), (apple, banana), (apple, apple)$\}$

d) $B \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ ■

We can see from Example 10 that, in general, $A \times B \neq B \times A$. The ordered pairs in $A \times B$ are not the same as the ordered pairs in $B \times A$ because (orange, 1) $\neq$ (1, orange).

In general, if a set A has m elements and a set B has n elements, then the number of ordered pairs in $A \times B$ will be $m \times n$. In Example 10, set A contains 3 elements and set B contains 2 elements. Notice that $A \times B$ contains 3×2 or 6 ordered pairs.

SECTION 2.3 Exercises

Warm Up Exercises

In exercises 1–8, fill in the blank with an appropriate word, phrase, or symbol(s).

1. The set of all the elements in the universal set that are not in set A is called the ________ of set A.

2. The set containing all the elements that are members of set A or of set B or of both sets is called the ________ of set A and set B.

3. The set containing all the elements that are common to both set A and set B is called the ________ of set A and set B.

4. The set of elements that belong to set A, but not to set B, is called the ________ of two sets A and B.

5. The set of all possible ordered pairs of the form (a, b), where $a \in A$ and $b \in$ B, is called the ________ product of set A and set B.

20. ***Racing Standings*** The following table shows the 2009 NASCAR Sprint Cup Series Final Standings, with the 10 drivers having the highest point total and the number of races won. Let the drivers in the table represent the universal set.

2009 NASCAR Sprint Cup Series Final Standings

Driver	Points	Wins
Jimmie Johnson	6652	7
Mark Martin	6511	5
Jeff Gordon	6473	1
Kurt Busch	6446	2
Denny Hamlin	6335	4
Tony Stewart	6309	4
Greg Biffle	6292	0
Juan Montoya	6252	0
Ryan Newman	6175	0
Kasey Kahne	6128	2

Source: NASCAR

Let A = the set of drivers with more than 6400 points

Let B = the set of drivers with more than 1 win

Construct a Venn diagram illustrating the sets. Use the driver's initials in the Venn diagram.

21. Let U represent the set of animals in U.S. zoos. Let A represent the set of animals in the San Diego zoo. Describe A'.

▲ *San Diego Zoo*

22. Let U represent the set of U.S. colleges and universities. Let A represent the set of U.S. colleges and universities in the state of Mississippi. Describe A'.

In Exercises 23–28,

U is the set of farms in the United States.

A is the set of farms that produce corn.

B is the set of farms that produce tomatoes.

Describe each of the following sets in words.

23. A'

24. B'

25. $A \cup B$

26. $A \cap B$

27. $A \cap B'$

28. $A \cup B'$

In Exercises 29–34,

U is the set of furniture stores.

A is the set of furniture stores that sell mattresses.

B is the set of furniture stores that sell outdoor furniture.

C is the set of furniture stores that sell leather furniture.

Describe the following sets.

29. $A \cup C$

30. $A \cap B$

31. $B' \cap C$

32. $A \cap B \cap C$

33. $A \cup B \cup C$

34. $A' \cup C'$

In Exercises 35–42, use the Venn diagram in Fig. 2.12 to list the set of elements in roster form.

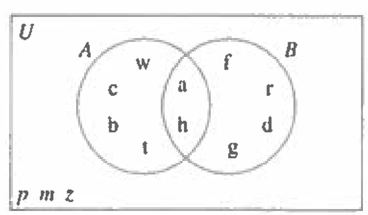


Figure 2.12

35. A

36. B

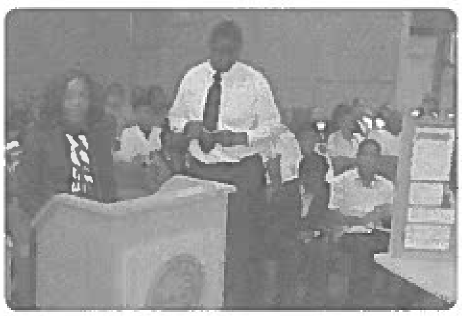
37. $A \cap B$

38. U

100. When will $n(A \cap B) = 0$? Explain and give an example.

101. *Pet Ownership* The results of a survey of customers at PetSmart showed that 27 owned dogs, 38 owned cats, and 16 owned both dogs and cats. How many people owned either a dog or a cat?

102. *Student Council and Intramurals* At Madison High School, 46 students participated in student council or intramurals, 30 participated in student council, and 4 participated in student council and intramurals. How many students participated in intramurals?



103. Consider the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

a) Show that this relation holds for $A = \{a, b, c, d\}$ and $B = \{b, d, e, f, g, h\}$.

b) Make up your own sets A and B, each consisting of at least six elements. Using these sets, show that the relation holds.

c) Use a Venn diagram and explain why the relation holds for any two sets A and B.

104. The Venn diagram in Fig. 2.14 shows a technique of labeling the regions to indicate membership of elements in a particular region. Define each of the four regions with a set statement. (*Hint:* $A \cap B'$ defines region I.)

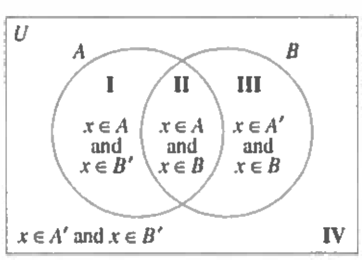


Figure 2.14

In Exercises 105–114, let $U = \{0, 1, 2, 3, 4, 5, \dots\}$, $A = \{1, 2, 3, 4, \dots\}$, $B = \{4, 8, 12, 16, \dots\}$, *and* $C = \{2, 4, 6, 8, \dots\}$. *Determine the following.*

105. $A \cup B$

106. $A \cap B$

107. $B \cup C$

108. $B \cap C$

109. $A \cap C$

110. $A' \cap C$

111. $B' \cap C$

112. $(B \cup C)' \cup C$

113. $(A \cap C) \cap B'$

114. $U' \cap (A \cup B)$

Challenge Problems/Group Activities

In Exercises 115–122, determine whether the answer is $\varnothing$, A, *or* U. *(Assume* $A \neq \varnothing, A \neq U$.*)*

115. $A \cap A'$

116. $A \cup A'$

117. $A \cup \varnothing$

118. $A \cap \varnothing$

119. $A' \cup U$

120. $A \cap U$

121. $A \cup U$

122. $A \cap A$

In Exercises 123–128, determine the relationship between set A and set B if

123. $A \cap B = B$.

