Smith and critical groups of the symplectic polar graph.

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Joint work with Peter Sin.

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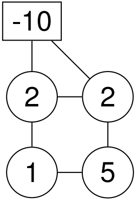
These groups are important invariants of a graph.
Consider the complete graph on n vertices, K_n. In this case $A = J - I$, and $L = (n-1)I - A$. Here I is the identity matrix and J the all-one matrix of size $n \times n$.The Smith group of this graph is $\mathbb{Z}/(n-1)\mathbb{Z}$, and the critical group being $(\mathbb{Z}/n\mathbb{Z})^{n-2}$.

Chip Firing Game



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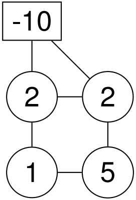

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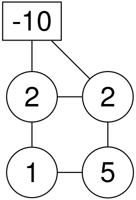

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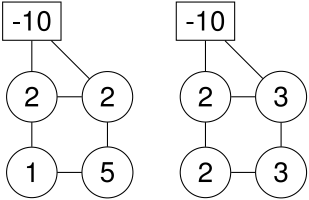

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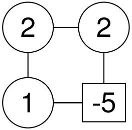
A round vertex v can be fired if $s(v) \geq d(v)$ The square vertex fires only when no other can fire.

Sample game



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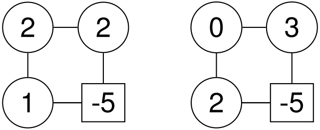

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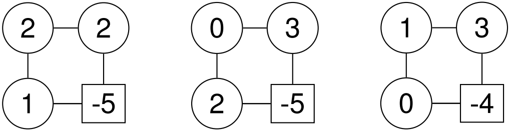

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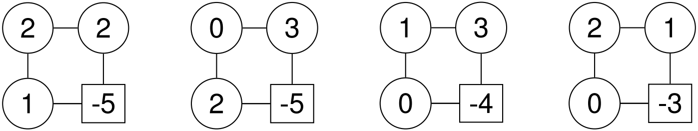

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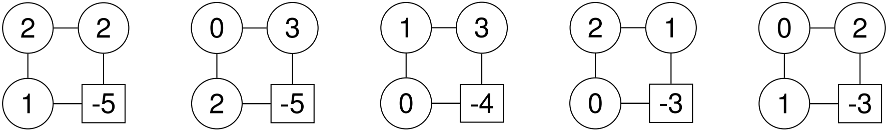

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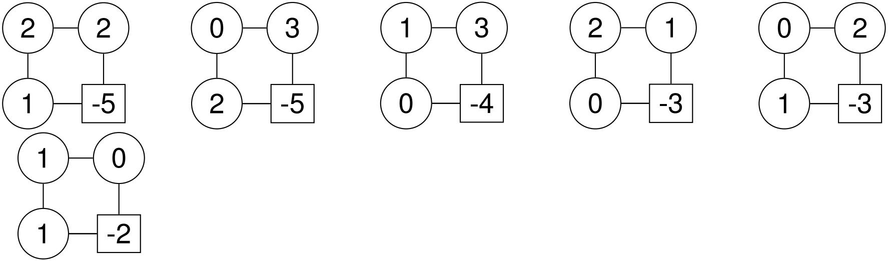

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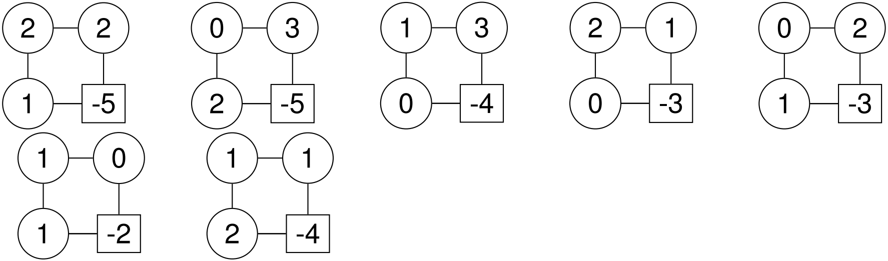

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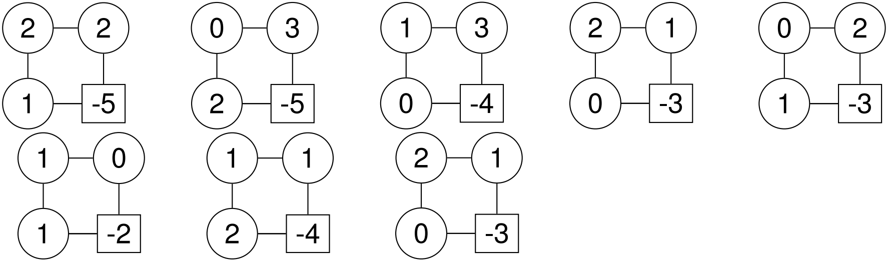

