1. Solve the problem

\[ \frac{\partial^2 u}{\partial x^2} = 3 \frac{\partial u}{\partial t}, \]
\[ u(0, t) = u(\pi, t) = 0, \quad t > 0 \]
\[ u(x, 0) = \sin 2x + 2 \sin 3x - 3 \sin 8x, \quad 0 < x < \pi \]

Solution. Set \( u(x, t) = X(x)T(t) \) and substitute: the result is

\[ \frac{X''}{X} = 3 \frac{T'}{T} = -k. \]

The initial conditions imply that \( X(0) = X(\pi) = 0 \), so we must have

\[ X(x) \propto \sin nx \quad n = 1, 2, 3 \ldots \quad k = n^2 \]

and

\[ T \propto e^{-(n^2/3)t}. \]

If we set

\[ u(x, t) = \sum_{n \geq 1} a_n T e^{-(n^2/3)t} \sin nx \]

we must have

\[ u(x, 0) = \sum_{n \geq 1} a_n T \propto \sin nx \]
\[ f(x) = \sin 2x + 2 \sin 3x - 3 \sin 8x \]

and comparing coefficients yields

\[ u(x, t) = e^{-(4/3)t} \sin 2t + 2e^{-3t} \sin 3t - 3e^{-(64/3)t} \sin 8t. \]
2. Solve the problem

\[
\frac{\partial^2 u}{\partial x^2} = 9 \frac{\partial^2 u}{\partial t^2},
\]

\[
u(0, t) = u(\pi, t) = 0, \quad t > 0
\]

\[
u(x, 0) = 0, \quad 0 < x < \pi
\]

\[
\frac{\partial u}{\partial t}(x, 0) = \sin 2x + 2 \sin 3x - 3 \sin 8x, \quad 0 < x < \pi
\]

**Solution.** Set \( u(x, t) = X(t)T(t) \) and substitute: we find

\[
\frac{X''}{X} = 9 \frac{T''}{T} = -k
\]

and the initial conditions yield \( X(0) = X(\pi) = 0 \). Then

\[
X \propto \sin nx \quad n = 1, 2, 3 \ldots \quad k = n^2
\]

and

\[
T(t) = C \cos(n/3)t + D \sin(n/3)t \quad n = 1, 2, 3 \ldots
\]

We therefore look for a solution of the form

\[
u(x, t) = \sum_{n \geq 1} a_n \cos(n/3)t \sin nx + \sum_{n \geq 1} b_n \sin(n/3)t \sin nx.
\]

The condition \( u(x, 0) = 0 \) implies \( a_n = 0 \) for all \( n \). Then

\[
\frac{\partial u}{\partial t}(x, 0) = \sum_{n \geq 1} (1/3) b_n \sin nx = \sin 2x + 2 \sin 3x - 3 \sin 8x
\]

yields \( b_2 = 3, \; b_3 = 6, \; b_8 = -9 \) and all other \( b_n = 0 \). The solution is therefore

\[
u(x, t) = 3 \cos 6t \sin 2x + 6 \cos 9t \sin 3t - 9 \cos 24t \sin 8x.
\]
3. Find a formal solution to the problem

$$\frac{\partial^2 u}{\partial x^2} = 9 \frac{\partial^2 u}{\partial t^2}$$

$$u(0, t) = u(\pi, t) = 0 \quad t > 0$$

$$u(x, 0) = \sin 4x + 7 \sin 5x$$

$$\frac{\partial u}{\partial t}(x, 0) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}.$$

**Solution.** The same argument as in the previous problem leads to a solution of the form

$$u(x, t) = \sum_{n \geq 1} a_n \cos(\frac{n}{3})t \sin nx + \sum_{n \geq 1} b_n \sin(\frac{n}{3})t \sin nx.$$

Since

$$u(x, 0) = \sum_{n \geq 1} a_n \sin nx = \sin 4x + 7 \sin 5x$$

we must have $a_4 = 1$, $a_5 = 7$ and all other $a_n = 0$. On the other hand

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n \geq 1} b_n \frac{3}{3} \sin nx = f(x) := \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

and the Fourier sine series of $f(x)$ has coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left( \int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right)$$

$$= \begin{cases} 4(-1)^{n-1}/n & n \text{ odd} \\ \pi n^2 & n \text{ even} \end{cases}.$$

Thus $b_{2k} = 0$ for all $k \geq 0$, and

$$b_{2k+1} = \frac{12(-1)^k}{\pi(2k+1)^2} \quad k \geq 0.$$

The formal solution is therefore

$$u(x, t) = \cos(\frac{4}{3})t \sin 4x + 7 \cos(\frac{5}{3})t \sin 7x + \sum_{k \geq 0} \frac{12(-1)^k}{\pi(2k+1)^2} \sin(\frac{n}{3})t \sin nx.$$
4. Find a formal solution to the problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \quad 0 < x < \pi, \ 0 < y < \pi, t > 0 \\
u(x,0,t) = u(x,\pi,t) = 0, \quad 0 < x < \pi, \ t > 0 \\
\frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(\pi,y,t) = 0, \quad 0 < y < \pi, \ t > 0 \\
u(x,y,0) = x \sin y.
\]

Solution. To separate variables we now let \(u(x,y,t) = X(x)Y(y)T(t)\) and substitute:

\[
\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{T} = -k
\]

so that

\[
\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{T} = -k
\]

and

\[
\frac{X''}{X} = -\frac{Y''}{Y} - k = -\ell.
\]