124. $A \cup B = B$.

125. $A \cap B = \varnothing$.

126. $A \cup B = A$.

127. $A \cap B = A$.

128. $A \cup B = \varnothing$.

PROCEDURE **GENERAL PROCEDURE FOR CONSTRUCTING VENN DIAGRAMS WITH THREE SETS, *A*, *B*, AND *C***

1. Determine the elements to be placed in region V by finding the elements that are common to all three sets, $A \cap B \cap C$.
2. Determine the elements to be placed in region II. Find the elements in $A \cap B$. The elements in this set belong in regions II and V. Place the elements in the set $A \cap B$ that are not listed in region V in region II. The elements in regions IV and VI are found in a similar manner.
3. Determine the elements to be placed in region I by determining the elements in set *A* that are not in regions II, IV, and V. The elements in regions III and VII are found in a similar manner.
4. Determine the elements to be placed in region VIII by finding the elements in the universal set that are not in regions I through VII.

Example 1 illustrates the general procedure.

Example 1 *Constructing a Venn Diagram for Three Sets*

Construct a Venn diagram illustrating the following sets.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$
$$A = \{1, 4, 8, 9, 10, 12\}$$
$$B = \{2, 4, 5, 9, 10, 13\}$$
$$C = \{1, 3, 4, 8, 9, 11\}$$

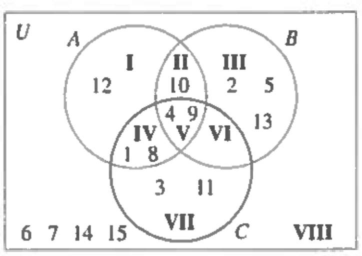


Figure 2.17

Solution First find the intersection of all three sets. Because the elements 4 and 9 are in all three sets, $A \cap B \cap C = \{4, 9\}$. The elements 4 and 9 are placed in region V in Fig.2.17. Next complete region II by determining the intersection of sets *A* and *B*.

$$A \cap B = \{4, 9, 10\}$$

$A \cap B$ consists of regions II and V. The elements 4 and 9 have already been placed in region V, so 10 must be placed in region II.

Now determine what numbers go in region IV.

$$A \cap C = \{1, 4, 8, 9\}$$

Since 4 and 9 have already been placed in region V, place the 1 and 8 in region IV. Now determine the numbers to go in region VI.

$$B \cap C = \{4, 9\}$$

Since both the 4 and 9 have been placed in region V, there are no numbers to be placed in region VI. Now complete set *A*. The only element of set *A* that has not previously been placed in regions II, IV, or V is 12. Therefore, place the element 12 in region I. The element 12 that is placed in region I is only in set *A* and not in set *B* or set *C*. Using set *B*, complete region III using the same general procedure used to determine the numbers in region I. Using set *C*, complete region VII by using the same procedure used to complete regions I and III. To determine the elements in region VIII, find the elements in *U* that have not been placed in regions I–VII. The elements 6, 7, 14, and 15 have not been placed in regions I–VII, so place them in region VIII. ■

Venn diagrams can be used to illustrate and analyze many everyday problems. One example follows.

Thus, we have proved that $A' \cup B \neq A' \cap B$ for all sets A and B by using a *counterexample*. A counterexample, as explained in Chapter 1, is an example that shows a statement is not true.

In Chapter 1, we explained that proofs involve the use of *deductive reasoning*. Recall that deductive reasoning begins with a general statement and works to a specific conclusion. To verify, or determine whether set statements are equal for any two sets selected, we use deductive reasoning with Venn diagrams. Venn diagrams are used because they can illustrate general cases. To determine if statements that contain sets, such as $(A \cup B)'$ and $A' \cap B'$, are equal for all sets A and B, we use the regions of Venn diagrams. If both statements represent the same regions of the Venn diagram, then the statements are equal for all sets A and B. See Example 3.

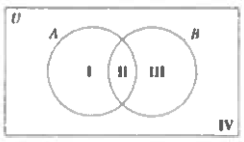


Figure 2.19

Example 3 *Equality of Sets*

Determine whether $(A \cup B)' = A' \cap B'$ for all sets A and B.

Solution Draw a Venn diagram with two sets A and B, as in Fig. 2.19. Label the regions as indicated.

Find $(A \cup B)'$

Set	Corresponding Regions
A	I, II
B	II, III
$A \cup B$	I, II, III
$(A \cup B)'$	IV

Find $A' \cap B'$

Set	Corresponding Regions
A'	III, IV
B'	I, IV
$A' \cap B'$	IV

Both statements are represented by the same region, IV, of the Venn diagram. Thus, $(A \cup B)' = A' \cap B'$ for all sets A and B. ∎

In Example 3, when we proved that $(A \cup B)' = A' \cap B'$, we started with two general sets and worked to the specific conclusion that both statements represented the same regions of the Venn diagram. We showed that $(A \cup B)' = A' \cap B'$ *for all sets A and B*. No matter what sets we choose for A and B, this statement will be true. For example, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{3, 4, 6, 10\}$, and $B = \{1, 2, 4, 5, 6, 8\}$.

$$\begin{aligned} (A \cup B)' &= A' \cap B' \\ \{1, 2, 3, 4, 5, 6, 8, 10\}' &= \{3, 4, 6, 10\}' \cap \{1, 2, 4, 5, 6, 8\}' \\ \{7, 9\} &= \{1, 2, 5, 7, 8, 9\} \cap \{3, 7, 9, 10\} \\ \{7, 9\} &= \{7, 9\} \end{aligned}$$

We can also use Venn diagrams to prove statements involving three sets.