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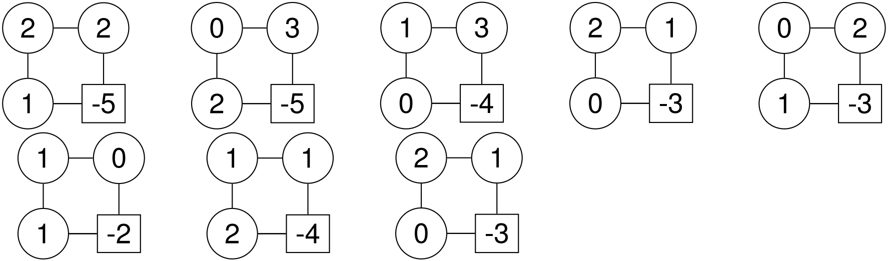

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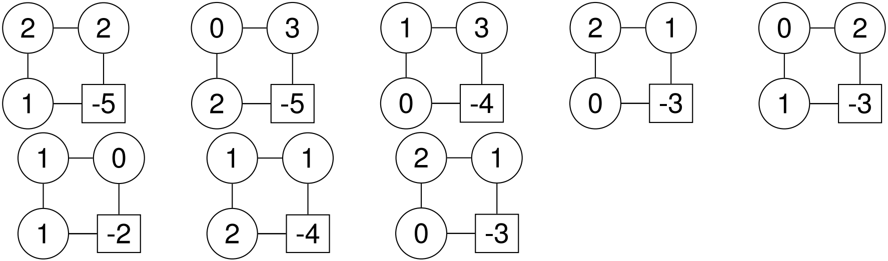

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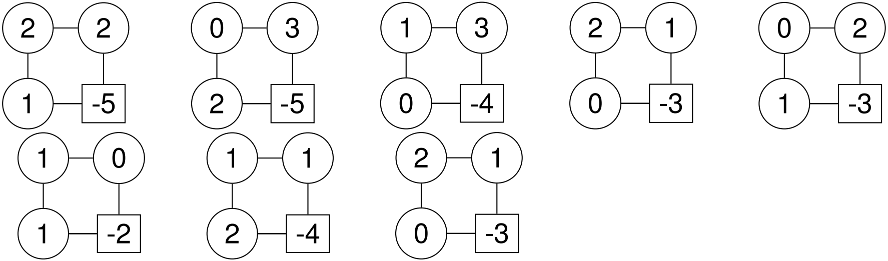

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- A configuration is said to be *stable* if there no round vertex can be fired
- A configuration is said to be *critical* if it is both *stable* and *recurrent*

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Theorem (Biggs 1997)

Any starting configuration of a graph G leads to a unique critical configuration.
The set of critical configuration has a natural group operation that is isomorphic to the critical group $K(G)$.

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- Grassman graph and complement (Sin-Ducey to appear)

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$\overline{M_i}'s$ are $\mathbb{F}_\ell Aut(G)$-submodules of the permutation action of $Aut(G)$ on V.

Integer eigenvalues

Now if C has an integer eigenvalue λ of multiplicity f. Let $v_\ell(\lambda) = a$. Treating C an element of $End_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell V)$, define V_λ to be the eigensubspace corresponding to λ. Then $V_\lambda \cap \mathbb{Z}_\ell V \subset M_a(C)$ and is a pure sublattice of rank f. Therefore we have $dim(\overline{M_a(C)}) \geq dim(\overline{V_\lambda \cap \mathbb{Z}_\ell V}) = f$.

Trivial application

Consider $G = K_n$(complete graph on n-vertices). Let $C = A = J - I$, with J being the all 1 matrix. C has eigenvalues $(n-1, -1)$ with multiplicities $(1, n-1)$. In this case, $|S(G)| = |det(A)| = n-1$. Assume $v_\ell(n-1) = a$. $n-1$ is an integer eigen value with multiplicity 1, so $\sum_{i \geq a} e_i = dim(\overline{M}_a) \geq 1$.

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So we have $e_a = 1$ and $e_i = 0$ for $i \neq a$. So the Smith group of the complete graph on n vertices is $\mathbb{Z}/(n-1)\mathbb{Z}$.

Strongly regular graph

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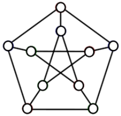

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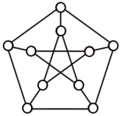

Let A be an adjacency matrix of a strongly regular graph with parameters (v, k, λ, μ). Then A satisfies

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where I is the $v \times v$ identity matrix and J is the $v \times v$ all one matrix.
We also have $(v - k - 1)\mu = k(k - \lambda - 1)$.

