F-isocrystals on the line

Richard Crew

The University of Florida

December 16, 2012

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- If X has relative dimension d over $\mathcal{V}$, $t_1, \ldots, t_d$ will usually denote local parameters at an (unspecified) point of X, so that $\Omega^1_{X/\mathcal{V}}$ has $dt_1, \ldots, dt_d$ at that point. Same for local parameters on the completion X_∞.

For the purposes of this workshop it's OK if you want to take $\mathcal{V} = \mathbb{Z}_p$, $K = \mathbb{Q}_p$, $\pi = p$ and

$$A = \mathcal{V}[X, (X - a_i)^{-1}, 1 \leq i \leq d]$$

where the a_i have distinct reduction modulo p, so that X is an open subset of $\mathbb{P}^1$. Then $d = 1$ and $t_1 = X - a$ is a local parameter at a.

Crystals and Isocrystals

A *coherent crystal* (M, ∇) on $X_\infty/\mathcal{V}$ is a coherent sheaf M on X_∞ endowed with an integrable, p-adically nilpotent connection ∇. In dimension one, of course, integrability is automatic. "p-adically nilpotent" means that for any set $t_1, \ldots, t_d$ of local parameters at any point of X_∞, the operators $\nabla(\partial_i^p)$ on M are topologically nilpotent; here $\partial_1, \ldots, \partial_d$ are the derivations dual to $t_1, \ldots, t_d$.

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An *isocrystal on* $X_\infty/\mathcal{V}$ is a coherent crystal on $X_\infty/\mathcal{V}$ up to isogeny, in other words a coherent sheaf of $\mathcal{O}_{X_\infty} \otimes \mathbb{Q}$-modules M endowed with a connection, such that $M = M_0 \otimes \mathbb{Q}$ for some coherent sheaf M_0 on X_∞ stable under the connection (more intrinsic definitions are possible). Any such M is in fact locally free.

If $e \leq p - 1$, i.e. if $\mathcal{V}$ is not too ramified, the category of crystals on $X_\infty/\mathcal{V}$ depends, up to canonical equivalence, only on X_0 and $\mathcal{V}$; this is a consequence of the nilpotence condition. We can thus speak of "crystals on $X_0/\mathcal{V}$."

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The same remarks hold for isocrystals.

Frobenius Structures

Recall now that $\phi : X_\infty \to X_\infty$ lifts the q^{th}-power Frobenius of X_0. An *F-crystal on X_∞* is a triple (M, ∇, F) where (M, ∇) is a crystal on X_∞ and Φ is a horizontal morphism

$$F : \phi^* M \to M$$

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Similarly, an *F-isocrystal on X_∞* is a triple (M, ∇, F) where (M, ∇) is an isocrystal on X_∞ and F is now an isomorphism $F : \phi^* M \xrightarrow{\sim} M$.

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The remarks in the previous section show that when $e \leq p - 1$, the categories of F-crystals and F-isocrystals on $X_\infty/\mathcal{V}$ only depend up to canonical equivalence on X and $\mathcal{V}$. As before, we will then speak of "F-crystals on $X/\mathcal{V}$."

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We will see later that in the case of F-isocrystals, the hypothesis on e can be dropped.

If (M, ∇, F) is an F-crystal (resp. F-isocrystal) on X_∞, the condition that F be horizontal is that

$$\begin{tikzcd}
\phi^* M \arrow[r, "\phi^* \nabla"] \arrow[d, "F"'] & \phi^* M \otimes \phi^* \Omega^1_{X_\infty} \arrow[d, "F \otimes \phi^*"] \\
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If M is a *free* $\mathcal{O}_{X_\infty}$-module, the Frobenius structure is given by a square matrix F (say) and the connection by a square matrix A of 1-forms. The above condition is the identity

$$dF + AF = F\phi^* A.$$

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The fact that the formal horizontal sections of an F-isocrystal (M, ∇, F) have p-adic radius of convergence one implies that the category of F-isocrystals depends up to canonical equivalence only on X_0 and K, *regardless* of the ramification index of $\mathcal{V}$ (without Frobenius, we had to assume $e \leq p - 1$).