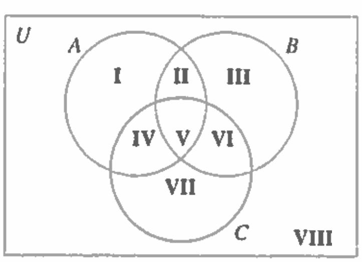


Figure 2.20

Example 4 *Equality of Sets*

Determine whether $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets, A, B, and C.

Solution Because the statements include three sets, A, B, and C, three circles must be used. The Venn diagram illustrating the eight regions is shown in Fig. 2.20.

First we will find the regions that correspond to $A \cup (B \cap C)$, and then we will find the regions that correspond to $(A \cup B) \cap (A \cup C)$. If both answers are the same, the statements are equal.

6. A Venn diagram contains three sets, A, B, and C, as in Fig. 2.15 on page 68. If region V contains 4 elements and there are 9 elements in $A \cap B$, how many elements belong in region II? Explain.

7. a) For $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 4, 5\}$, and $B = \{1, 4, 5\}$, does $A \cup B = A \cap B$?

b) By observing the answer to part (a), can we conclude that $A \cup B = A \cap B$ for all sets A and B? Explain.

c) Using a Venn diagram, determine if $A \cup B = A \cap B$ for all sets A and B.

8. Construct a Venn diagram illustrating the following sets.

$U = \{a, b, c, d, e, f, g, h, i, j\}$
$A = \{c, d, e, g, h, i\}$
$B = \{a, c, d, g\}$
$C = \{c, f, i, j\}$

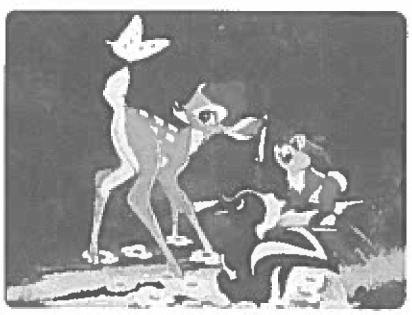
9. Construct a Venn diagram illustrating the following sets.

U = {*Cinderella, Pinocchio, Ratatouille, Fantasia, Dumbo, Bambi, Pocahontas, Hercules, Mulan, Tarzan, Cars*}

A = {*Bambi, Hercules, Pocahontas, Tarzan*}

B = {*Ratatouille, Bambi, Mulan, Hercules*}

C = {*Pocahontas, Cinderella, Bambi, Ratatouille, Fantasia*}



▲ *Bambi*

10. Construct a Venn diagram illustrating the following sets.

U = {microwave oven, freezer, dishwasher, refrigerator, washer, dryer, toaster, blender, food processor, iron}

A = {toaster, blender, iron, dishwasher, washer, dryer}

B = {dishwasher, iron, freezer}

C = {washer, dryer, iron, freezer, microwave oven}

11. Construct a Venn diagram illustrating the following sets.

U = {American Eagle, Best Buy, Wal-Mart, Kmart, Target, Sears, JCPenney, Costco, Kohl's, Gap, Gap Kids, Foot Locker, Old Navy, Macy's}

A = {American Eagle, Wal-Mart, Target, JCPenney, Old Navy}

B = {Best Buy, Target, Costco, Old Navy, Macy's}

C = {Target, Sears, Kohl's, Gap, JCPenney}

12. Construct a Venn diagram illustrating the following sets.

U = {Louis Armstrong, Glenn Miller, Stan Kenton, Charlie Parker, Duke Ellington, Benny Goodman, Count Basie, John Coltrane, Dizzy Gillespie, Miles Davis, Thelonius Monk}

A = {Stan Kenton, Count Basie, Dizzy Gillespie, Duke Ellington, Thelonius Monk}

B = {Louis Armstrong, Glenn Miller, Count Basie, Duke Ellington, Miles Davis}

C = {Count Basie, Miles Davis, Stan Kenton, Charlie Parker, Duke Ellington}

13. ***Olympic Medals*** Consider the following table, which shows countries that won at least 25 medals in the 2008 Summer Olympics. Let the countries in the table represent the universal set.

Country	Gold Medals	Silver Medals	Bronze Medals	Total Medals
United States	36	38	36	110
China	51	21	28	100
Russia	23	21	28	72
Great Britain	19	13	15	47
Australia	14	15	17	46
Germany	16	10	15	41
France	7	16	17	40
South Korea	13	10	8	31
Italy	8	10	10	28
Ukraine	7	5	15	27
Japan	9	6	10	25

Source: United States Olympic Committee.

Let A = set of countries that won at least 30 gold medals.
Let B = set of countries that won at least 15 silver medals.
Let C = set of countries that won at least 10 bronze medals.

Construct a Venn diagram that illustrates the sets A, B, and C.

14. ***Popular TV Shows*** Construct a Venn diagram illustrating the following sets.

U = {*American Idol (AI), CSI, Dancing with the Stars (DWS), Family Guy (FG), Gossip Girl (GG), Monday Night Football (MNF), NCIS, Sunday Night Football (SNF), Survivor (S)*}.

