Research Statement
Venkata Raghu Tej Pantangi

My area of research is algebraic combinatorics. I apply representation theory of finite groups to compute some numerical invariants of combinatorial structures. My current focus is on computing the Smith and critical groups of families of Strongly Regular Graphs. The Smith group of a graph is the abelian group with the same invariant factors as the smith normal form of its Adjacency matrix. The critical group of a graph is the finite part of the abelian group with the same invariant factors as the Smith normal form of its Laplacian matrix.

The critical groups of various graphs arise in combinatorics in the context of chip firing games (cf. [2]), as the abelian sandpile group in statistical mechanics (cf. [5]), and also in arithmetic geometry. One may refer to [13] for a discussion on these connections. It is therefore of some interest to compute the Smith groups and critical groups of graphs.

Background

Let \( \Gamma = (\tilde{V}, \tilde{E}) \) be a simple undirected graph. Fix an ordering on the set of vertices \( \tilde{V} \), and let \( A \) be the adjacency matrix with respect to this ordering. Let \( D \) be the diagonal matrix with \( D_{ii} \) being the degree of the \( i \)th vertex of \( \Gamma \). Then \( L := D - A \) is called the Laplacian matrix of \( \Gamma \).

The matrices \( A \) and \( L \) are integer matrices. Let \( M \) be any integer matrix, and let \( \text{Rank}(M) = r \). Then as a consequence of the structure theorem of finitely generated modules over PIDs, we can find unimodular integer matrices \( P \) and \( Q \) and non-zero integers \( \alpha_1, \alpha_2, \ldots, \alpha_r \) with \( \alpha_i | \alpha_{i+1} \) for \( 1 \leq i < r \) such that \( PMQ \) is

\[
\begin{bmatrix}
Y & 0_{(m-r\times r)} \\
0_{(r\times r)} & 0_{(n-r\times n-r)}
\end{bmatrix}
\]

where \( Y = \text{diag}(\alpha_1, \ldots, \alpha_r) \). This diagonal matrix is called the Smith normal form of \( M \). We shall denote this by \( \text{SNF}(M) \).

The \( \text{SNF}(A) \) and \( \text{SNF}(L) \) are important invariants of the graph. These Smith normal forms determine the Smith group and Critical group of \( \Gamma \).

Chip Firing Games

Given a vertex \( w \) of \( \Gamma \), let \( d(v) \) denotes its degree. Fix a vertex \( B \) of \( \Gamma \). Place \( s(v) \) chips at every vertex \( v \neq B \) and define \( s(B) := -\sum_{v \neq B} s(v) \) to \( B \). The function \( s : \tilde{V} \rightarrow \mathbb{Z} \) is called a configuration of \( \Gamma \). We say a vertex \( v \neq B \) is ready if and only if \( s(v) \geq d(v) \). The vertex \( B \) is ready when no other vertex is ready. A ready vertex can fire by donating one chip to each of its neighbours. The Chip firing game is played by going from one configuration to other other via a sequence of firings. The following is a sample game on the 4-cycle.

A configuration is said to be recurrent if there is a sequence of firings that lead back to the configuration. A configuration is said to be stable if only \( B \) can be fired. A configuration is said to be critical if it is both stable and recurrent.

In a sequence of firings let vertex \( w \) fire \( e(w) \) times. Then the final configuration \( t \) is given by \( t = s + Le \). In [2] it was shown that the set of critical configurations is in one-one correspondence with the critical group \( K(\Gamma) \) of \( \Gamma \).
Some families of graphs with known Critical Groups.

Critical group has been computed for relatively few families of graphs. Using the fact that the order of the critical group is the number of spanning trees, one can see that the critical group of any tree is trivial. Some elementary computations can show that the critical group of the complete graph on \( n \) vertices is \( (\mathbb{Z}/n\mathbb{Z})^{n-2} \).

In [2], the critical groups of the family of Wheel graphs was computed. This was done by finding explicit critical configurations that generate the critical group. In [6] the critical groups of the families of Square Rook graphs and their complements were computed by finding explicit critical configurations that generate the critical group.

The critical groups of complete multipartite graphs (c.f. [9]) and those of the family of hypercube graphs (c.f. [1]) were computed by finding the Smith normal forms of the Laplacian by explicit unimodular matrix operations.

Some families of graphs with known Critical Groups.

Critical group has been studied in [12], [11], [10], and [18]. Using this information, I and P. Sin were able to determine the Smith group of automorphisms of \( V \) has been studied in [12], [11], [10], and [18]. Using this information, I and P. Sin were able to determine the Smith group and critical group of the family of Polar graphs. Polar graphs are Strongly Regular collinearity spaces. Let \( \tilde{\text{End}}(\Gamma) \) be the diagonal matrix with \( \ell \) in the elementary divisor representation of \( \Gamma \).

