Homological Algebra
Lecture 1

Richard Crew

Additive Categories

Categories of modules over a ring have many special features that categories in general do not have. For example the Hom sets are actually abelian groups. Products and coproducts are representable, and one can form kernels and cokernels. The notation of an *abelian category* axiomatizes this structure. This is useful when one wants to perform module-like constructions on categories that are not module categories, but have all the requisite structure.

We approach this concept in stages. A *preadditive category* is one in which one can add morphisms in a way compatible with the category structure. An *additive category* is a preadditive category in which finite coproducts are representable and have an "identity object." A *preabelian category* is an additive category in which kernels and cokernels exist, and finally an *abelian category* is one in which they behave sensibly.

Definition

A *preadditive category* is a category $\mathcal{C}$ for which each Hom set has an abelian group structure satisfying the following conditions:

- For all morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{C}$ the maps

  $$\mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y), \qquad \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y')$$

  induced by f and g are homomorphisms.
- The composition maps

  $$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z) \quad (g, f) \mapsto g \circ f$$

  are bilinear.

The group law on the Hom sets will always be written additively, so the last condition means that

$$(f + g) \circ h = (f \circ h) + (g \circ h), \quad f \circ (g + h) = (f \circ g) + (f \circ h).$$

We denote by 0 the identity of any Hom set, so the bilinearity of composition implies that

$$f \circ 0 = 0 \circ f = 0$$

for any morphism f in $\mathcal{C}$.

Definition

Suppose $\mathcal{C}$ is a preadditive category. A *zero object* of $\mathcal{C}$ is an object Z such that $\mathrm{Hom}_{\mathcal{C}}(Z, Z)$ is the trivial group.

If Z is a zero object, $1_Z = 0$. Conversely if Z is any object such that $1_Z = 0$,

$$f = f \circ 1_Z = f \circ 0 = 0$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(Z, Z)$. Thus Z is zero if and only if $1_Z = 0$.

Lemma

If Z is an object of the preadditive category $\mathcal{C}$ the following are equivalent:

- *Z is initial.*
- *Z is final.*
- *Z is zero.*

Proof: If Z is initial (resp. final) then $\mathrm{Hom}(Z, X)$ (resp. $\mathrm{Hom}(X, Z)$) is a singleton for all X in $\mathcal{C}$, and therefore the trivial group. In either case then $\mathrm{Hom}(Z, Z) = 0$ and Z is zero. Conversely if Z is zero and $f : X \to Z$ is a morphism,

$$f = 1_Z \circ f = 0 \circ f = 0$$

and thus $\mathrm{Hom}(X, Z) = 0$, i.e. Z is final. The same argument shows that Z is initial. ■

In particular any two zero objects are isomorphic. We will usual use 0 to denote any (unspecified) zero object.

If $\mathcal{C}$ is a preabelian category the functor

$$h_X(T) = \mathrm{Hom}_{\mathcal{C}}(T, X)$$

can be viewed as a functor $h_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ where $\mathbf{Ab}$ is the category of abelian groups. Then $X \mapsto h_X$ defines a functor

$$h : \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$$

and the "preabelian" version of Yoneda's lemma says that this functor is fully faithful. The argument is the same as in the case of normal categories. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ is *representable* if it is in the essential image of h, i.e. if there is an isomorphism $\alpha : h_X \xrightarrow{\sim} F$ for some object X of $\mathcal{C}$. In this situation the *universal element* is

$$\xi = \alpha_X(1_X)$$

and in terms of this element α can be computed as

$$f \in \mathrm{Hom}_{\mathcal{C}}(T, X) = h_X(T) \mapsto F(f)(\xi) \in F(T).$$

Definition

An *additive category* is a preadditive category $\mathcal{C}$ such that

- finite coproducts are representable;
- $\mathcal{C}$ has a zero object 0.

If $X \oplus Y$ is representable we denote by $i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$ the canonical morphisms. If $f : X \to T$ and $g : Y \to T$ are morphisms we denote by $[f, g] : X \oplus Y \to T$ morphism induced by f and g, i.e. the unique morphism making

$$\begin{tikzcd}
X \arrow[d, "i_1"'] \arrow[dr, bend left, "f"] & \\
X \oplus Y \arrow[r, "{[f,g]}"] & T \\
Y \arrow[u, "i_2"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

commutative.

If $f : X \to X'$ and $g : Y \to Y'$ we denote by $f \oplus g$ the morphism $X \oplus Y \to X' \oplus Y'$ induced by functoriality.