A = {*AI, CSI, DWS, SNF, NCIS*}

B = {*AI, DWS, SNF, NCIS, MNF*}

C = {*AI, CSI, SNF, NCIS, MNF, S*}

In Exercises 53–60, use Venn diagrams to determine whether the following statements are equal for all sets A and B.

53. $(A \cap B)'$, $A' \cup B'$

54. $(A \cap B)'$, $A' \cup B$

55. $A' \cup B'$, $A \cap B$

56. $(A \cup B)'$, $(A \cap B)'$

57. $A' \cup B'$, $(A \cup B)'$

58. $A' \cap B'$, $A \cup B'$

59. $(A' \cap B)'$, $A \cup B'$

60. $A' \cap B'$, $(A' \cap B')'$

In Exercises 61–70, use Venn diagrams to determine whether the following statements are equal for all sets A, B, and C.

61. $A \cap (B \cup C)$, $(A \cap B) \cup C$

62. $A \cup (B \cap C)$, $(B \cap C) \cup A$

63. $A \cap (B \cup C)$, $(B \cup C) \cap A$

64. $A \cup (B \cap C)'$, $A' \cap (B' \cup C)$

65. $A \cap (B \cup C)$, $(A \cap B) \cup (A \cap C)$

66. $A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$

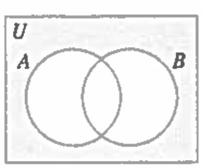
67. $A \cup (B \cup C)'$, $A \cup (B' \cap C')$

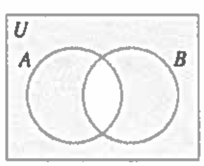
68. $(A \cup B) \cap (B \cup C)$, $B \cup (A \cap C)$

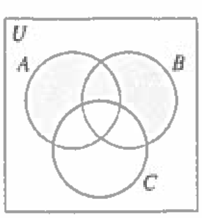
69. $(A \cup B)' \cap C$, $(A' \cup C') \cap (B' \cup C)$

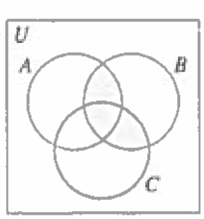
70. $(C \cap B)' \cup (A \cap B)'$, $A \cap (B \cap C)$

In Exercises 71–74, use set statements to write a description of the shaded area. Use union, intersection and complement as necessary. More than one answer may be possible.

71.


72.


73.


74.


75. Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 6, 7\}$$
$$C = \{6, 7, 9\}$$

a) Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for these sets.

b) Make up your own sets A, B, and C. Verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for your sets A, B, and C.

c) Use Venn diagrams to verify that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ for all sets A, B, and C.

76. Let

$$U = \{a, b, c, d, e, f, g, h, i\}$$
$$A = \{a, c, d, e, f\}$$
$$B = \{c, d\}$$
$$C = \{a, b, c, d, e\}$$

a) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for these sets.

b) Make up your own sets, A, B, and C. Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for your sets.

c) Determine whether $(A \cup C)' \cap B = (A \cap C)' \cap B$ for all sets A, B, and C.

77. ***Blood Types*** A hematology text gives the following information on percentages of the different types of blood worldwide.

Type	Positive Blood, %	Negative Blood, %
A	37	6
O	32	6.5
B	11	2
AB	5	0.5

Construct a Venn diagram similar to the one in Example 2 and place the correct percentage in each of the eight regions.

78. Define each of the eight regions in Fig. 2.25 using sets A, B, and C and a set operation. (*Hint:* $A \cap B' \cap C'$ defines region I.)

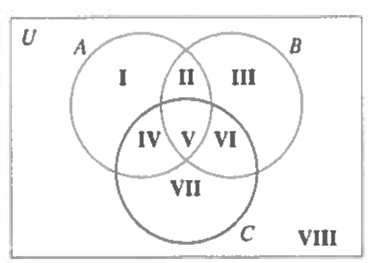


Figure 2.25

this section contain two or three sets of elements, which can be represented in a Venn diagram. Our plan will generally include drawing a Venn diagram, labeling the diagram, and filling in the regions of the diagram.

Whenever possible, follow the procedure in Section 2.4 for completing the Venn diagram and then answer the questions. *Remember: When drawing Venn diagrams, we generally start with the intersection of the sets and work outward.*

Example 1 *Yogurt Taste Test*

A yogurt company wishes to introduce a new yogurt flavor. The company is considering two flavors: raspberry cheesecake (R) and orange creme (O). In a survey of 250 people it was found that

180 people liked raspberry cheesecake.

139 people liked orange creme.

82 people liked both flavors.

Of those surveyed, how many people

a) did not like either raspberry cheesecake or orange creme?
b) liked raspberry cheesecake, but not orange creme?
c) liked orange creme, but not raspberry cheesecake?
d) liked either raspberry cheesecake or orange creme?

Solution The problem provides the following information.

The number of people surveyed is 250: $n(U) = 250$.

The number of people surveyed who liked raspberry cheesecake is 180: $n(R) = 180$.

The number of people surveyed who liked orange creme is 139: $n(O) = 139$.

The number of people surveyed who liked both raspberry cheesecake and orange creme is 82: $n(R \cap O) = 82$.

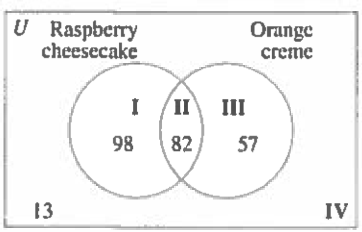


Figure 2.26

We illustrate this information on the Venn diagram shown in Fig. 2.26. We already know that $R \cap O$ corresponds to region II. Because $n(R \cap O) = 82$, we write 82 in region II. Set R consists of regions I and II. We know that set R, the number of people who liked raspberry cheesecake, contains 180 people. Therefore, region I contains $180 - 82 = 98$ people. We write the number 98 in region I. Set O consists of regions II and III. Because $n(O) = 139$, the total in these two regions must be 139. Region II contains 82, leaving $139 - 82$, or 57, for region III. We write 57 in region III.

