Research Statement

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My area of research is algebraic combinatorics. I apply representation theory of finite groups to compute some numerical invariants of combinatorial structures. My current focus is on computing the Smith and critical groups of families of Strongly Regular Graphs. The Smith group of a graph is the abelian group with the same invariant factors as the smith normal form of its Adjacency matrix. The *critical* group of a graph is the finite part of the abelian group with the same invariant factors as the Smith normal form of its Laplacian matrix. The critical groups of various graphs arise in combinatorics in the context of chip firing games (cf. [2]), as the abelian sandpile group in statistical mechanics (cf. [6]), and also in arithmetic geometry. One may refer to [15] for a discussion on these connections. It is therefore of some interest to compute the Smith groups and critical groups of graphs.

Background

Let $\Gamma = (\tilde{V}, \tilde{E})$ be a simple connected graph on vertex set $\tilde{V}$ and edge set $\tilde{E}$. Let A be the adjacency matrix with respect to some arbitrary but fixed order of V. Then the Laplacian matrix of Γ is the matrix $L := D - A$, where D is the diagonal matrix whose ith diagonal entry is the valency of the ith vertex. By abuse of notation we may consider A and L to be elements of $\mathrm{End}(\mathbb{Z}V)$. The abelian group $\dfrac{\mathbb{Z}V}{A(\mathbb{Z}V)}$ is called the Smith group $S(\Gamma)$ of Γ. The torsion of the abelian group $\dfrac{\mathbb{Z}V}{L(\mathbb{Z}V)}$ is called the *critical* group $\mathcal{K}(\Gamma)$ of Γ. These groups are important invariants of the graph Γ. The Kirchoff Matrix-Tree theorem implies that the order of $\mathcal{K}(\Gamma)$ is equal to the number of spanning trees of Γ (see for eg. [22]).

The Smith normal forms of the integer matrices A and L completely determine the Smith and *critical* groups respectively.

Some Families of graphs with known *critical* groups.

An early author on the *critical* group was Vince, who in [24] computed them for Wheel graphs and complete bipartite graphs. In the same paper, it was shown that the group depends only on the cycle matroid of the graph. There are relatively few classes of graphs with known Smith and *critical* groups. Other papers that include computation of *critical* groups of families of graphs include [5], [1], [11], [7], [4], and [17].

The methodology described in the next section was used to compute the *critical* groups of the following graphs: Paley Graphs (c.f. [5]); Peisert Graphs (c.f. [19]); Grassmann graph of lines in finite projective space and of its complement (c.f. [4] and [9]); Polar Graphs (c.f. [17]); Kneser graph on 2-subsets of an n-element set (c.f [8]); and van Lint-Schrijver cyclotomic Strongly Regular Graphs [16].

Recent work

As a part of my Ph.D. thesis under the supervision of Prof.Peter Sin (Univ. of Florida), I and P. Sin were able to determine the Smith group and critical group of the family of Polar graphs. Polar graphs are Strongly Regular collinearity graphs of finite classical polar spaces. These graphs admit certain finite classical groups as automorphisms. The action of finite classical groups on Polar spaces is a rank 3 permutation actions. The permutation modules corresponding to this action have been studied in [14], [13], [12] ,and [20]. In [17], we used these results to determine the Smith and Critical groups of the Polar graphs.

Methodology

Let $\Gamma = (\tilde{V}, \tilde{E})$ be a simple undirected graph. Fix an ordering on the set of vertices $\tilde{V}$, and let A be the adjacency matrix with respect to this ordering.Let D be the diagonal matrix with $D_i i$ being the degree of the ith vertex of Γ. Then $L := DA$ is called the Laplacian matrix of Γ. With some abuse of notation we may assume that A and L are elements of $End_{\mathbb{Z}}(\mathbb{Z}\tilde{V})$. The cokernal of A is the Smith group S of , and the finite part of the cokernal of L is the critical group $\mathcal{K}$ of Γ. Let ℓ be a prime number, and $\mathbb{Z}_\ell$ be the ring of ℓ-adic integers. We may assume that A and L are elements of $End_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell\tilde{V})$. Given $i \in \mathbb{N}$, define $M_i(A) := \{x \in \mathbb{Z}_\ell\tilde{V} | \, Ax \in \ell^i\mathbb{Z}_\ell\tilde{V}\}$, and define $M_i(L)$ in a similar fashion. Given any submodule M of $\mathbb{Z}_\ell\tilde{V}$, define $\overline{M} := (M + \ell\mathbb{Z}_\ell\tilde{V})/\mathbb{Z}_\ell\tilde{V}$. We observe that M is a subspace of the vector space $\mathbb{F}_\ell\tilde{V}$. Using some elementary Linear algebra, we can show the following result.