We observe that \( \text{Conf}(\Gamma) \) is the critical group of \( \Gamma \) and the finite part of the cokernal of \( \tilde{L} \) is the critical group \( K \) of \( \Gamma \).

1. Let \( e_i \) be the multiplicity of \( (\mathbb{Z}/\ell \mathbb{Z})^i \) in the elementary divisor representation of \( S \).

Then \( \text{dim}(\tilde{M}_i(A))/\tilde{M}_{i+1}(A)) = e_i \).

2. Let \( e_i \) be the multiplicity of \( (\mathbb{Z}/\ell \mathbb{Z})^i \) in the elementary divisor representation of \( K \).

Then \( \text{dim}(\tilde{M}_i(L))/\tilde{M}_{i+1}(L)) = e_i \).

Thus we reduce the problem of computing \( S \) and \( K \) to finding dimensions of certain vector spaces.

Let \( G \) be a group of automorphisms of \( \Gamma \). Then \( A \) and \( L \) commute with the action of \( G \) on \( \Gamma \). So the vector spaces \( \tilde{M}_i(A)(\Gamma) \)'s and \( \tilde{M}_i(L)(\Gamma) \)'s are also \( G \)-submodules of \( F_{\ell} \tilde{V} \). Therefore we may use the \( G \)-submodule structure of \( F_{\ell} \tilde{V} \) to determine \( S \) and \( K \).

Let \( V \) be a finite vector space endowed with a non degenerate quadratic form \( q \) or a non degenerate symplectic space. Let \( \tilde{V} \) be the set of isotropic one dimensional subspaces of \( V \). A polar graph is a graph \( \Gamma(V) \) on \( \tilde{V} \), in which two elements of \( \tilde{V} \) are connected if and only if they are perpendicular with respect to the form on \( V \). Let \( G(V) \) be the group of automorphisms of \( V \). The submodule structure of the permutation module corresponding to the action of \( G(V) \) on \( \tilde{V} \) has been studied in [12], [11], [10], and [18]. Using this information, I and P. Sin were able to determine the Smith group and critical group of \( \Gamma(V) \). (cf. [16])
Future Work

I plan to find the Smith and critical groups of other families of Strongly Regular Graphs using the methodology described in the previous section. I am currently interested in computing the Smith and critical groups of the following graphs.

1. Strongly regular graphs arising from Latin square designs.
2. The complements of Polar graphs.
3. The families of Strongly regular graphs arising from rank 3 permutation action of certain finite classical groups on the set of non-isotropic points.

My current focus is to find the Smith and critical groups of strongly regular graphs arising from Latin squares. A Latin square is an $n \times n$ array filled with $n$ different symbols, each occurring exactly once in each row and exactly ones in each column. The multiplication table of a finite group is a Latin square. Let $G$ be a Loop of size $n$, and $L(G)$ be the Latin square representing the multiplication table of $G$. Define $\Gamma(G)$ to be the graph on $G \times G$, such that $(g, h)$ is connected to $(g', h')$ if and only if either $g = g'$, or $h = h'$, or $gh = gh'$. Let $A$ be an elementary abelian group, and let $L(A)$ be the Latin square representing the multiplication table of $A$. Then $\text{Aut}(A)$, $A$, and $S_3$ are groups of automorphisms of $L(A)$. Let $p$ be a prime and $A = (\mathbb{Z}/p\mathbb{Z})^n$. Then $\text{Aut}(A) \cong \text{Gl}(n, p)$. I plan to use the very well known representation theory of $\text{Gl}(n, p)$ to compute the Smith and critical groups of $\Gamma(A)$.

I am also interested in computing the Smith normal forms of other incidence matrices arising in combinatorics. I believe that representation theory of the automorphism group of an incidence structure may shed some light on the invariant factors of the corresponding incidence matrix. In particular, I am interested in the incidence matrices of Hadamard-2-designs. The invariants factors of these are closely related to the invariant factors of Hadamard matrices. A Hadamard matrix of order $n$ is an $n \times n$ matrix $H$ whose entries are either 1 or $-1$ and $HH^T = nI$. It is well known that the only values for $n$ are 1, 2, and a multiple of 4. Using a well know argument (cf. [15]), it can be shown that if $s_1 \ldots s_n$ are the invariant factors of $H$, then $s_1 = 1$, $s_2 = 2$ and $s_i s_{n-i+1} = n$. If $n = 4m$, and $s_2 = \ldots = s_{2m} = 2$, the Hadamard matrix $H$ is said to have a standard type of Smith normal form. In a few cases (cf. [15] & [14]), the matrix $H$ had Smith normal form of standard type. There are plenty of examples of Hadamard matrices whose Smith normal forms are not of standard type. I am interested in determining the Smith normal forms of other families of Hadamard matrices.

I am open to working on any problems in the areas of combinatorics and representation theory.

References