The total number of people surveyed who liked raspberry cheesecake or orange creme is found by adding the numbers in regions I, II, and III. Therefore $n(R \cup O) = 98 + 82 + 57 = 237$. The number in region IV is the difference between $n(U)$ and $n(R \cup O)$. There are $250 - 237$, or 13, members in region IV.

a) The people surveyed who did not like either raspberry cheesecake or orange creme are those people in the universal set who are not contained in set R or set O. The 13 people in region IV did not like raspberry cheesecake or orange creme.
b) The 98 people in region I are those people surveyed who liked raspberry cheesecake, but not orange creme.
c) The 57 people in region III are those people surveyed who liked orange creme, but not raspberry cheesecake.
d) The people in regions I, II, or III are those people surveyed who liked either raspberry cheesecake or orange creme. Thus, $98 + 82 + 57$, or 237, people surveyed liked either raspberry cheesecake or orange creme. Notice that the 82 people in region II who like both flavors are included in those people surveyed who liked either raspberry cheesecake or orange creme. ■

Similar problems involving three sets can be solved, as illustrated in Example 2.

TIMELY TIP

When constructing a Venn diagram, the most common mistake made by students is forgetting to subtract the number in region V from the respective values in determining the numbers to be placed in regions II, IV, and VI.

b) Region I represents those customers who purchased only games. Thus, 200 customers purchased only games.

c) The words *at least one* mean "one or more." All those in regions I through VII purchased at least one of the types of software. The sum of the numbers in regions I through VII is 791, so 791 customers purchased at least one of the types of software.

d) The customers in regions II, IV, and VI purchased exactly two of the types of software. Summing the numbers in these regions $140 + 48 + 39$ we find that 227 customers purchased exactly two of these types of software. Notice that we did not include the customers in region V. Those customers purchased all three types of software. ■

The procedure to work problems like those given in Example 2 is generally the same. Start by completing region V. Next complete regions II, IV, and VI. Then complete regions I, III, and VII. Finally, complete region VIII. When you are constructing Venn diagrams, be sure to check your work carefully.

Example 3 *Travel Packages*

Liberty Travel surveyed 125 potential customers. The following information was obtained.

68 wished to travel to Hawaii.
53 wished to travel to Las Vegas.
47 wished to travel to Disney World.
34 wished to travel to Hawaii and Las Vegas.
26 wished to travel to Las Vegas and Disney World.
23 wished to travel to Hawaii and Disney World.
18 wished to travel to all three destinations.

▲ *Hawaii*

Use a Venn diagram to answer the following questions. How many of those surveyed

a) did not wish to travel to any of these destinations?
b) wished to travel only to Hawaii?
c) wished to travel to Disney World *and* Las Vegas, but not to Hawaii?
d) wished to travel to Disney World *or* Las Vegas, but not to Hawaii?
e) wished to travel to exactly one of these destinations?

Solution The Venn diagram is constructed using the procedures we outlined in Example 2. The diagram is illustrated in Fig. 2.28. We suggest you construct the diagram by yourself now and check your diagram with Fig. 2.28.

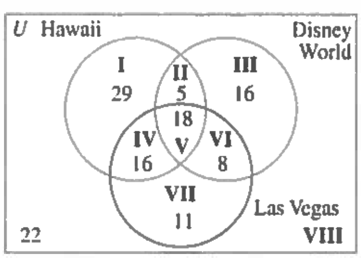


Figure 2.28

a) Twenty-two potential customers did not wish to travel to any of these destinations (see region VIII).

b) Twenty-nine potential customers wished to travel only to Hawaii (see region I).

c) Those potential customers in region VI wished to travel to Disney World *and* Las Vegas, but not to Hawaii. Therefore, eight customers satisfied the criteria.

d) The word *or* in this type of problem means one or the other or both. All the potential customers in regions II, III, IV, V, VI, and VII wished to travel to Disney World or Las Vegas. Those in regions II, IV, and V also wished to travel to Hawaii. The potential customers that wished to travel to Disney World or Las Vegas, but not to Hawaii, are found by adding the numbers in regions III, VI, and VII. There are $16 + 8 + 11 = 35$ potential customers who satisfy the criteria.

e) Those potential customers in regions I, III, and VII wished to travel to exactly one of the destinations. Therefore, $29 + 16 + 11 = 56$ customers wished to travel to exactly one of these destinations. ■

How many of the amusement parks surveyed had

a) only water slides?

b) exactly one of these features?

c) at least one of these features?

d) exactly two of these features?

e) none of these features?

7. *Book Purchases* A survey of 85 customers was taken at Barnes & Noble regarding the types of books purchased. The survey found that

44 purchased mysteries.
33 purchased science fiction.
29 purchased romance novels.
13 purchased mysteries and science fiction.
5 purchased science fiction and romance novels.
11 purchased mysteries and romance novels.
2 purchased all three types of books.

How many of the customers surveyed purchased

a) only mysteries?

b) mysteries and science fiction, but not romance novels?

c) mysteries or science fiction?

d) mysteries or science fiction, but not romance novels?

e) exactly two types?

8. *Movies* A survey of 350 customers was taken at Regal Cinemas in Austin, Texas, regarding the type of movies customers liked. The following information was determined.

196 liked dramas.
153 liked comedies.
88 liked science fiction.
59 liked dramas and comedies.
37 liked dramas and science fiction.
32 liked comedies and science fiction.
21 liked all three types of movies.

Of the customers surveyed, how many liked

a) none of these types of movies?

b) only dramas?

c) exactly one of these types of movies?

d) exactly two of these types of movies?

e) dramas or comedies?