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Γ is an SRG with parameters
$(v, k, \lambda, \mu) = \left(\begin{bmatrix} 2m \\ 1 \end{bmatrix}_q, \ q \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q (1 + q^{m-1}), \ \begin{bmatrix} 2m-2 \\ 1 \end{bmatrix}_q - 2, \ \begin{bmatrix} 2m-2 \\ 1 \end{bmatrix}_q \right)$.

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$|S| = |det(A)| = |kr^f s^g|$ and $|K| = t^f u^g / v$(by Kirchhoff's matrix-tree theorem.)

Description of S

Theorem

Let $\ell \mid |S|$, then

1. *If ℓ is odd prime with $v_\ell(1+q^{m-1}) = a > 0$, then $e_a(\ell) = g+1$ and $e_i(\ell) = 0$ for $i \neq a$.*
2. *If ℓ is an odd prime with $v_\ell(\left[{m-1 \atop 1}\right]_q) = a$ and $v_\ell(q-1) = b$, we have*
    1. *If $a > 0, b > 0$, $e_{a+b}(\ell) = f$, $e_a(\ell) = 1$ and $e_i(\ell) = 0$ for $i \neq 0, a+b, a$*
    2. *If $b = 0$, $e_a = f+1$ and $e_i(\ell) = 0$ for $i \neq 0, a$*
    3. *If $a = 0$, $e_b = f$ and $e_i(\ell) = 0$ for $i \neq 0, b$*
3. *If $\ell \mid q$, $e_{v_\ell(q)}(\ell) = 1$, and $e_i(\ell) = 0$ for $i \neq v_\ell(q)$.*
4. *If $\ell = 2$ and q is odd,*
    1. *If m is even, $e_a(2) = f-g-1$ and $e_{a+b}(2) = g+1$ and $e_i(2) = 0$ for all other i's. Where $a = v_2(q-1)$ and $b = v_2(q^{m-1}+1)$.*
    2. *If m is odd, $e_{a+b+1}(2) = g+1$, $e_{a+b}(2) = f-g-1$, $e_a(2) = 1$, and $e_i(2) = 0$ for all other i's. Here, $v_2(\left[{m-1 \atop 1}\right]_q) = a$, $v_2(q-1) = b$.*

Description of K

Theorem

Let $\ell \mid |K|$, then

1. *If ℓ is odd prime with $v_\ell(\left[{m \atop 1}\right]_q) = a$ $v_\ell(1+q^{m-1}) = b$, then*
   1. *If $a > 0, b > 0$ $e_{a+b}(\ell) = g-1$, $e_b(\ell) = 1$ and $e_i(\ell) = 0$ for $i \neq a$.*
   2. *If $a = 0$, $e_b = g$ and $e_i = 0$ for all other i.*
   3. *If $b = 0$, $e_a = g-1$ and $e_i = 0$ for all other i.*
2. *If ℓ is an odd prime with $v_\ell(\left[{m-1 \atop 1}\right]_q) = a > 0$ and $v_\ell(q^m+1) = b > 0$, we have*
   1. *If $a > 0, b > 0$ $e_{a+b}(\ell) = f-1$, $e_a(\ell) = 1$ and $e_i(\ell) = 0$ for $i \neq a+b, a$.*
   2. *If $b = 0$, $e_a = f$ and $e_i = 0$ for all other i.*
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   2. *If m is odd, $v_\ell(\left[{m-1 \atop 1}\right]_q) = a > 0$, and $v_\ell(q^m+1) = b > 0$, we have $e_{a+b}(2) = f-g-1$, $e_{a+b+1}(2) = g+1$ and $e_a(2) = 1$.*

Numerical Example

Let $q = 9$ and $m = 3$. Then Γ is an SRG with parameter $(66430, 7380, 818, 820)$. The eigenvalues of A are $(7380, 80, -82)$ with multiplicities $(1, 33579, 32850)$. The eigenvalues of L are $(0, 7300, 7462)$.

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$$S = \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/41\mathbb{Z})^{32581} \times (\mathbb{Z}/5\mathbb{Z})^{33580} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/16\mathbb{Z})^{728} \times (\mathbb{Z}/32\mathbb{Z})^{32851}$$

$$K = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})^{728} \times (\mathbb{Z}/8\mathbb{Z})^{32851} \times (\mathbb{Z}/41\mathbb{Z}) \times (\mathbb{Z}/91\mathbb{Z})^{32580} \times (\mathbb{Z}/25\mathbb{Z})^{33578} \times (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/73\mathbb{Z})^{33579}$$

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There are about 4×10^{81} atoms in the observable universe. This graph has more spanning trees than the number of atoms in the observable universe!