Lemma 1. *Let e_i be the multiplicity of $\mathbb{Z}/\ell^i\mathbb{Z}$ in the elementary divisor representation of S (respectively $\mathcal{K}$) . Then* $\dim\left(\overline{M_i(A)}/\overline{M_{i+1}(A)}\right) = e_i$ *(respectively* $\dim\left(\overline{M_i(L)}/\overline{M_{i+1}(L)}\right) = e_i$*).*

As S and $\mathcal{K}$ are abelian groups, they are the products of their Sylow subgroups. Thus the above Lemma reduces computing these groups to finding the dimensions of certain finite vector spaces. More over if G is a group of automorphisms of Γ, then $\overline{M_i(A)}$'s and $\overline{M_i(L)}$'s are $\mathbb{F}_\ell$G-submodules of the permutation module $\mathbb{F}_\ell\tilde{V}$.

Results

Let V be a finite vector space endowed with a non degenerate quadratic form q or a non degenerate symplectic space. Let $\hat{V}$ be the set of isotropic one dimensional subspaces of V. A polar graph is a graph (V) on $\hat{V}$, in which two elements of $\hat{V}$ are connected if and only if they are perpendicular with respect to the form on V. Let $\mathrm{G}(V)$ be the group of automorphisms of V. This action of $\mathrm{G}(V)$ on $\hat{V}$ is a rank 3 permutation action (cf. [14] and [13]) and thus $\Gamma(V)$ is a strongly regular graph. Let $\mathcal{K}$ be the critical group of $\Gamma(V)$.

Let p be the characteristic of the underlying field, then $v_p(|\mathcal{K}|) = 1$. Thus the p-part of $\mathcal{K}$ is cyclic. Let $\ell \neq p$ be a prime dividing $|\mathcal{K}|$. By the discussion above, the dimensions of $\mathbb{F}_\ell\mathrm{G}(V)$-modules $\overline{M_i(L)}$'s yield the ℓ-part of $\mathcal{K}$. These modules are submodules of the cross characteristic permutation module $\mathbb{F}_\ell\hat{V}$. The submodule structure of the cross characteristic permutation module corresponding to the action of $G(V)$ on $\hat{V}$ has been studied in [14], [13], [12], and [20]. This submodule structure was used to determine the dimensions of $\overline{M_i(L)}$'s. Using those dimensions and Lemma 1 we obtained the ℓ-part of $\mathcal{K}$. The following describes $\mathcal{K}$ when V is a symplectic space.

Theorem 2 (Pantangi-Sin 2017)**.** *Let V be a symplectic space of dimension $2m$ over a finite field $\mathbb{F}_q$, and $\mathcal{K}$ be the critical group of $\Gamma(V)$. Given $\ell \mid |\mathcal{K}|$ is a prime, let e_i denote the multiplicity of $\mathbb{Z}/\ell^i\mathbb{Z}$ as an elementary divisor of $\mathcal{K}$. The following are true.*

