MAS 5312 Section 17110
Introduction to Algebra 2

Exam 2 – March 27, 2020

Time: 2 hours. All problems will be weighted equally. You may make use of all theorems proven in text if you cite them (correctly).

---

1. Which of the following polynomials are irreducible in the given ring? Explain, and if the polynomial is reducible then give an irreducible factorization.

(i) $X^5 + 2X^2 + X + 1 \in \mathbb{Z}[X]$ (hint: reduce).

(ii) $X^3 - X^2 - X - 2 \in \mathbb{Q}[X]$.

(iii) $X^4 + 2 \in \mathbb{F}_7[X]$

(iv) $X^4 + (2 + i)X^2 + (1 - 2i)X + 5 \in \mathbb{Q}[i][X]$.

2. Suppose R is a ring with identity, M is a (left) R-module, $N \subseteq M$. Denote by $p : M \to M/N$ the canonical homomorphism, and by $i : N \to M$ the (evident) inclusion homomorphism.

(i) Show that if there is an R-module homomorphism $s : M/N \to M$ such that $p \circ s = id_{M/N}$ then $M \simeq N \oplus M/N$.

(ii) Show that if there is an R-module homomorphism $s : M \to N$ such that $s \circ i = id_N$ then $M \simeq N \oplus M/N$.

(iii) Give an example of a ring R with identity, an R-module M and a submodule N such that $M \not\simeq N \oplus M/N$.

3. Suppose K is a field, m and n are relatively prime positive integers and $R = K[X,Y]/(X^m - Y^n)$. Show that R is an integral domain.

4. Suppose K is a field, V is a vector space of finite dimension, and W, $W' \subseteq V$ are subspaces of V. (i) Show that

$$\dim(W + W') = \dim(W) + \dim(W') - \dim(W \cap W').$$

(ii) Use this (or any other argument you like) to show that $V \simeq W \oplus W'$ if and only if $W \cap W' = 0$ and $\dim(W) + \dim(W') = \dim(V)$.