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In particular, for any point x of X_0, the fiber M_x of an F-isocrystal is canonically defined without hypotheses on the ramification index of $\mathcal{V}$.

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The first is that the formal horizontal sections of a unit-root F-crystal are *p-adically integral*. In particular, they converge and are bounded on the open unit polydisk; this is also true of the formal horizontal sections of a unit-root F-isocrystal. In Grothendieck's terminology, a unit-root F-crystal extends to a stratification; this is a very strong property. One consequence of this is that unit-root F-crystals are locally free.

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In the unit-root case F^{-1} is integral, and since $\phi^* A$ is divisible by p, this equation can be solved by successive iteration.

One can show that if $\mathbb{F}_q \subseteq k$, the category of unit-root F-crystals on $X_\infty/\mathcal{V}$ is equivalent to the category of continuous representations of the fundamental group $\pi_1(X_0)$ on finite free $\mathcal{V}^\sigma$-modules; here recall that σ lifts the q^{th}-power Frobenius of $\mathcal{V}$, and $\mathcal{V}^\sigma$ is the subring of σ-fixed elements; it is the integer ring of the local field K^σ.

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The proof, roughly consists in showing that for any $n \geq 0$, the reduction of M modulo π^{n+1} has a basis of F-fixed vectors in the pullback of M to some étale cover $Y_n \to X_n$ (recall $X_n = X \otimes \mathcal{V}/\pi^{n+1}$).

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From this we see (Katz, RC) that the category of unit-root F-isocrystals on $X_\infty/\mathcal{V}$ is equivalent to the category of continuous representations of the fundamental group $\pi_1(X_0)$ on finite-dimensional K^σ-vector spaces (again assuming $\mathbb{F}_q \subseteq k$).

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When k is algebraically closed, Dieudonné and Manin proved the following structure theorem for F-isocrystals on K: any such F-isocrystal (V, F) is a direct sum

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where V^λ is defined as follows: if $\lambda = r/s$ in lowest terms, then M^λ has a basis $e_1, \ldots, e_s$ such that

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We say V is *isopentic* of slope λ if $n_\mu = 0$ for $\mu \neq \lambda$.

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If $D(\lambda)$ is the division algebra over K^{σ} with invariant λ, then the category of isopentic V of slope λ is equivalent to the category of representations of the absolute Galois group of k on left $D(\lambda)$-vector spaces.

Suppose (M, ∇, F) is an F-isocrystal on X_∞. If $x \to X_0$ is any geometric point of X_0, the the Frobenius structure F induces a Frobenius structure F_x on M_x; this is another consequence of the fact that the category of F-isocrystals on a formal scheme only depends on its reduction.

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- If x is a specialization of y, the the Newton polygon at x is on or above the Newton polygon at y, and both polygons have the same endpoint.

If the Newton polygon of (M, ∇, F) is *constant*, there is a filtration of M by F-isocrystals, with isopentic quotients (this does not seem to appear in the literature, however).

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Overconvergence

So far in our discussion of F-isocrystals we have not imposed any conditions "at infinity." There is one condition that is extremely important, being a necessary condition for the existence of a reasonable cohomology theory.

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We say that an isocrystal (M, ∇) is *overconvergent* if there is a (rigid-analytic) open neighborhood $X_\infty \subset U \subset \mathbb{P}^1$, which for each $s \in S$ contains some annulus $r < |t_s| < 1$, such that (M, ∇) extends to a U.

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Similarly, an F-isocrystal (M, ∇, F) is overconvergent if M, ∇ and F all extend to some such neighborhood U.

Overconvergence is a very restrictive condition. For example:

- If (M, ∇, F) is an overconvergent unit-root F-isocrystal in $X_0 \subset \mathbb{P}^1$, then the corresponding p-adic representation of $\pi_1(X_0)$ has finite inertia at every point at infinity. Conversely, if a unit-root F-isocrystal has this property of finite inertia, it is overconvergent.