Thus \(X, Y\) and \(T\) satisfy the conditions

\[
X'' + \ell X = 0, \quad X'(0) = X'(\pi) = 0 \\
Y'' + (k - \ell)Y = 0 \quad Y(0) = Y(\pi) = 0 \\
T' + kT = 0
\]

The boundary conditions imply

\[
\ell = m^2, \ m = 0, 1, 2, 3 \ldots, \quad X \propto \cos mx \\
k - \ell = n^2, \ n = 1, 2, 3 \ldots, \quad Y \propto \sin ny
\]

and thus \(k = m^2 + n^2\) and \(T \propto e^{-(m^2+n^2)t}\). We thus look for a solution of the form

\[
u(x,y,t) = \sum_{m \geq 0, \ n \geq 1} a_{mn} e^{-(m^2+n^2)t} \cos mx \sin ny.
\]

For \(t = 0\) we must have

\[
u(x,y,0) = \sum_{m \geq 0, \ n \geq 1} a_{mn} \cos mx \sin ny = x \sin y
\]

and therefore \(a_{mn} = 0\) for \(n \neq 1\). The remaining \(a_{m1}\) are determined by

\[
x = \sum_{m \geq 0} a_{m1} \cos mx
\]

and the formula for coefficients in a Fourier cosine series shows that

\[
a_{01} = \frac{1}{\pi} \int_0^\pi x \ dx = \frac{\pi}{2}
\]

and

\[
a_{m1} = \frac{2}{\pi} \int_0^\pi x \cos mx \ dx = 2\frac{(-1)^m - 1}{\pi m^2}.
\]
Thus $a_{m_1} = 0$ for even $m$ and
\[ a_{(2k+1),1} = \frac{4}{(2k+1)^2 \pi} \quad k \geq 0. \]
The formal solution is
\[ u(x, y, t) = \frac{\pi}{2} e^{-t} + \sum_{k \geq 0} \frac{4}{(2k+1)^2 \pi} e^{-((2k+1)^2+1)t} \cos((2k+1)x) \sin y. \]

5. Solve the Dirichlet problem
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \\
0 \leq r < 2, \quad -\pi \leq \theta \leq \pi \\
u(2, \theta) = \cos^2 \theta.
\]

**Solution.** Set $u(r, \theta) = R(r)\Theta(\theta)$ and substitute:
\[
r^2 \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta'}{\Theta} = k.
\]
Since $\Theta$ must be periodic with period $2\pi$ we must have $k = n^2$ for some integer $n$, and the $R$ is a solution of
\[
r^2 R'' + r R' - n^2 R = 0.
\]
We may take $n$ to be positive, in which case the only solution bounded as $r \to 0$ is $R \propto r^n$. We therefore look for a solution of the form
\[ u(r, \theta) = \sum_{n \geq 0} a_n r^n \cos n\theta + \sum_{n \geq 0} b_n r^n \sin n\theta. \]
Set $r = 2$: we have
\[ u(2, \theta) = \sum_{n \geq 0} a_n 2^n \cos n\theta + \sum_{n \geq 0} b_n 2^n \sin \theta = \cos^2 \theta = \frac{1 + \cos 2\theta}{2}. \]
From this we get $a_0 = 1/2$, $a_2 = 1/8$, all other $a_n = 0$ and all $b_n = 0$. Thus the solution is
\[ u(r, \theta) = \frac{1}{2} + \frac{1}{8} r^2 \cos 2\theta. \]
6. The equation
\[ \frac{\partial^2 u}{\partial x^2} = a \frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} + cu \]
is known as the equation of telegraphy; here \( a \) and \( b \) are positive constants depending on the physical properties of a telegraph cable. (i) Use the method of separation of variables to find solutions of the form
\[ u(x,t) = X(x)T(t) \]
satisfying
\[ u(0,t) = u(L,t) = 0 \quad t \geq 0 \]
and \( a = b = c = 1 \). (ii) Use part (i) to give a formal solution to the problem
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \]
where \( L > 0 \) and \( f(x) \) is a given function on \([0,L]\).

Solution. (i) If we set \( u(x,t) = X(x)T(t) \) and substitute we find
\[ \frac{X''}{X} = \frac{T''}{T} + \frac{T'}{T} + 1 = -k \]
and so for \( X(t) \) we get the equations
\[ X'' + kX = 0, \quad X(0) = X(L) = 0 \]
which have the solutions
\[ X \propto \sin \frac{n\pi}{L} nx, \quad k = \left( \frac{n\pi}{L} \right)^2. \]
The equation for \( T(t) \) is
\[ T'' + T' + (k + 1)T = 0 \]
with \( k \) as above. Now the roots of
\[ \lambda^2 + \lambda + (k + 1) = 0 \]
are
\[ \lambda = \frac{-1 \pm \sqrt{1 - 4(k + 1)}}{2} = \frac{-1 \pm \sqrt{4k + 3}(2i)}{2} \]
and we will write this
\[ \lambda = -\frac{1}{2} \pm \omega_n i, \quad \omega_n = \sqrt{\frac{n^2\pi^2}{L^2} + \frac{3}{4}} \]
so that the general solution of the equation for \( T \) is
\[ T = Ce^{-(1/2)t} \cos \omega_n t + De^{-(1/2)t} \sin \omega_n t. \]

Thus we get solutions of the form
\[ u_c^c(x,t) = e^{-(1/2)t} \cos \omega_n t \sin \frac{n\pi}{L} x \]
\[ u_c^s(x,t) = e^{-(1/2)t} \sin \omega_n t \sin \frac{n\pi}{L} x \]
for \( n \geq 1 \).

(ii) ...