9. *Jobs at a Restaurant* Panera Bread compiled the following information regarding 30 of its employees. The following was determined.

8 cooked food.
9 washed dishes.
18 operated the cash register.
4 cooked food and washed dishes.
5 washed dishes and operated the cash register.
3 cooked food and operated the cash register.
2 did all three jobs.

How many of the employees

a) only cooked food ?

b) only operated the cash register?

c) washed dishes and operated the cash register but did not cook food?

d) washed dishes or operated the cash register but did not cook food?

e) did at least two of these jobs?

10. *Electronic Devices* In a survey of college students, it was found that

356 owned an iPod.
293 owned a laptop.
285 owned a gaming system.
193 owned an iPod and a laptop.
200 owned an iPod and a gaming system.
139 owned a laptop and a gaming system.
68 owned an iPod, a laptop, and a gaming system.
26 owned none of these devices.

a) How many college students were surveyed?

Of the college students surveyed, how many owned

b) an iPod and a gaming system, but not a laptop?

c) a laptop, but neither an iPod nor a gaming system?

d) exactly two of these devices?

e) at least one of these devices?

11. *Homeowners' Insurance Policies* A committee of the Florida legislature decided to analyze 350 homeowners' insurance policies to determine if the consumers' homes

16. ***Family Reunion*** When the Montesano family members discussed where their annual reunion should take place, they found that of all the family members,
    8 would not go to a park.
    7 would not go to a beach.
    11 would not go to the family cottage.
    3 would go to neither a park nor a beach.
    4 would go to neither a beach nor the family cottage.
    6 would go to neither a park nor the family cottage.
    2 would not go to a park or a beach or to the family cottage.
    1 would go to all three places.

    What is the total number of family members?

Recreational Mathematics

17. ***Number of Elements*** A universal set U consists of 12 elements. If sets A, B, and C are proper subsets of U and $n(U) = 12$, $n(A \cap B) = n(A \cap C) = n(B \cap C) = 6$, $n(A \cap B \cap C) = 4$, and $n(A \cup B \cup C) = 10$, determine
    a) $n(A \cup B)$
    b) $n(A' \cup C)$
    c) $n(A \cap B)'$

SECTION 2.6 Infinite Sets

▲ *Georg Cantor, founder of set theory*

Which set is larger, the set of integers or the set of even integers? One might argue that because the set of even integers is a subset of the set of integers, the set of integers must be larger than the set of even integers. Yet both sets are infinite sets, so how can we determine which set is larger? This question puzzled mathematicians for centuries until 1874, when Georg Cantor developed a method of determining the cardinal number of an infinite set. In this section, we will discuss infinite sets and how to determine the number of elements in an infinite set.

Why ***This Is Important*** The concept of infinity and which sets contain more elements has led to the expansion and understanding of many mathematical and scientific concepts.

On page 45, we state that a finite set is a set in which the number of elements is zero or the number of elements can be expressed as a natural number. On page 46, we define a one-to-one correspondence. To determine the number of elements in a finite set, we can place the set in a one-to-one correspondence with a subset of the set of counting numbers. For example, the set $A = \{\#, ?, \$\}$ can be placed in one-to-one correspondence with set $B = \{1, 2, 3\}$, a subset of the set of counting numbers.

$$\begin{aligned} A &= \{\#,\ ?,\ \$\} \\ &\quad\ \downarrow\ \ \downarrow\ \ \downarrow \\ B &= \{1,\ 2,\ 3\} \end{aligned}$$

Because the cardinal number of set B is 3, the cardinal number of set A is also 3. Any two sets, such as set A and set B, that can be placed in a one-to-one correspondence must have the same number of elements (therefore the same cardinality) and must be equivalent sets. Note that $n(A)$ and $n(B)$ both equal 3.

German mathematician Georg Cantor (1845–1918), known as the father of set theory, thought about sets that were not bounded. He called an unbounded set an *infinite set* and provided the following definition.

Profile in Mathematics

Leopold Kronecker

Mathematician Leopold Kronecker (1823–1891), Cantor's former mentor, ridiculed Cantor's theories and prevented Cantor from gaining a position at the University of Berlin. Although Cantor's work on infinite sets is now considered a masterpiece, it generated heated controversy when originally published. Cantor's claim that the infinite set was unbounded offended the religious views of the time that God had created a complete universe that could not be wholly comprehended by humans. Eventually Cantor was given the recognition due him, but by then the criticism had taken its toll on his health. He had several nervous breakdowns and spent his last days in a mental hospital. See the Profile in Mathematics on page 43 for more information on Cantor.

Solution First create a proper subset of the set of even counting numbers by removing the first number from the set. Then establish a one-to-one correspondence.

$$\begin{array}{ll} \text{Even counting numbers:} & \{2, 4, 6, 8, \ldots, \quad 2n, \ldots\} \\ & \ \downarrow\ \downarrow\ \downarrow\ \downarrow \qquad\quad \downarrow \\ \text{Proper subset:} & \{4, 6, 8, 10, \ldots, 2n+2, \ldots\} \end{array}$$

A one-to-one correspondence exists between the two sets, so the set of even counting numbers is infinite. ■

Example 3 *The Set of Multiples of Five*

Show that the set $\{5, 10, 15, 20, \ldots, 5n, \ldots\}$ is an infinite set.