1. *If ℓ is odd prime with $v_\ell(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a$ $v_\ell(1 + q^{m-1}) = b$, then*
   - (a) *If $a > 0, b > 0$ $e_{a+b} = g - 1$, $e_b = 1$ and $e_i = 0$ for $i \neq a$.*
   - (b) *If $a = 0$, $e_b = g$ and $e_i = 0$ for all other i.*
   - (c) *If $b = 0$, $e_a = g - 1$ and $e_i = 0$ for all other i.*
2. *If ℓ is an odd prime with $v_\ell(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a > 0$ and $v_\ell(q^m + 1) = b > 0$, we have*
   - (a) *If $a > 0, b > 0$ $e_{a+b} = f - 1$, $e_a = 1$ and $e_i = 0$ for $i \neq a + b, a$.*
   - (b) *If $b = 0$, $e_a = f$, and $e_i = 0$ for all other i.*
   - (c) *If $a = 0$, $e_b = f - 1$, and $e_i = 0$ for all other i.*
3. *If $\ell = 2$ and q is odd,*
   - (a) *If m is even, $v_\ell(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a > 0$ and $v_\ell(q^{m-1} + 1) = b > 0$, we have $e_{a+b+1} = g - 1$, $e_{b+1} = 1$,$e_1 = f - g - 1$, and $e_i = 0$ for all other i.*
   - (b) *If m is odd, $v_\ell(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a > 0$ and $v_\ell(q^m + 1) = b > 0$, we have $e_{a+b}(2) = f - g - 1$, $e_{a+b+1}(2) = g + 1$, $e_a(2) = 1$, , and $e_i = 0$ for all other i.*

In [17], we obtained similar descriptions for the *critical* groups of other Polar graphs.

In [16], I obtained a description of the *critical* groups of the van Lint-Schrijver *cyclotomic Strongly Regular Graphs*. Let $q = p^f$ be a prime power, let $N > 1$ be a divisor of $q-1$. Let D be a subgroup of multiplicative group of the finite field $\mathbb{F}_q$. Let $\mathrm{Cay}(\mathbb{F}_q, D)$ be the Cayley graph on the additive group of $\mathbb{F}_q$ with connection set D. If $\mathrm{Cay}(G, D)$ is a strongly regular graph, then we speak of a *cyclotomic strongly regular graph*. The Paley graph is a well known example of a *cyclotomic strongly regular graph*. Extensive scholarship on these graphs include [23], [3], [18], and [10].

In [23], van Lint and Schrijver define a family of *cyclotomic Strongly Regular Graphs* whose construction is similar to that of the Paley Graph. Let p and ℓ be primes,with $\ell > 2$ and p primitive $\pmod{\ell}$. Let $t \in \mathbb{N}$ and $q = p^{(\ell-1)t}$. Consider the field $K = \mathbb{F}_q$, and let S be the unique subgroup of K^* of order $k = (q-1)/\ell$. Then by $G(p, \ell, t)$ we denote the graph with vertex set K and edge set $\{\{x, y\} \mid x, y \in K \;\&\; x - y \in S\}$. This is the undirected Cayley graph associated with (K, S). We call these families of graphs, *Van Lint-Schrijver cylclotomic Strongly Regular Graphs*.

The additive group K, and multiplicative groups S and $K*$ are automorphism groups for $G(p, \ell, t)$. Let R be the ring of integers of the unique unramified extension of degree $(\ell - 1)t$ over $\mathbb{Q}_P$. Let R^K be the permutation module over R, associated with the action of S on vertex set K. Let T be the Teichmller character of the multiplicative group K^*. Given $x \in K$, by $[x]$ we denote the basis element of R^K corresponding to x. Let $f_i = \sum_{x \in K^*} T^i(x^{-1})[x]$ We obtained the decomposition $R^K = \oplus_{i=0}^{k-1} N_i$, where each N_i is a submodule of R^K with basis $\{f_{i+mk} | \; 0 \leq m \leq \ell - 1\}$. Each N_i is an isotopic component of for the character $T^i|_S$. By L_i we denote the restriction $L|_{N_i}$ of the Laplacian L of $G(p, \ell, t)$. With respect to the basis of L_i describe above, the matrix of L_i has certain Jacobi sums as entries.

Computing the Smith Normal forms over R of the smaller matrices L_i, give us a description of the SNF over R of L. This information was used to compute the p-part of the *critical* group C of $G(p, \ell, t)$. The p'-part was obtained by conjugating L by $\frac{1}{q}X$, where X is the complex character table K. I obtained the following descriptions of the *critical* group of C in [16].

Theorem 3. *Consider the graph $G(p, \ell, t)$ with $p^{(\ell-1)t/2} \neq \ell - 1$ whenever t is odd. Given integers a, b not divisible by $p^{(\ell-1)t} - 1$, let $c(a, b)$ denote the number of carries when adding the p-adic expansions of a and b $\pmod{q-1}$. Let L be the Laplacian matrix and C be the* critical *group of $G(p, \ell, t)$. For $1 \leq i < k - 1$, let*

$$c(i) = \min\left(\{c(i + mk, nk) | 0 \leq m \leq \ell - 1 \;\&\; 0 \leq n \leq \ell - 1\}\right).$$

Let e_j be the multiplicity of $\mathbb{Z}/p^j\mathbb{Z}$ as an elementary divisor of C. Then we have the following.

