1. Find the Taylor series expansion about 0 of the following functions:

(i) \( f(z) = \ln \frac{1-z}{1+z} \)

(ii) \( g(z) = \frac{1-\cos z}{z^2} \)

(iii) \( h(z) = \int_0^z \frac{1-e^{-x}}{x} \, dx \)

(Hint: don’t use Taylor’s theorem, use known Taylor series).

Solution. (i) Since

\[ \ln \left( \frac{1-z}{1+z} \right) = \ln(1-z) - \ln(1+z) \]

and

\[ \ln(1-z) = -\sum_{n \geq 1} \frac{z^n}{n}, \quad \ln(1+z) = \sum_{n \geq 1} (-1)^{n+1} \frac{z^n}{n} \]

we find, on adding the two series,

\[ \ln \left( \frac{1-z}{1+z} \right) = \sum_{n \geq 1} (-1 - (-1)^{n+1}) \frac{z^n}{n}. \]

Now

\[ -1 - (-1)^{n+1} = -1 + (-1)^n = \begin{cases} 0 & n \text{ even} \\ -2 & n \text{ odd} \end{cases} \]

and so we may set \( n = 2k + 1 \) in the sum, resulting in

\[ \ln \left( \frac{1-z}{1+z} \right) = -2 \sum_{k \geq 1} \frac{z^{2k+1}}{2k+1}. \]

(ii) Since

\[ \cos z = \sum_{n \geq 0} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \sum_{n \geq 1} (-1)^{n+1} \frac{z^{2n}}{(2n)!} \]

we have

\[ 1 - \cos z = \sum_{n \geq 1} (-1)^{n+1} \frac{z^{2n}}{(2n)!} \]

and thus

\[ \frac{1 - \cos z}{z^2} = \sum_{n \geq 1} (-1)^{n+1} \frac{z^{2n-2}}{(2n)!} \]

or, on replacing \( n \) by \( n + 1 \),

\[ \frac{1 - \cos z}{z^2} = \sum_{n \geq 0} (-1)^n \frac{z^{2n}}{(2n + 2)!}. \]
(iii) Since
\[ e^x = \sum_{n \geq 0} \frac{x^n}{n!} \]
the same argument as in (ii) leads to
\[ \frac{1 - e^{-x}}{x} = \sum_{n \geq 0} (-1)^n \frac{x^n}{(n+1)!} \]
and integrating this series yields
\[ \int_0^z \frac{1 - e^{-x}}{x} \, dx = \sum_{n \geq 0} (-1)^n \frac{x^{n+1}}{(n+1) \cdot (n+1)!} \].

2. Find all solutions of the differential equation
\[ u'' + 2zu' + u = 0 \]
as power series in \( z \).

Solution. Substituting \( u = \sum_{n \geq 0} a_n z^n \) into the equation yields
\[ \sum_{n \geq 2} n(n-1)a_n z^{n-2} + \sum_{n \geq 0} 2na_n z^n + \sum_{n \geq 0} a_n z^n = 0. \]
Replace \( n \mapsto n + 2 \) in the first sum: the result is
\[ \sum_{n \geq 0} (n+2)(n+1)a_{n+2} z^n + \sum_{n \geq 0} 2na_n z^n + \sum_{n \geq 0} a_n z^n = 0 \]
or
\[ \sum_{n \geq 0} [(n+2)(n+1)a_{n+2} + (2n+1)a_n] z^n = 0. \]
Equating the coefficients to zero yields
\[ (n+2)(n+1)a_{n+2} + (2n+1)a_n = 0 \]
for all \( n \geq 0 \), or
\[ a_{n+2} = -\frac{(2n+1)a_n}{(n+2)(n+1)} \quad n \geq 0. \]
Now \( a_0 \) and \( a_1 \) are undetermined, and the above relation shows that \( a_n \) will be a multiple of \( a_0 \) (resp. \( a_1 \)) if \( n \) is even (resp. odd). Substituting \( n = 2k \) in this yields
\[ a_{2(k+1)} = -\frac{(4k+1)}{2(k+1)(2k+1)} a_{2k} \quad k \geq 0. \]
If we set \( b_k = a_{2k} \) this says
\[ b_{k+1} = -\frac{(4k+1)}{2(k+1)(2k+1)} b_k \quad k \geq 0. \]
and thus
\[ b_k = (-1)^k \frac{1 \cdot 5 \cdot 9 \cdots (4k - 3)}{2^k k! \cdot 1 \cdot 3 \cdot 5 \cdots (2k - 1)} a_0 \hspace{1em} k \geq 1. \]

Similarly, substituting \( n = 2k + 1 \) yields
\[ a_{2(k+1)+1} = -\frac{(4k + 3)}{2(2k + 3)(k + 1)} a_{2k+1} \hspace{1em} k \geq 0. \]

or, if we put \( c_k = a_{2k+1} \),
\[ c_{k+1} = -\frac{(4k + 3)}{2(2k + 3)(k + 1)} c_k \hspace{1em} k \geq 0. \]