Some arithmetic

Let $\ell \mid |S| = |kr^f s^g|$ be a prime. We have $|s| - |r| = (1 + q^{m-1}) - (q^{m-1} - 1) = 2$, $k = q\frac{r}{q-1}s$ and $\mu = \frac{r}{q-1}s$. A satisfies $(A - rI)(A - sI) = \mu J$

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Same problems arise for the critical group when $\ell = 2$.

ℓ-part of S when $\ell \mid r$ and $\ell \nmid s$

Let $v_\ell(q-1) = b > 0$, $v_\ell(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a > 0$, then $v_\ell(r) = a + b$ and $v_\ell(k) = v_\ell(\mu) = a$. Then $e_{a+b} = f$, $e_a = 1$ and $e_i = 0$ for all other positive i's.

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2-parts of S and K

In this case, $\ell = 2$. The vector space $\mathbb{F}_2\Gamma$ is a permutation module for $Sp(V)$. The vector spaces $\overline{M_i(A)}$, and $\overline{M_i(L)}$ are $Sp(V)$– submodules of $\mathbb{F}_2\Gamma$.

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Let (,) be the symmetric bilinear form on $\mathbb{F}_2\Gamma$ with vertices of Γ being an orthonormal basis. If W is any subspace of V, $[W] := \sum_{<v> \in \mathbb{P}^1(W)} < v >$.

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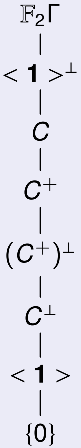
Let $\mathbf{1} = [V]$, and $C = \langle \{ [W] | \ W \text{ is a maximal totally isotropic subspace} \} \rangle$
$C^+ = \langle \{ [W] - [W'] | \ W, W' \text{ are maximal totally isotropic subspace} \} \rangle$.
Let $M \subset \mathbb{F}_2\Gamma$, then $M^\perp$ is the orthogonal complement of M with respect to $(\, , \,)$.

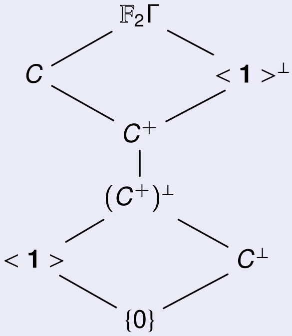
Theorem (Lattile Sin Tiep 2003)

The submodule structure for $\mathbb{F}_2\Gamma$ *is given by the following Hasse diagrams:*

m is even



m is odd



$dim(C) = f + 1$, $dim(C^{+}) = f$, $dim((C^{+})^{\perp}) = g + 1$, $\& dim(C^{\perp}) = g$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f-g-1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\begin{array}{c} \mathbb{F}_2\Gamma \\ | \\ <\mathbf{1}>^{\perp} \\ | \\ C \\ | \\ C^{+} \\ | \\ (C^{+})^{\perp} \\ | \\ C^{\perp} \\ | \\ <\mathbf{1}> \\ | \\ \{0\} \end{array}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g + 1$ and $e_i = 0$ for other i.

$\mathbb{F}_2\Gamma$

|

$< \mathbf{1} >^{\perp}$

|

C

|

$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$

|

$(C^{+})^{\perp}$

|

$C^{\perp}$

|

$< \mathbf{1} >$

|

$\{0\}$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g + 1$ and $e_i = 0$ for other i.

$$\mathbb{F}_2\Gamma$$
$$|$$
$$<\mathbf{1}>^{\perp}$$
$$|$$
$$C$$
$$|$$
$$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$$
$$|$$
$$(C^{+})^{\perp}$$
$$|$$
$$C^{\perp}$$
$$|$$
$$<\mathbf{1}> \qquad \subsetneq Im(\overline{A})$$
$$|$$
$$\{0\}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\mathbb{F}_2\Gamma$$
$$|$$
$$<\mathbf{1}>^{\perp}$$
$$|$$
$$C$$
$$|$$
$$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$$
$$|$$
$$(C^{+})^{\perp}$$
$$|$$
$$C^{\perp} \qquad \neq Im(\overline{A})$$
$$|$$
$$<\mathbf{1}> \qquad \subsetneq Im(\overline{A})$$
$$|$$
$$\{0\}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f-g-1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\mathbb{F}_2\Gamma$$
$$|$$
$$<\mathbf{1}>^{\perp}$$
$$|$$
$$C$$
$$|$$
$$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$$
$$|$$
$$(C^{+})^{\perp} \qquad Im(\overline{A}) \subset \overline{M_{a+b}}$$
$$|$$
$$C^{\perp} \qquad \neq Im(\overline{A})$$
$$|$$
$$<\mathbf{1}> \qquad \subsetneq Im(\overline{A})$$
$$|$$
$$\{0\}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$\mathbb{F}_2\Gamma$