1. $e_0 = |\{i \mid 1 \leq i \leq k-1 \;\& c(i) = 0\}| + 2$ *and* $e_{(\ell-1)t+d} = e_0 = |\{i \mid c(i) = 0\}|$.
2. $e_j = |\{i \mid 1 \leq i \leq k-1 \;\& c(i) = j\}|$ *for* $0 < j < \frac{(\ell-1)t}{2}$.
3. $e_j = e_{(\ell-1)t+d-j}$ *for* $0 < j < \frac{(\ell-1)t}{2}$.
4. *If* $p \nmid \ell - 1$, *then* $e_{\frac{(\ell-1)t}{2}} = q + 1 - 2\sum_{j<t} e_j$.
5. *If* $p \mid \ell - 1$, *then*
   (a) $e_{\frac{(\ell-1)t}{2}+d} = k + 2 - \sum_{j<t} e_j$ *and*
   (b) $e_{\frac{(\ell-1)t}{2}} = (\ell - 1)k - \sum_{j<t} e_j$.
6. $e_j = 0$ *for all other* j.

In the case of $G(p, 3, t)$, using the transfer matrix method (cf. Section 4.7 of [21]) we were able to determine a closed form for the p-rank (i.e e_0 in the context of the Theorem above) of the Laplacian. The following theorem gives a quick recursive algorithm to compute other p-elementary divisors.

Let $P = \left(\left(\frac{p+1}{3}\right)^2 (x^2y^2 + x^2y + xy^2 + x + y + 1) + \left(\frac{p-2}{3}\right)^2 3xy\right)$, $R = p^2x^3y^3$ and

$Q = \left(\left(\frac{p+1}{3}\right)^2 (xy)(x^2y^2 + x^2y + xy^2 + x + y + 1) + \left(\frac{2p-1}{3}\right)^2 3x^2y^2\right)$. We define the polynomial $C(2t) \in \mathbb{C}[x, y]$ recursively as follows:

$$\begin{aligned} C(2) &= 2P \\ C(4) &= 2(P^2 - 2Q), \\ C(6) &= 6R + 2(P^3 - 2QP) - 2PQ, \\ \text{and } C(2t) &= PC(2t-2) - QC(2t-4) + RC(2t-6) \text{ for } t > 3. \end{aligned} \tag{1}$$

Theorem 4. *Let C_p be the p-part of the critical group of the graph $G(p, 3, t)$ (with $(p, t) \neq (2, 1)$). Let e_j denote the multiplicity of $\mathbb{Z}/p^j\mathbb{Z}$ in the elementary factor form of C_p. Let E_{ab} be the coefficient of $x^a y^b$ in $C(2t)$. Then we have the following.*

1. $e_0 = e_{2t+\delta_{2,p}} + 2 = \left(\frac{(p+1)}{3}\right)^{2t} (2^{t+1} - 2)$.
2. *For $a < t$, we have* $e_a = e_{2t+\delta_{2,p}-a} = \sum_{a<b\leq t} E_{ab}$
3. $e_{t+\delta_{2,p}} = (k + 2 - \sum_{j<t} e_j) + (1 - \delta_{2,p})(2k - \sum_{j<t} e_j)$.
4. $e_t = (1 - \delta_{2,p})(k + 2 - \sum_{j<t} e_j) + (2k - \sum_{j<t} e_j)$.
5. $e_a = 0$ *for all other a.*

Future Work

I plan to find the Smith and critical groups of other families of Strongly Regular Graphs using the methodology described in the previous section. I am currently interested in computing the critical groups of the following graphs.

1. Cyclotomic Strongly Regular graphs.
2. Strongly regular graphs arising from Latin square designs.
3. The complements of Polar graphs.
4. The families of Strongly regular graphs arising from rank 3 permutation action of certain finite classical groups on the set of non-isotropic points.