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- If (M, ∇, F) is an overconvergent F-isocrystal on X_∞ with constant Newton polygon, the quotients for the slope filtration are not usually overconvergent. The simplest example is the rank two F-isocrystal coming from the the relative crystalline H^1 of an ordinary elliptic curve on X. The slope filtration has a rank one unit-root F-isocrystal; if the j-invariant is not constant, this is not overconvergent.

Rigid Cohomology

When (M, ∇) is an overconvergent isocrystal on $X \subset \mathbb{P}^1$, the *rigid cohomology* $H^i(X, M)$ is defined as follows. First, for all sufficiently large $r < 1$ we choose a rigid-analytic open U_r with $X_\infty \subset U_r \subset \mathbb{P}^1$ containing the annulus $r < |t_s| < 1$ for all s, as above. Denote by A_r the ring of global rigid-analytic functions on U_r.

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We can suppose that (M, ∇) extends to U_r for sufficiently large r in a manner compatible with the inclusions $U_r \to U_s$ for $r < s < 1$. Since U_r is Stein, we can identify this extension with its A_r-module global sections M_r.

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$$M^\dagger = \varinjlim_r M_r \qquad A^\dagger = \varinjlim_r A_r$$

then the rigid cohomology of M is defined by

$$H^i(X_0, M) = \begin{cases} \mathrm{Ker}(\nabla : M^\dagger \to \Omega^1_{A/K}) \otimes M & i = 0 \\ \mathrm{Coker}(\nabla : M^\dagger \to \Omega^1_{A/K}) \otimes M & i = 1 \\ 0 & i > 1. \end{cases}$$

To define cohomology with supports we introduce the *Robba ring* $\mathcal{R}(s)$ at a point $s \in S$; it is the ring of formal Laurent series $\sum_{n\in\mathbb{Z}} a_n t_s^n$ convergent on some annulus $r < |t_s| < 1$.

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If $C(M)$ is the cokernel of this map, the rigid cohomology with supports is defined by

$$H^i_c(X_0, M) = \begin{cases} \mathrm{Ker}(\nabla : C(M) \to C(\Omega^1_{A/K} \otimes M) & i = 1 \\ \mathrm{Coker}(\nabla : C(M) \to C(\Omega^1_{A/K} \otimes M) & i = 2 \\ 0 & i \neq 1,\ 2. \end{cases}$$

The motive for replacing M by $M^\dagger$ is that this is necessary for a de Rham type cohomology theory to have finite dimension; it is easy to see that the de Rham H^1 of (M, ∇) on X_∞ is not of finite dimension even in the simplest cases.

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One can show that if a is not p-adic Liouville, then (M, ∇) is a coherent module over Berthelot's ring $\mathcal{D}^\dagger$ of arithmetic differential operators.

The Trace Formula

If $k = \mathbb{F}_q$ is a finite field and (M, ∇, F) is an overconvergent F-isocrystal on X_0, the sum of the Frobenius traces is computed by the Lefschetz formula

$$\sum_{x \in X_0(\mathbb{F}_{q^n})} (\mathrm{Tr} F^n | M_x) = \sum_i (-1)^i \mathrm{Tr}(F^n)^* | H^i_c(X_0, M)$$

(Dwork, Monsky, Reich).

The Monodromy Theorem

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The *Robba ring* $\mathcal{R}$ is the ring of formal Laurent series $\sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in K$, convergent in some annulus $r < |t| < 1$; the rings $\mathcal{R}(s)$ considered just now are examples. We can define an "overconvergent isocrystal on $\mathcal{R}$" to be a finite free $\mathcal{R}$-module M endowed with a connection ∇; likewise a "overconvergent F-isocrystal" (M, ∇, F) is an overconvergent isocrystal (M, ∇) endowed with a Frobenius structure.