Therefore
\[ c_k = (-1)^k \frac{3 \cdot 7 \cdot 10 \cdots (4k - 1)}{2^k k! \cdot 3 \cdot 5 \cdot 7 \cdots (2k + 1)} a_1. \]

Put this all together and remember that \( b_k \) is the coefficient of \( z^{2k} \) in the original series, while \( c_k \) is the coefficient of \( z^{2k+1} \): we find
\[
\begin{align*}
u & = a_0 \left( 1 + \sum_{k \geq 0} (-1)^k \frac{1 \cdot 5 \cdot 9 \cdots (4k - 3)}{2^k k! \cdot 1 \cdot 3 \cdot 5 \cdots (2k - 1)} z^{2k} \right) \\
& \quad + a_1 \left( 1 + \sum_{k \geq 0} (-1)^k \frac{3 \cdot 7 \cdot 10 \cdots (4k - 1)}{2^k k! \cdot 3 \cdot 5 \cdot 7 \cdots (2k + 1)} z^{2k+1} \right).
\end{align*}
\]

3. Find the first four nonvanishing terms of a series solution \( u = \sum_{n \geq 0} a_n z^n \) of the initial value problem
\[ u'' - e^{-2z} u = 0, \hspace{1em} u(0) = 1, \hspace{1em} u'(0) = 0. \]

**Solution.** The initial conditions imply that \( a_0 = 1 \) and \( a_1 = 0 \). The Taylor series for \( e^{-2z} \) around \( z = 0 \) is
\[ e^{-2z} = \sum_{n \geq 0} \frac{2^n z^n}{n!} \]
so substituting \( u = \sum_{n \geq 0} a_n z^n \) into the equation yields
\[
\begin{align*}
2a_2 + 6a_3 z + 12a_4 z^2 + 20a_5 z^3 + 30a_6 z^4 + \cdots \\
&= (1 - 2z + 2z^2 - (4/3)z^3 + (2/3)z^4 + \cdots)(1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)
\end{align*}
\]
or
\[
\begin{align*}
2a_2 + 6a_3 z + 12a_4 z^2 + 20a_5 z^3 + 30a_6 z^4 + \cdots \\
&= 1 - 2z + (a_2 + 2)z^2 + (a_3 - 2a_2 - (4/3))z^3 + \cdots.
\end{align*}
\]
Equating coefficients we find

\[ 2a_2 = 1 \]
\[ 6a_3 = -2 \]
\[ 12a_4 = 2 + a_2 \]
\[ 20a_5 = a_3 - 2a_2 - (4/3) \]

and thus

\[ a_0 = 1 \]
\[ a_1 = 0 \]
\[ a_2 = 1/2 \]
\[ a_3 = -1/3 \]
\[ a_4 = 5/24. \]

4. The nonlinear equation

\[ y'' + \frac{2}{z} y' + y^n = 0 \]

is known as Emden’s equation and arises in the theory of stellar atmospheres. Here \( n \) is a constant, usually real. Find the first three nonvanishing terms of a series solution \( y = \sum_{n \geq 0} a_n z^n \).

**Solution.** If \( y = \sum_{n \geq 0} a_n z^n \) has \( a_1 \neq 0 \), this will yield a term in \( 1/z \) on the left hand side which cannot be cancelled by the right hand side. We therefore try a solution of the form

\[ y = 1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots. \]

Substituting, we find

\[
(2a_2 + 6a_3 z + 12a_4 z^2 + \cdots) + 2(2a_2 + 3a_3 z + 4a_4 z^2 + \cdots)
\]

\[ = -(1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^n \]

or

\[ 6a_2 + 12a_3 z + 20a_4 z^2 + \cdots = (1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^n. \]

To calculate the right hand side we write

\[ (1 + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)^n = -(1 + z^2(a_2 z + a_3 z^3 + a_4 z^4 + \cdots))^n \]

and use the binomial formula

\[ (1 + x)^a = \sum_{n \geq 0} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2}a^2 + \cdots. \]

Thus if we just keep the terms through \( z^3 \) we find

\[ 6a_2 + 12a_3 z + 20a_4 z^2 + \cdots = -1 - na_2 z^2 - na_3 z^3 + \cdots. \]
Therefore
\[ 6a_2 = -1, \quad a_3 = 0, \quad 20a_4 = -n \]
and thus
\[ y(z) = 1 - \frac{z}{6} - \frac{n}{20} z^4 + \cdots. \]

5. Find two independent solutions of the equation
\[ 3x^2 u'' + 8xu' + (x - 2)u = 0 \]
as series in powers of \( x \), using the method of Frobenius. The first solution should give an explicit formula for the coefficients. For the second, just find the first three nonzero terms.