$|$

$< \mathbf{1} >^{\perp}$

$|$

C

$|$

$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$

$|$

$(C^{+})^{\perp} \qquad Im(\overline{A}) \subset \overline{M_{a+b}}$

$|$

$C^{\perp} \qquad \neq Im(\overline{A})$

$|$

$< \mathbf{1} > \qquad \subsetneq Im(\overline{A})$

$|$

$\{0\}$

So we have

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f-g-1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\mathbb{F}_2\Gamma$$
$$|$$
$$<\mathbf{1}>^{\perp}$$
$$|$$
$$C$$
$$|$$
$$C^{+} \qquad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$$
$$|$$
$$(C^{+})^{\perp} \qquad Im(\overline{A}) \subset \overline{M_{a+b}}$$
$$|$$
$$C^{\perp} \qquad \neq Im(\overline{A})$$
$$|$$
$$<\mathbf{1}> \qquad \subsetneq Im(\overline{A})$$
$$|$$
$$\{0\}$$

So we have

$$v_2(|S|) = b + gb + fa$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g + 1$ and $e_i = 0$ for other i.

$\mathbb{F}_2\Gamma$

|

$< \mathbf{1} >^{\perp}$

|

C

|

C^{+} $\quad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$

|

$(C^{+})^{\perp}$ $\quad Im(\overline{A}) \subset \overline{M_{a+b}}$

|

$C^{\perp}$ $\quad \neq Im(\overline{A})$

|

$< \mathbf{1} >$ $\quad \subsetneq Im(\overline{A})$

|

$\{0\}$

So we have

$$\begin{aligned} v_2(|S|) &= b + gb + fa \\ &= \sum_{i>0} ie_i \end{aligned}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g + 1$ and $e_i = 0$ for other i.

$$
\begin{array}{lc}
\mathbb{F}_2\Gamma & \\
| & \\
<\mathbf{1}>^{\perp} & \\
| & \\
C & \\
| & \\
C^{+} & \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a} \\
| & \\
(C^{+})^{\perp} & Im(\overline{A}) \subset \overline{M_{a+b}} \\
| & \\
C^{\perp} & \neq Im(\overline{A}) \\
| & \\
<\mathbf{1}> & \subsetneq Im(\overline{A}) \\
| & \\
\{0\} &
\end{array}
$$

So we have

$$
\begin{aligned}
v_2(|S|) &= b + gb + fa \\
&= \sum_{i>0} ie_i \\
&\geq \sum_{a+b>i\geq a} ie_i + \sum_{i\geq a+b} ie_i
\end{aligned}
$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f-g-1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\begin{array}{ll}
\mathbb{F}_2\Gamma & \\
| & \\
<\mathbf{1}>^{\perp} & \\
| & \\
C & \\
| & \\
C^+ & \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a} \\
| & \\
(C^+)^{\perp} & Im(\overline{A}) \subset \overline{M_{a+b}} \\
| & \\
C^{\perp} & \neq Im(\overline{A}) \\
| & \\
<\mathbf{1}> & \subsetneq Im(\overline{A}) \\
| & \\
\{0\} &
\end{array}$$

So we have

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&\geq a \sum_{a+b>i\geq a} e_i + (a+b) \sum_{i\geq a+b} e_i
\end{aligned}$$

2-part of S, when m is even.

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$\mathbb{F}_2\Gamma$

|

$<\mathbf{1}>^{\perp}$

|

C

|

$C^{+} \quad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$

|

$(C^{+})^{\perp} \quad Im(\overline{A}) \subset \overline{M_{a+b}}$

|

$C^{\perp} \quad \neq Im(\overline{A})$

|

$<\mathbf{1}> \quad \subsetneq Im(\overline{A})$

|

$\{0\}$

So we have

$$\begin{aligned}
v_2(|S|) &= b + gb + fa \\
&= \sum_{i>0} ie_i \\
&\geq \sum_{a+b>i\geq a} ie_i + \sum_{i\geq a+b} ie_i \\
&\geq a \sum_{a+b>i\geq a} e_i + (a+b) \sum_{i\geq a+b} e_i \\
&\geq a(dim(\overline{M}_a(A)) - dim(\overline{M}_{a+b}(A))) \\
&+ a + b(dim(\overline{M}_{a+b}(A)))
\end{aligned}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$$\begin{array}{ll}
\mathbb{F}_2\Gamma & \\
\vert & \\
<\mathbf{1}>^{\perp} & \\
\vert & \\
C & \\
\vert & \\
C^{+} & \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a} \\
\vert & \\
(C^{+})^{\perp} & Im(\overline{A}) \subset \overline{M_{a+b}} \\
\vert & \\
C^{\perp} & \neq Im(\overline{A}) \\
\vert & \\
<\mathbf{1}> & \subsetneq Im(\overline{A}) \\
\vert & \\
\{0\} &
\end{array}$$