Given morphisms f, $f' : X \to T$ and g, $g' : Y \to T$, the bilinearity of composition shows that

$$\begin{aligned}([f,g] + [f',g']) \circ i_1 &= [f,g] \circ i_1 + [f',g'] \circ i_1 \\ &= f + f' \\ &= [f + f', g + g'] \circ i_1\end{aligned}$$

and similarly

$$([f,g] + [f',g']) \circ i_2 = [f + f', g + g'] \circ i_2.$$

Since $[f,g]$ is determined by the compositions $[f,g] \circ i_1$ and $[f,g] \circ i_2$ we conclude that

$$[f,g] + [f',g'] = [f + f', g + g']. \tag{1}$$

In the category **Ab** of abelian groups $X \oplus Y$ is the usual direct sum of the groups X and Y, and $i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$ are $i_1(x) = (x, 0)$ and $i_2(y) = (0, y)$. Given $f : X \to T$ and $g : Y \to T$, $[f, g] : X \oplus Y \to T$ is the map

$$(x, y) \mapsto f(x) + g(y).$$

On the other hand if $f : X \to X'$ and $g : Y \to Y'$ are morphisms, the induced morphism $f \oplus g : X \oplus Y \to X' \oplus Y'$ is

$$(f \oplus g)(x, y) = (f(x), g(y)).$$

The category **Ab** is itself and additive category. In **Ab** the product $X \times Y$ of two groups is as it were "accidentally" isomorphic to the coproduct (direct sum) $X \oplus Y$. This turns out to be the case in any additive category (in particular, it's not an accident).

We first note that if X and Y are objects of the additive category $\mathcal{C}$ there are morphisms

$$\mathrm{Hom}(X, T) \to \mathrm{Hom}(X, T) \times \mathrm{Hom}(Y, T) \simeq \mathrm{Hom}(X \oplus Y, T)$$
$$\mathrm{Hom}(Y, T) \to \mathrm{Hom}(Y, T) \times \mathrm{Hom}(Y, T) \simeq \mathrm{Hom}(X \oplus Y, T)$$

where on each line the first maps are

$$f \mapsto (f, 0) \qquad g \mapsto (0, g)$$

respectively. Since these are functorial in T, Yoneda shows that they arise from morphisms

$$p_1 : X \oplus Y \to X, \qquad p_2 : X \oplus Y \to Y$$

in $\mathcal{C}$. It is easily checked that in **Ab**, p_1 and p_2 are the usual projections

$$p_1(x, y) = x, \qquad p_2(x, y) = y.$$

From this we immediately deduce the identities

$$\begin{aligned} p_1 i_1 &= 1_X & p_2 i_2 &= 1_Y \\ p_1 i_2 &= 0 & p_2 i_1 &= 0 \end{aligned}$$

and for any $f : X \to X'$, $g : Y; \to Y'$

$$f \oplus g = i_1' f p_1 + i_2' g p_2 \tag{2}$$

where $i_1' : X' \to X' \oplus Y'$ and $i_2' : Y' \to X' \oplus Y'$ are the canonical morphisms. As usual these are proven by (i) checking them in **Ab**, and (ii) Yoneda. Let's carry out (i) for the formula (2): for any $(x, y) \in X \oplus Y$,

$$\begin{aligned} (i_1' f p_1 + i_2' g p_2)(x, y) &= i_1' f p_1(x, y) + i_2' g p_2(x, y) \\ &= i_1' f(x) + i_2' g(y) \\ &= (f(x), 0) + (0, g(y)) \\ &= (f(x), g(y)) = (f \oplus g)(x, y). \end{aligned}$$

Note that applying (2) with $f = 1_X$, $g = 1_Y$ and $[1_X, 1_Y] = 1_{X\oplus Y}$ we find

$$i_1 p_1 + i_2 p_2 = 1_{X\oplus Y}. \tag{3}$$

Theorem

For any two objects X and Y of a additive category $\mathcal{C}$ the triple $(X \oplus Y, p_1, p_2)$ represents the product of X and Y.