Solution

$$\begin{array}{ll} \text{Given set:} & \{5, 10, 15, 20, 25, \ldots, \quad 5n, \ldots\} \\ & \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \qquad\quad \downarrow \\ \text{Proper subset:} & \{10, 15, 20, 25, 30, \ldots, 5n+5, \ldots\} \end{array}$$

Therefore, the given set is an infinite set. ■

Countable Sets

In his work with infinite sets, Cantor developed ideas on how to determine the cardinal number of an infinite set. He called the cardinal number of infinite sets "transfinite cardinal numbers" or "transfinite powers." He defined a set as ***countable*** if it is finite or if it can be placed in a one-to-one correspondence with the set of counting numbers. All infinite sets that can be placed in a one-to-one correspondence with the set of counting numbers have cardinal number ***aleph-null***, symbolized $\aleph_0$ (the first Hebrew letter, aleph, with a zero subscript, read "null").

Example 4 *The Cardinal Number of the Set of Even Numbers*

Show that the set of even counting numbers has cardinal number $\aleph_0$.

Solution In Example 2, we showed that the set of even counting numbers is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

Now we will show that it is countable and has cardinality $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the set of even counting numbers.

$$\begin{array}{ll} \text{Counting numbers:} & N = \{1, 2, 3, 4, \ldots, \ n, \ldots\} \\ & \qquad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow \qquad \downarrow \\ \text{Even counting numbers:} & E = \{2, 4, 6, 8, \ldots, 2n, \ldots\} \end{array}$$

For each number n in the set of counting numbers, its corresponding number is $2n$. Since we found a one-to-one correspondence between the set of counting numbers and the set of even counting numbers, the set of even counting numbers is countable. Thus, the cardinal number of the set of even counting numbers is $\aleph_0$; that is, $n(E) = \aleph_0$. As we mentioned earlier, the set of even counting numbers is an infinite set, since it can be placed in a one-to-one correspondence with a proper subset of itself. Therefore, the set of even counting numbers is both infinite and countable. ■

Definition: Cardinal Number of Infinite Sets

Any set that can be placed in a one-to-one correspondence with the set of counting numbers has **cardinal number** (or cardinality) $\aleph_0$ and is infinite and is countable.

In Exercises 13–22, show that the set has cardinal number $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set. Be sure to show the pairing of the general terms in the sets.

13. $\{3, 6, 9, 12, 15, \ldots\}$

14. $\{40, 41, 42, 43, 44, \ldots\}$

15. $\{4, 6, 8, 10, 12, \ldots\}$

16. $\{0, 2, 4, 6, 8, \ldots\}$

17. $\{2, 5, 8, 11, 14, \ldots\}$

18. $\{7, 11, 15, 19, 23, \ldots\}$

19. $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \ldots\}$

20. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\}$

21. $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\}$

22. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$

Challenge Problems/Group Activities

In Exercises 23–26, show that the set has cardinality $\aleph_0$ by establishing a one-to-one correspondence between the set of counting numbers and the given set.

23. $\{1, 4, 9, 16, 25, \ldots\}$

24. $\{2, 4, 8, 16, 32, \ldots\}$

25. $\{3, 9, 27, 81, 243, \ldots\}$

26. $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \ldots\}$

Recreational Mathematics

In Exercises 27–31, insert the symbol $<$, $>$, or = in the shaded area to make a true statement.

27. $\aleph_0 \quad \aleph_0 + \aleph_0$

28. $2\aleph_0 \quad \aleph_0 + \aleph_0$

29. $2\aleph_0 \quad \aleph_0$

30. $\aleph_0 + 5 \quad \aleph_0 + 3$

31. $n(N) \quad \aleph_0$

32. There are a number of paradoxes (a statement that appears to be true and false at the same time) associated with infinite sets and the concept of infinity. One of these, called ***Zeno's Paradox***, is named after the mathematician Zeno, born about 496 B.C. in Italy. According to Zeno's paradox, suppose Achelles starts out 1 meter behind a tortoise. Also, suppose Achelles walks 10 times as fast as the tortoise crawls. When Achelles reaches the point where the tortoise started, the tortoise is 1/10 of a meter ahead of Achelles; when Achelles reaches the point where the tortoise was 1/10 of a meter ahead, the tortoise is now 1/100 of a meter ahead; and so on. According to Zeno's Paradox, Achelles gets closer and closer to the tortoise but never catches up to the tortoise.

a) Do you believe the reasoning process is sound? If not, explain why not.

b) In actuality, if this situation were real, would Achelles ever pass the tortoise?

Internet/Research Activities

33. Do research to explain how Cantor proved that the set of rational numbers has cardinal number $\aleph_0$.

34. Do research to explain how it can be shown that the real numbers do not have cardinal number $\aleph_0$.

CHAPTER 2 *Summary*

Important Facts and Concepts		Examples and Discussion
Section 2.1		
Methods Used to Indicate a Set		
Description		Example 1, page 43
Roster Form		Examples 2–3, 5–7 pages 44, 45
Set-Builder Notation		Examples 4–6, pages 44–45
Symbol	*Meaning*	
$\in$	is an element of	Examples 4–6, pages 44–45
$\notin$	is not an element of	
$n(A)$	number of elements in set A	
$\varnothing$ or { }	the empty set	
U	the universal set	

In Exercises 15–18, express each set in roster form.

15. Set A is the set of odd natural numbers between 5 and 16.

16. Set B is the set of states that border Kansas.

17. $C = \{x \mid x \in N \text{ and } x < 162\}$

18. $D = \{x \mid x \in N \text{ and } 8 < x \leq 80\}$

In Exercises 19–22, express each set in set-builder notation.

19. Set A is the set of natural numbers between 50 and 150.

20. Set B is the set of natural numbers greater than 42.

21. Set C is the set of natural numbers less than 7.

22. Set D is the set of natural numbers between 27 and 51, inclusive.

In Exercises 23–26, express each set with a written description.