My current focus is to try to obtain the *critical* groups of some other families of Cyclotomic Strongly Regular graphs. The methods I used to find the *critical* groups of the van Lint-Schrijver family were extensions of to those used in [5] to compute the same for Paley Graphs. I believe these methods can be generalized to other families of Cyclotomic Strongly Regular Graphs.

References

[1] Hua Bai. On the critical group of the n-cube. *Linear algebra and its applications*, 369:251–261, 2003.

[2] N.L. Biggs. Chip-firing and the critical group of a graph. *Journal of Algebraic Combinatorics*, 9(1):25–45, Jan 1999.

[3] AE Brouwer, RM Wilson, and Qing Xiang. Cyclotomy and strongly regular graphs. *Journal of Algebraic Combinatorics*, 10(1):25–28, 1999.

[4] Andries Brouwer, Joshua Ducey, and Peter Sin. The elementary divisors of the incidence matrix of skew lines in PG$(3, q)$. *Proceedings of the American Mathematical Society*, 140(8):2561–2573, 2012.

[5] David B. Chandler, Peter Sin, and Qing Xiang. The smith and critical groups of paley graphs. *Journal of Algebraic Combinatorics*, 41(4):1013–1022, Jun 2015.

[6] Deepak Dhar. Self-organized critical state of sandpile automaton models. *Physical Review Letters*, 64(14):1613, 1990.

[7] Joshua E Ducey, Jonathan Gerhard, and Noah Watson. The smith and critical groups of the square rook's graph and its complement. *arXiv preprint arXiv:1507.06583*, 2015.

[8] Joshua E Ducey, Ian Hill, and Peter Sin. The critical group of the kneser graph on 2-subsets of an n-element set. *arXiv preprint arXiv:1707.09115*, 2017.

[9] Joshua E Ducey and Peter Sin. The smith group and the critical group of the grassmann graph of lines in finite projective space and of its complement. *arXiv preprint arXiv:1706.01294*, 2017.

[10] Tao Feng, Koji Momihara, and Qing Xiang. Constructions of strongly regular cayley graphs and skew hadamard difference sets from cyclotomic classes. *Combinatorica*, 35(4):413–434, 2015.

[11] Brian Jacobson, Andrew Niedermaier, and Victor Reiner. Critical groups for complete multipartite graphs and cartesian products of complete graphs. *Journal of Graph Theory*, 44(3):231–250, 2003.

[12] JM Lataille, Peter Sin, and Pham Huu Tiep. The modulo 2 structure of rank 3 permutation modules for odd characteristic symplectic groups. *Journal of Algebra*, 268(2):463–483, 2003.

[13] Martin W Liebeck. Permutation modules for rank 3 unitary groups. *Journal of Algebra*, 88(2):317–329, 1984.

[14] Martin W Liebeck. Permutation modules for rank 3 symplectic and orthogonal groups. *Journal of Algebra*, 92(1):9–15, 1985.

[15] Dino Lorenzini. Smith normal form and laplacians. *Journal of Combinatorial Theory, Series B*, 98(6):1271 – 1300, 2008.

[16] Venkata Raghu Tej Pantangi. Critical group of van lint-schrijver cyclotomic strongly regular graphs. 2018.

[17] Venkata Raghu Tej Pantangi and Peter Sin. Smith and critical groups of polar graphs. *arXiv preprint arXiv:1706.08175*, 2017.

[18] Bernhard Schmidt and Clinton White. All two-weight irreducible cyclic codes? *Finite Fields and Their Applications*, 8(1):1–17, 2002.

[19] Peter Sin. The critical groups of the peisert graphs $\mathrm{P}^*(q)$. *arXiv preprint arXiv:1606.00870*, 2016.

[20] Peter Sin and Pham Huu Tiep. Rank 3 permutation modules of the finite classical groups. *Journal of Algebra*, 291(2):551–606, 2005.

[21] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, New York, NY, USA, 2nd edition, 2011.

[22] Richard P Stanley. Smith normal form in combinatorics. *Journal of Combinatorial Theory, Series A*, 144:476–495, 2016.

[23] Jacobus H van Lint and Alexander Schrijver. Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields. *Combinatorica*, 1(1):63–73, 1981.

[24] A. Vince. Elementary divisors of graphs and matroids. *European Journal of Combinatorics*, 12(5):445 – 453, 1991.