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We can view $\mathcal{R}$ as attached to the field $F = k((t))$ of Laurent series in characteristic $p > 0$, and write it $\mathcal{R}_F$. If L/F is a finite extension there is a natural K-algebra homorphism $\mathcal{R}_F \to \mathcal{R}_L$. If in addition L/F is Galois with group G, then G acts on $\mathcal{R}_L$, and the quotient is $\mathcal{R}_F$.

An isocrystal (M, ∇) on $\mathcal{R} = \mathcal{R}_F$ is *quasi-unipotent* if there is a finite separable extension L/F such that $M \otimes_{\mathcal{R}_F} \mathcal{R}_L$ with its induced connection is a successive extension of trivial one-dimensional isocrystals on $\mathcal{R}_L$.

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One can show (RC) that if (M, ∇) is quasi-unipotent then $H^i(M)$ and $H^i_c(M)$ have finite dimension. Matsuda showed that quasi-unipotent isocrystals have a canonical extension (in the sense of Katz) to $\mathbb{P}^1$.

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The local monodromy theorem asserts that any F-isocrystal on $\mathcal{R}$ is quasi-unipotent. This was conjectured by RC and (independently) N. Tsuzuki, and proven in the summer of 2002 by Y. André, Z. Mebkout and K. Kedlaya (independently and by different methods). If (M, ∇, F) is unit-root, this had been proven by Tsuzuki.

Monodromy Groups

We have seen that the category of unit-root F-isocrystals on X_0 is equivalent to the category of representations of $\pi_1(X_0)$. For the category of F-isocrystals on X_0, isocrystals on X_0 there is a similar description by means of differential Galois theory.

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Suppose for the moment that (M, ∇) is any locally free module on, say a smooth variety X in characteristic 0. If x is a point of X we define an algebraic group $\mathrm{DGal}(M, x)$ as follows: it is the subgroup of $GL(M_x)$ fixing the fiber at x of any horizontal submodule of any tensor product $M^{\otimes m} \otimes (M^\vee)^{\otimes n}$.

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The same definition can be for a convergent isocrystal on X_0. The resulting algebraic group over K is the *differential Galois group* or *monodromy group* of M.

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If (M, ∇) is an overconvergent isocrystal on X_0 and $\hat{M}$ is M viewed simply as a convergent isocrystal, there is a natural closed immersion

$$\mathrm{DGal}(\hat{M}) \hookrightarrow \mathrm{DGal}(M).$$

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It is not an isomorphism, usually.

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Suppose (M, ∇, F) is an F-isocrystal (of either sort). To the isocrystal (M, ∇) we can attach an algebraic group $\mathrm{DGal}(M, x)$ as before. On the other hand, the Frobenius structure induces one F_x on the fiber M_x, so we get an F-isocrystal (M_x, F_x) on K (this is assuming x is a k-rational point of X).

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Now one can show that pullback by geometric Frobenius induces a σ-linear isomorphism

$$\Phi : \mathrm{DGal}(M, x)^{(\sigma)} \xrightarrow{\sim} \mathrm{DGal}(M, x)$$

i.e. a "Frobenius structure on the group" $\mathrm{DGal}(M, x)$.

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We could call the latter objects "representations of $(\mathrm{DGal}(M, x), \Phi_x)$ on the F-isocrystal (M_x, Ψ_x)."

Suppose (M, ∇, F) is a unit root F-iscrystal, and let $\rho : \pi_1(X_0) \to GL(V)$ be the corresponding representation of the fundamental group. One can show that $DGal(M)$ is isomorphic to the Zariski closure of the image of $\pi_1(X_0 \times \overline{k})$, where $\overline{k}$ is the algebraic closure of k.

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One prove (RC) an analogue of Grothendieck's *global monodromy theorem* for these monodromy groups: if k is the perfection of an absolutely generated field and (M, ∇, F) is an overconvergent isocrystal on X_0, then the radical of DGal(M) is unipotent.

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The theorem is proven by successive reduction to the case of a curve, to the rank one case, and then to the unit-root case, in which case it follows from the finite local monodromy of unit-root F-crystals, together with results of Katz and Lang on the structure of $\pi_1(X_0)$ when k is absolutely finitely generated.