Solution. The indicial equation is
\[ 3r(r - 1) + 8r - 2 = 3r^2 + 5r - 2 = 0 \]
and we find that the roots are 1/3 and −2. We can therefore use the usual method for both solutions. For \( r = 1/3 \) the solution has the form
\[ u(z) = \sum_{n \geq 0} a_n x^{n+1/3} \]
and substituting this into the equation yields
\[
3 \sum_{n \geq 0} (n + 1/3)(n - 2/3)a_n x^{n+1/3} + 8 \sum_{n \geq 0} (n + 1/3)a_n x^{n+1/3} \\
- 2 \sum_{n \geq 0} a_n x^{n+1/3} + \sum_{n \geq 0} a_n x^{n+4/3} = 0.
\]
We replace \( n \mapsto n - 1 \) in the last sum and simplify; the result is
\[
\sum_{n \geq 1} [(3n(n + 7)a_n + a_{n-1}] = 0
\]
(note that when the first three sums are grouped together, the \( n = 0 \) term vanishes – this is the indicial equation). Thus
\[ a_n = -\frac{a_{n-1}}{3n(n + 7)} \quad \text{for} \quad n \geq 1. \]
and we find
\[ a_n = (-1)^n \frac{a_0}{3^n n! \cdot 8 \cdot 9 \cdots (n + 7)} \quad n \geq 1. \]
and
\[ u = a_0 \left( 1 + \sum_{n \geq 1} (-1)^n \frac{x^n}{3^n n! \cdot 8 \cdot 9 \cdots (n + 7)} \right). \]

We may use the same procedure for \( r = -2 \); the solution is \( u_1 = \sum_{n \geq 0} b_n x^{n-2} \), and substituting yields
\[
\sum_{n \geq 0} 3(n - 2)(n - 3)b_n x^{n-2} + \sum_{n \geq 0} 8(n - 2)b_n x^{n-2} \\
- \sum_{n \geq 0} 2b_n x^{n-2} + \sum_{n \geq 0} b_n x^{n-1} = 0.
\]
Simplifying and shifting the summation in the last sum results in
\[
\sum_{n \geq 1} [n(3n - 7)b_n + b_{n-1}] = 0
\]
and we find that
\[
b_n = -\frac{b_{n-1}}{n(3n - 7)} \quad n \geq 1.
\]
We find that
\[
b_n = (-1)^n \frac{b_0}{n!(-4)(-1) \cdot 2 \cdot 5 \cdots (3n - 7)}
\]
and
\[
u_1 = b_0 \left(1 + \sum_{n \geq 1} \frac{(-1)^n}{n!(-4)(-1) \cdot 2 \cdot 5 \cdots (3n - 7)} x^n\right).
\]

6. Same as the previous problem, for the equation
\[
z(z + 1)u'' + 2(1 - z)u' - 4u = 0.
\]

Solution. The indicial equation is
\[
r(r - 1) + 2r = r^2 + r = 0
\]
and the exponents (roots) are \(r = 0, -1\). There is therefore a solution of the form \(u_1 = \sum_{n \geq 0} a_n z^n\), and substituting this into the equation yields
\[
\sum_{n \geq 0} (n - 4)(n + 1)a_n z^n + \sum_{n \geq 0} n(n + 1) a_n z^{n-1} = 0
\]
after a certain amount of arithmetic. The second sum can be started at \(n = 1\), and the reindexing and combining yields
\[
\sum_{n \geq 0} [(n - 4)(n + 1)a_n + (n + 1)(n + 2)a_{n+1}] z^n = 0.
\]
Therefore
\[
a_{n+1} = \frac{n - 4}{n + 2} a_n.
\]
If we take \(a_0 = 1\) we find
\[
a_1 = -2, \ a_2 = 2, \ a_3 = -1, \ a_4 = 1/5, \ a_n = 0 \quad n \geq 5
\]
and thus
\[
u_1(z) = 1 - 2z + 2z^2 - z^3 + (1/5)z^4.
\]
The second solution has the form
\[
u_2 = C u_1(z) \ln z + \sum_{n \geq 0} b_n z^{n-1}
\]
(to be continued....)
7. Which of the following equations have a bounded solution near \( z = 0 \)? which have all of their solutions bounded near \( z = 0 \)? For which are all solutions single-valued in a neighborhood of \( z = 0 \)?

(i) \( 3z u'' + 2(1 - z)u' - 4u = 0 \).

(ii) \( z u'' + (z + 2)u' - u = 0 \).

**Solution.** (i) The indicial equation is

\[ 3r(r - 1) - 2r = 3r^2 - 5r = 0 \]

and the roots are 0 and 5/3. Both are nonnegative, so all solutions are bounded. Since one root is not an integer, there are solutions that are not single-valued near \( z = 0 \). (ii) The indicial equation is

\[ r(r - 1) + 2r = r^2 + r = 0 \]

and the roots are 0 and -1. The solution corresponding to \( r = 0 \) is bounded near \( z = 0 \), while the other is not. Since both roots are integers, all solutions are single-valued near \( z = 0 \).

8. For which values of the (possibly) complex number \( a \) does the equation

\[ zu'' + (1 - z)u' + au = 0 \]

have a polynomial solution?

9. The **confluent hypergeometric equation** is

\[ zu'' + (c - z)u'' - au = 0 \]

for constants \( a, c \). Show that if \( c \) is not an integer, the functions

\[ u_1(z) = 1_F(\alpha; c; z) = \sum_{n \geq 0} \frac{(\alpha)_n}{n!(c)_n} z^n \]

\[ u_2(z) = z^{1-c} 1_F(\alpha - c + 1; 2 - c; z) \]

are independent solutions.