So we have

$$\begin{aligned}
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&= \sum_{i>0} ie_i \\
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&\geq a \sum_{a+b>i\geq a} e_i + (a+b) \sum_{i\geq a+b} e_i \\
&\geq a(dim(\overline{M}_a(A)) - dim(\overline{M}_{a+b}(A))) \\
&+ a + b(dim(\overline{M}_{a+b}(A))) \\
&= a(dim(\overline{M}_a(A)) \\
&+ b(dim(\overline{M}_{a+b}(A)))
\end{aligned}$$

2-part of S, when m is even.

Let $a = v_2(q-1) = v_2(q^{m-1}-1)$ and $b = v_2(q^{m-1}+1)$. Then $e_a = f - g - 1$ and $e_{a+b} = g+1$ and $e_i = 0$ for other i.

$\mathbb{F}_2\Gamma$

$|$

$< \mathbf{1} >^{\perp}$

$|$

C

$|$

$C^+ \quad \subset \overline{V_r \cap \mathbb{Z}_\ell^v} \subset \overline{M_a}$

$|$

$(C^+)^{\perp} \quad Im(\overline{A}) \subset \overline{M_{a+b}}$

$|$

$C^{\perp} \quad \neq Im(\overline{A})$

$|$

$< \mathbf{1} > \quad \subsetneq Im(\overline{A})$

$|$

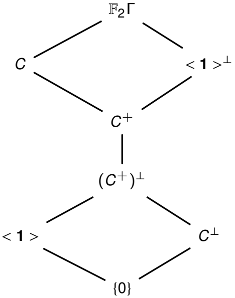
$\{0\}$

So we have

$$\begin{aligned}
v_2(|S|) &= b + gb + fa \\
&= \sum_{i>0} ie_i \\
&\geq \sum_{a+b>i\geq a} ie_i + \sum_{i\geq a+b} ie_i \\
&\geq a \sum_{a+b>i\geq a} e_i + (a+b) \sum_{i\geq a+b} e_i \\
&\geq a(dim(\overline{M}_a(A)) - dim(\overline{M}_{a+b}(A))) \\
&+ a + b(dim(\overline{M}_{a+b}(A))) \\
&= a(dim(\overline{M}_a(A)) \\
&+ b(dim(\overline{M}_{a+b}(A))) \\
&\geq af + b(g+1)
\end{aligned}$$

2-part of S, when m is odd.

We assume $v_2(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a$, $v_2(q-1) = b$, $v_2(r) = a+b$, $v_2(s) = 1$, $v_2(k) = v_2(\mu) = a+1$.We get $e_{a+b+1}(2) = g+1$, $e_{a+b}(2) = f-g-1$, $e_a(2) = 1$, and $e_i(2) = 0$



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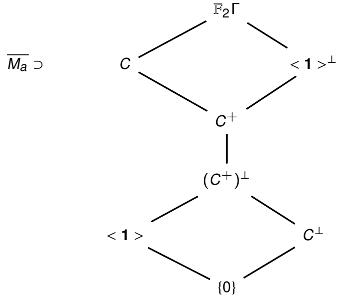
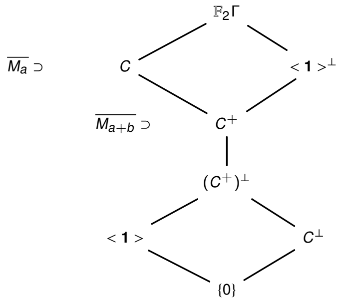


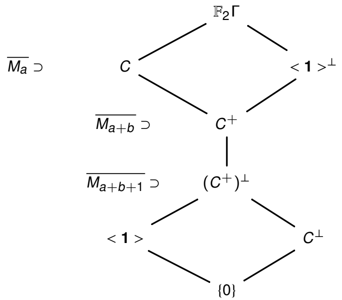
2-part of S, when m is odd.

We assume $v_2(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a$, $v_2(q-1) = b$, $v_2(r) = a+b$, $v_2(s) = 1$, $v_2(k) = v_2(\mu) = a+1$.We get $e_{a+b+1}(2) = g+1$, $e_{a+b}(2) = f-g-1$, $e_a(2) = 1$, and $e_i(2) = 0$



2-part of S, when m is odd.

We assume $v_2(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a$, $v_2(q-1) = b$, $v_2(r) = a+b$, $v_2(s) = 1$, $v_2(k) = v_2(\mu) = a+1$.We get $e_{a+b+1}(2) = g+1$, $e_{a+b}(2) = f-g-1$, $e_a(2) = 1$, and $e_i(2) = 0$



2-part of S, when m is odd.