From the global monodromy theorem one can derive the theory of "determinantal weights" used by Deligne to construct the monodromy weight filtration in ℓ-adic cohomology. The corresponding p-adic construction was used by Kedlaya in his proof of the p-adic Weil conjectures for pure overconvergent F-isocrystals. n

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One can also prove equidistribution results for the Frobenius eigenvalues, following the method of Deligne.

Local Monodromy

The local monodromy theorem also has a formulation in terms of monodromy groups. We need to introduce the *ring of p-adic hyperfunctions* $\mathcal{B}$.

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As before we denote by $\mathcal{R}$ the Robba ring, with t, say, as the local parameter. We recall that of L is any finite separable extension of $F = k((t))$, the morphism $F \to L$ lifts, in a sense, to a morphism $\mathcal{R} \to \mathcal{R}_L$ of Robba rings. We denote by $\overline{\mathcal{R}}$ the direct limit, over all such extensions, of the corresponding Robba rings.

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The ring of p-adic hyperfunctions $\mathcal{B}$ is the polynomial ring $\overline{\mathcal{R}}[\log t]$ where for the moment "$\log t$" is viewed as a formal variable.

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- Finally, any lifting of Frobenius to $\mathcal{R}$ extends to $\mathcal{B}$. A standard choice is to take $\phi(t) = t^p$, $\phi(\log t) = p \log t$.The endomorphisms ϕ, $\mathcal{N}$ satisfy

$$\mathcal{N}\phi = p\phi\mathcal{N}.$$

If (M, ∇, F) is an overconvergent F-isocrystal on $\mathcal{R}$, the local monodromy theorem can interpreted as saying that $M \otimes_{\mathcal{R}} \mathcal{B}$ is a free $\mathcal{B}$-module with a basis of horizontal sections for ∇.

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Furthermore the endomorphisms F, N satisfy

$$NF = pFN$$

Finally, the F-isocrystal (M, ∇, F) can be reconstructed from the data $(\mathbb{V}, N, F)$ and the Galois action. Thus the category of overconvergent F-isocrystals on $\mathcal{R}$ is equivalent to the category of $(\mathbb{V}, N, F, \text{galois action})$.

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The construction $(M, \nabla, F) \mapsto \mathbb{V}(M)$ can serve as the (missing) "fiber at $t = 0$" of the F-isocrystal (M, ∇, F), which allows us to define a monodromy group $\mathrm{DGal}(M)$ with a Frobenius structure.

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One can then show that the category of overconvergent F-isocrystals on $\mathcal{R}$ is equivalent to the category of $\overline{K}$-representations of the pro-algebraic group with Frobenius structure $(G_k \times \mathbb{G}_a, \Phi)$, where the action of Φ on the absolute Galois group is the canonical one induced by functoriality, and on the additive group $\mathbb{G}_a$ is multiplication by p.

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As a consequence, when k is a *finite* field, the category of overconvergent F-isocrystals on $\mathcal{R}$ is equivalent to the category of K-representations of the usual Deligne-Weil group of k.

Computations: Katz's congruences

Actually computing a Frobenius structure can be rather difficult, as one sees from the work of Dwork (for example). In geometric cases, the Frobenius eigenvalues on rigid cohomology can in principle be recovered from the trace sums over rational points, but this is not computationally effective.

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This technique was first used by Katz to prove some congruences for Gauss sums that were conjectured by Honda and used by Koblitz and Gross. Others have treated some examples of higher dimension.

Since X_0 is liftable, the Hodge spectral sequence degenerates at E_1, so that every global i-form is closed. This means that the Cartier operator is defined as an operation on $H^0(X_0, \Omega^i_{X_0/k})$, and we will assume that is an *isomorphism*.

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With these hypotheses on X_0, we can identify $H^i_{cris}(X_0/W)$ with the de Rham cohomology $H^i_{DR}(X/W)$. We denote by Q_i the part of $H^i_{cris}(X_0/W)$ with slope i. Since $H^0(X, \Omega^i_{X/W}) \subset H^i_{DR}(X/W)$, we get a map

$$H^0(X, \Omega^i_{X/W}) \to Q_i$$

which our hypotheses show to be an isomorphism. In particular, they have the same dimension $h^{i,0}$.