23. $A = \{x \mid x$ is a capital letter of the English alphabet from E through M inclusive$\}$

24. $B = \{$penny, nickel, dime, quarter, half-dollar$\}$

25. $C = \{a, b, c\}$

26. $D = \{x \mid 3 \leq x < 9\}$

In Exercises 27–36, let

$$U = \{1, 2, 3, 4, \ldots, 10\}$$
$$A = \{1, 3, 5, 7\}$$
$$B = \{3, 7, 9, 10\}$$
$$C = \{1, 7, 10\}$$

Determine the following.

27. $A \cap B$

28. $A \cup B'$

29. $A' \cap B$

30. $(A \cup B)' \cup C$

31. $A - B$

32. $A - C'$

33. $A \times C$

34. $B \times A$

35. The number of subsets of set B

36. The number of proper subsets of set A

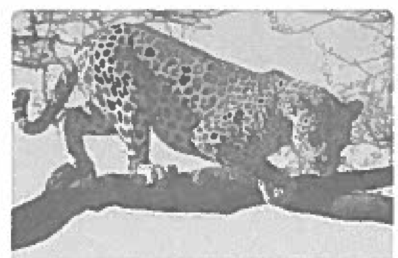
37. For the following sets, construct a Venn diagram and place the elements in the proper region.

$U = \{$lion, tiger, leopard, cheetah, puma, lynx, panther, jaguar$\}$

$A = \{$tiger, puma, lynx$\}$

$B = \{$lion, tiger, jaguar, panther$\}$

$C = \{$tiger, lynx, cheetah, panther$\}$



In Exercises 38–43, use Fig. 2.29 to determine the sets.

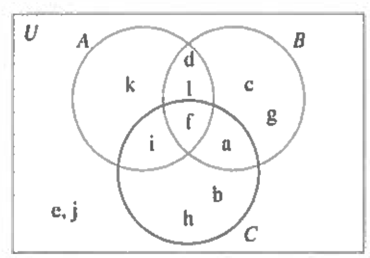


Figure 2.29

38. $A \cup B$

39. $A \cap B'$

40. $A \cup B \cup C$

41. $A \cap B \cap C$

42. $(A \cup B) \cap C$

43. $(A \cap B) \cup C$

Construct a Venn diagram to determine whether the following statements are true for all sets A, B, and C.

44. $(A' \cup B')' = A \cap B$

45. $(A \cup B') \cup (A \cup C') = A \cup (B \cap C)'$

2.6

In Exercises 56 and 57, show that the sets are infinite by placing each set in a one-to-one correspondence with a proper subset of itself.

56. $\{2, 4, 6, 8, 10, \dots\}$

57. $\{3, 5, 7, 9, 11, \dots\}$

In Exercises 58 and 59, show that each set has cardinal number $\aleph_0$ by setting up a one-to-one correspondence between the set of counting numbers and the given set.

58. $\{5, 8, 11, 14, 17, \dots\}$

59. $\{4, 9, 14, 19, 24, \dots\}$

CHAPTER 2 *Test*

In Exercises 1–8, state whether each is true or false. If the statement is false, explain why.

1. $\{2, y, \triangle, \$\}$ is equivalent to $\{p, \#, 4, \square\}$.

2. $\{3, 5, 9, h\} = \{9, 5, 3, j\}$

3. $\{\text{star, moon, sun}\} \subset \{\text{star, moon, sun, planet}\}$

4. $\{7\} \subseteq \{x \mid x \in N \text{ and } x < 7\}$

5. $\{p, q, r, s\}$ has 15 subsets.

6. If $A \cap B = \{\ \}$, then A and B are disjoint sets.

7. For any set A, $A \cup A' = \{\ \}$.

8. For any set A, $A \cap U = A$.

In Exercises 9 and 10, use set

$$A = \{x \mid x \in N \text{ and } x < 10\}$$

9. Write set A in roster form.

10. Write a description of set A.

In Exercises 11–16, use the following information.

$$U = \{3, 5, 7, 9, 11, 13, 15\}$$
$$A = \{3, 5, 7, 9\}$$
$$B = \{7, 9, 11, 13\}$$
$$C = \{3, 11, 15\}$$

Determine the following.

11. $A \cap B$

12. $A \cup C'$

13. $A \cap (B \cap C')$

14. $n(A \cap B')$

15. $A - B$

16. $A \times C$

17. Using the sets provided for Exercises 11–16, draw a Venn diagram illustrating the relationship among the sets.

18. Use a Venn diagram to determine whether

$$A \cap (B \cup C') = (A \cap B) \cup (A \cap C')$$

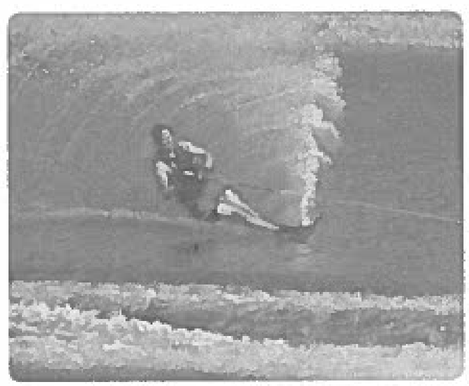
for all sets A, B, and C. Show your work.

19. ***Water Activities*** A survey of 155 residents of Lake Placid were asked what kind of water activities they participated in on a daily basis during the summer months. The following information was determined.

107 swam.
90 sailed.
76 water skied.
57 swam and sailed.
54 swam and water skied.
52 sailed and water skied.
35 swam, sailed, and water skied.

Construct a Venn diagram and then determine the number of residents who participated in

a) exactly one of these activities.
b) none of these activities.
c) at least two of these activities.
d) swimming and sailing, but not water skiing.
e) swimming or sailing, but not water skiing.
f) only water skiing.



20. Show that the following set is infinite by setting up a one-to-one correspondence between the set and a proper subset of itself.

$$\{7, 8, 9, 10, \dots\}$$