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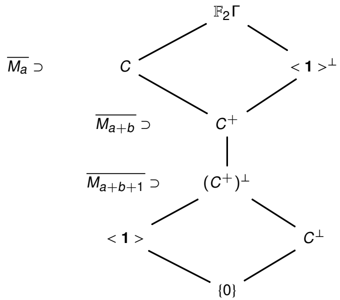

$$
\begin{aligned}
v_2(|S|) &= \sum_{i>0} ie_i \\
&\geq \sum_{a+b>i\geq a} ie_i + (a+b)(e_{a+b}) + \sum_{i\geq a+b+1} ie_i \\
&\geq a \sum_{a+b>i\geq a} e_i + (a+b)e_{a+b} \\
&+ (a+b+1) \sum_{i\geq a+b+1} e_i \\
&\geq a(dim_{\mathbb{F}_2}(\overline{M}_a(A)) - dim_{\mathbb{F}_2}(\overline{M}_{a+b}(A))) \\
&+ a + b(dim_{\mathbb{F}_2}(\overline{M}_{a+b}(A)) - dim_{\mathbb{F}_2}(\overline{M}_{a+b+1}(A))) \\
&+ (a+b+1)(dim_{\mathbb{F}_2}(\overline{M}_{a+b+1}(A))) \\
&\geq a + (a+b)f + g + 1 = v_2(|S|)
\end{aligned}
$$

2-part of K, when m is even.

As $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$ is even , $v_2(q^m - 1) > 1$, and thus $v_2(q^m + 1) = 1$. Assume $v_2(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a$ and $v_2(q^{m-1} + 1) = b$. Then we have

$e_{a+b+1} = g - 1$, $e_{b+1} = 1$, $e_1 = f - g - 1$

$\mathbb{F}_2\Gamma$

|

$< \mathbf{1} >^{\perp}$

|

C

|

C^{+}

|

$(C^{+})^{\perp}$

|

$C^{\perp}$

|

$< \mathbf{1} >$

|

$\{0\}$

2-part of K, when m is even.

As $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$ is even , $v_2(q^m - 1) > 1$, and thus $v_2(q^m + 1) = 1$. Assume $v_2(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a$ and $v_2(q^{m-1} + 1) = b$. Then we have

$e_{a+b+1} = g - 1, e_{b+1} = 1, e_1 = f - g - 1$

$\mathbb{F}_2\Gamma$

|

$< \mathbf{1} >^\perp$

|

C

|

C^+ $\subset \overline{M_1}$

|

$(C^+)^\perp$

|

$C^\perp$

|

$< \mathbf{1} >$

|

$\{0\}$

2-part of K, when m is even.

As $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$ is even , $v_2(q^m - 1) > 1$, and thus $v_2(q^m + 1) = 1$. Assume $v_2(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a$ and $v_2(q^{m-1} + 1) = b$. Then we have

$e_{a+b+1} = g - 1, e_{b+1} = 1, e_1 = f - g - 1$

$\mathbb{F}_2\Gamma$

|

$< \mathbf{1} >^\perp$

|

C

|

C^+ $\subset \overline{M_1}$

|

$(C^+)^\perp$ $\subset \overline{M_{b+1}}$

|

$C^\perp$

|

$< \mathbf{1} >$

|

$\{0\}$

2-part of K, when m is even.

As $\begin{bmatrix} m \\ 1 \end{bmatrix}_q$ is even , $v_2(q^m - 1) > 1$, and thus $v_2(q^m + 1) = 1$. Assume $v_2(\begin{bmatrix} m \\ 1 \end{bmatrix}_q) = a$ and $v_2(q^{m-1} + 1) = b$. Then we have

$e_{a+b+1} = g - 1, e_{b+1} = 1, e_1 = f - g - 1$

$\mathbb{F}_2\Gamma$

$|$

$< \mathbf{1} >^{\perp}$

$|$

C

$|$

$C^{+} \quad \subset \overline{M_1}$

$|$

$(C^{+})^{\perp} \quad \subset \overline{M_{b+1}}$

$|$

$C^{\perp} \quad \subset \overline{M_{a+b+1}}$

$|$

$< \mathbf{1} >$

$|$

$\{0\}$

2-part of K, when m is even.