On the other hand if x is a W-point of X and $t_1, \ldots, t_n$ are local parameters at x, there is a "formal expansion map"

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and if ω is a global i-form we write

$$\omega = \sum_K \sum_W a(\omega, K, W) t^W dt_K / t_K$$

where $K \subseteq \{1, \ldots, n\}$ and $dt_K/t_K = \prod_{j \in K} \frac{dt_j}{t_j}$.

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We can choose $h^{i,0}$ pairs of indices (K_v, W_v) such that the map

$$H^0(X, \Omega^i_{X/W}) \to W^{h^{i,0}} \qquad \omega \mapsto (a(\omega, K_v, W_v))$$

is an isomorphism. Let ω_α, $1 \leq \alpha \leq h^{i,0}$ be the basis of $H^0(\Omega^i)$ corresponding to the standard basis of $W^{h^{i,0}}$.

For $m \geq 1$ we define a square matrix $E(m)$ by

$$E(m)_{v,\alpha} = a(\omega_\alpha, p^m K_v, p^m W_v).$$

This is an invertible matrix since its reduction modulo p is the m^{th} iterate of the Cartier operator.

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Katz shows that for all integers $m \geq 1$ there are congruences

$$F \equiv p^i E(m+1)^{-1} E(m)^\sigma \mod p^{m+1}$$

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Taking the limit as $m \to \infty$, we find

$$F = \lim_{m \to \infty} E(m+1)^{-1} E(m)^\sigma.$$

The argument for the congruence formula amounts to showing that the formal expansion map

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has as its kernel the part of $H^i_{cris}(X_0/W)$ with slopes in the interval $[0, i-1]$. Then Q_i injects into $H^i_{DR}(\mathrm{Spf}(W[[t_1, \ldots, t_n]])/W)$, and the computation of F can be done there.

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Instead of taking X to be a smooth projective W-scheme, we can take it to be a smooth projective S-scheme where S is itself a smooth W-scheme. In this case, we can also recover the "slope i piece" of the Gauss-Manin connection for the family X/S.

As an example, let's consider the case where X is smooth projective W-scheme with trivial canonical bundle, and we assume it is partially ordinary in dimension n. Since $\Omega^n_{X/W}$ is generated by a single nonvanishing n-form ω, we may take $K = \{1, \ldots, n\}$ and $W = (1, \ldots, 1)$: ω has nonzero "constant term" $a(1, \ldots, 1)$.

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When W reduces to an ordinary elliptic curve, this was proven by Dwork in ancient times.

Lauder's Method

We conclude with a method of computing the Frobenius matrix in specific cases due to Alan Lauder. We recall that if (M, ∇, F) is an F-isocrystal on X_0, then

$$dF + AF = F\phi^* A.$$

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Note that the equation for F can be solved by first solving the Gauss Manin system. Suppose for example that X_0 is an open subset of $\mathbb{P}^1$, s_0 is $t = 0$ for some parameter on $\mathbb{P}^1$, and the Gauss-Manin system is actually free on X_0, with connection matrix A as before, so that the system is

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for some matrix C with entries that are power series in t, satisfying $C(0) = I$. The power series expansion of F at $t = 0$ is then

$$F = C^{-1}F(0)\phi(C).$$

We then continue this power series solution analytically, and evaluate at s_1.

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$$\sum_{1 \leq i \leq n} a_i X_i^d + th(X_1, \ldots, X_n) = 0$$

with a_i, $y \in \mathbb{F}_q$ and $h \in \mathbb{F}_q[X_1, \ldots, X_n]$ has no diagonal terms; furthermore the a_i are all nonzero, $p > 2$ and p does not divide d.

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In fact, Lauder computes explicitly the computational complexity of this problem, which turns out to be of polynomial time.

Envoi

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Happy Beethoven's Birthday!