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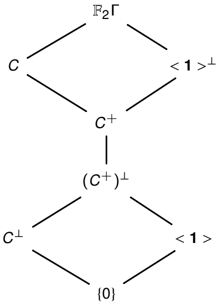
$\mathbb{F}_2\Gamma$
|
$< \mathbf{1} >^{\perp}$
|
C
|
$C^+ \qquad \subset \overline{M_1}$
|
$(C^+)^{\perp} \qquad \subset \overline{M_{b+1}}$
|
$C^{\perp} \qquad \subset \overline{M_{a+b+1}}$
|
$< \mathbf{1} >$
|
$\{0\}$

So we have

$$\begin{aligned} v_2(|K|) = f + (a+b)g - (a+1) &= \sum_{i>0} ie_i \\ &= \sum_{b+1>i>0} ie_i + \sum_{a+b+1>i\geq b+1} ie_i + \sum_{i\geq a+b+1} ie_i \\ &\geq \sum_{b+1>i>0} e_i + b \sum_{a+b+1>i\geq b+1} e_i + \\ &(a+b+1) \sum_{i\geq a+b+1} e_i \\ &= dim_{\mathbb{F}_2}(\overline{M_1(L)}) - \\ &dim_{\mathbb{F}_2}(\overline{M_{b+1}(L)} + (b+1)(dim_{\mathbb{F}_2}(\overline{M_{b+1}(L)}) \\ &- dim_{\mathbb{F}_2}(\overline{M_{a+b+1}(L)}) + \\ &(a+b+1)(dim_{\mathbb{F}_2}(\overline{M_{a+b+1}(L)} - dim_{\mathbb{F}_2}(\overline{Ker(L)})) \\ &\geq f - (g+1) + b + 1(a+b+1)(g-1) \\ &= (f + (a+b)g - (a+1)). \end{aligned}$$

2-part of K when m is odd.

As $\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q$ is even, $v_2(-1+q^{m-1}) > 1$ and thus $v_2(1+q^{m-1}) = 1$. Assume $v_2(q^m+1) = b$ and $v_2(\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q) = a$. Then $e_{a+b+1} = g+1$, $e_{a+b} = f-g-1$, $e_a = 1$, and $e_i = 0$ for all other i.



2-part of K when m is odd.

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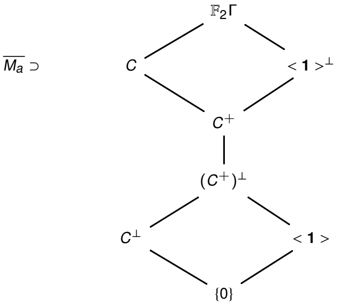

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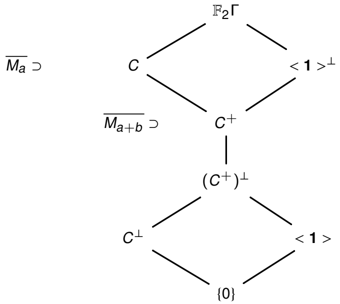

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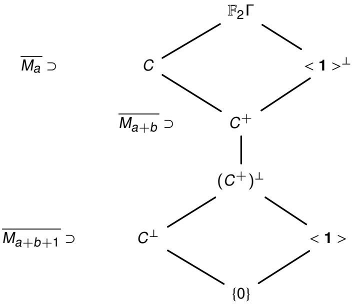

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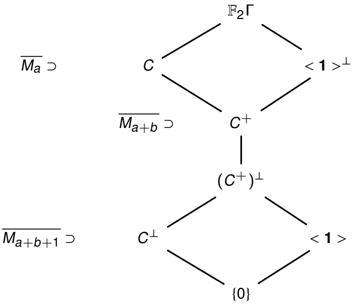

$$
\begin{aligned}
v_2(|K|) &= \sum_{i>0} ie_i \\
&\geq \sum_{a+b>i\geq a} ie_i + (a+b)(e_{a+b}) + \sum_{i\geq a+b+1} ie_i \\
&\geq a \sum_{a+b>i\geq a} e_i + (a+b)e_{a+b} \\
&+ (a+b+1) \sum_{i\geq a+b+1} e_i \\
&\geq a(dim_{\mathbb{F}_2}(\overline{M_a(L)}) - dim_{\mathbb{F}_2}(\overline{M_{a+b}(L)})) \\
&+ a + b(dim_{\mathbb{F}_2}(\overline{M_{a+b}(L)}) - dim_{\mathbb{F}_2}(\overline{M_{a+b+1}(L)})) \\
&+ (a+b+1)(dim_{\mathbb{F}_2}(\overline{M_{a+b+1}(L)} \\
&- dim_{\mathbb{F}_2}(\overline{ker(L)})) \\
&\geq (a+b)f + g - b = v_2(|K|)
\end{aligned}
$$

Other Results

Consider the graph on a non-degenerate quadric in $\mathbb{P}^n(q)$, in which two points are adjacent if and only if they are orthogonal.

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Using the submodule structure given in *Rank 3 permutation modules of the finite classical groups* , J. Algebra 291 (2005) 551-606 by Sin and Tiep, we were able to determine the Smith and Critical groups of the above family of graphs.

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Thank You!