Proof: We must show that for any T in $\mathcal{C}$ and morphisms $f : T \to X$, $g : T \to Y$ there is a unique morphism $h : T \to X \oplus Y$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$:

$$\begin{tikzcd}
T \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \oplus Y \arrow[r, "p_2"'] \arrow[d, "p_1"] & Y \\
& X &
\end{tikzcd}$$

Denote by $i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$ the canonical morphisms. It is easy to see what h must be if it exists: if $p_1 h = f$ and $p_2 h = g$, (3) shows that

$$\begin{aligned} h &= 1_{X \oplus Y} \circ h \\ &= (i_1 p_1 + i_2 p_2) h \\ &= i_1 p_1 h + i_2 p_2 h \\ &= i_1 f + i_2 g. \end{aligned}$$

Thus h is unique. Conversely, given f and g we can define h by this formula, and then

$$\begin{aligned} p_1 h = p_1(i_1 f + i_2 g) = p_1 i_1 f + p_1 i_2 g = 1_X f + 0g = f \\ p_2 h = p_2(i_1 f + i_2 g) = p_2 i_1 f + p_1 i_2 g = 0f + 1_Y g = g \end{aligned}$$

as required. ■

Corollary

If $\mathcal{C}$ is additive, finite products are representable in $\mathcal{C}$.

Given $f : X \to X'$ and $g : Y \to Y'$ there is functorial morphism $X \times Y \to X' \times Y'$, which we denote by $f \times g$. However the same Yoneda-lemma argument as before shows that

$$f \times g = i_1' f p_1 + i_2' g p_2$$

and since we can identify $X \times Y = X \oplus Y$, we find that

$$f \times g = f \oplus g.$$

Remark: One can show, conversely that if $\mathcal{C}$ is a preadditive category with zero objects and finite products are representable then so are finite coproducts. The argument is entirely dual to the preceding one. Given objects X and Y of $\mathcal{C}$ we denote by

$$i_1 : X \to X \times Y, \qquad i_2 : Y \to X \times Y$$

the morphisms arising via Yoneda from the functorial morphisms

$$\begin{aligned} \mathrm{Hom}(T, X) &\to \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y) \simeq \mathrm{Hom}(T, X \times Y) \\ \mathrm{Hom}(T, Y) &\to \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y) \simeq \mathrm{Hom}(T, X \times Y) \end{aligned}$$

given by $f \mapsto (f, 0)$ and $g \mapsto (0, g)$. With this definition the formula (3) still holds. Then $(X \times Y, i_1, i_2)$ represents the coproduct of X and Y: given morphisms $f : X \to T$ and $g : Y \to T$ in $\mathcal{C}$, the morphism

$$h = fp_1 + gp_2 : X \times Y \to T$$

is the unique morphism such that $hi_1 = f$ and $hi_2 = g$.

Suppose $\mathcal{C}$ is preadditive and let $\mathcal{C}^{\mathrm{op}}$ be the opposite category, i.e. the category with the same objects and

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

This endows the Hom sets of $\mathcal{C}^{\mathrm{op}}$ with an abelian group structure and it is easily checked that $\mathcal{C}^{\mathrm{op}}$ is preabelian. One says in these circumsances that the notion of a preabelian category is *self-dual*.

If $\mathcal{C}$ is preabelian the classes of zero objects, initial objects and final objects are the same. Therefore any zero object of $\mathcal{C}$ is a zero object of $\mathcal{C}^{\mathrm{op}}$ and conversely.

If the product (resp. coproduct) of any two objects of $\mathcal{C}$ is representable then the product (resp. coproduct) is a coproduct (resp. product) in $\mathcal{C}^{\mathrm{op}}$. Thus if $\mathcal{C}$ is an additive category, $\mathcal{C}^{\mathrm{op}}$ has zero objects and finite products, and is therefore an additive category by the previous remark. Conversely if $\mathcal{C}^{\mathrm{op}}$ is additive, so is $\mathcal{C}$. In other words the notion of additive category is self-dual.

We have seen that an additive category has finite products and coproducts, but we do not know anything about the representability of more general limits or colimits (e.g. equalizers and coequalizers).

Suppose $\mathcal{A}$ is an additive category and $f : X \to Y$ is a morphism in $\mathcal{A}$. A morphism $i : K \to X$ is a *kernel* of f if

$$0 \to \mathrm{Hom}(T, K) \to \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$$

is an exact sequence for all T in $\mathcal{C}$ (here Hom is $\mathrm{Hom}_{\mathcal{A}}$). We say that f *has a kernel* if such a morphism $i : K \to X$ exists.

Evidently $f : X \to Y$ has a kernel if and only if the functor

$$T \mapsto \mathrm{Ker}(\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y))$$

is representable. If it is the "universal object" is none other than an object K and a morphism $i : K \to X$ making the previous diagram an exact sequence.

Observe that if $i : K \to X$ is a kernel of $f : X \to Y$ then $\mathrm{Hom}(T, K) \to \mathrm{Hom}(T, X)$ is injective for all T, i.e. $i : K \to X$ is a monomorphism. In other words $K \to X$ is a *subobject* of X. Since objects representing a functor are unique up to unique isomorphism, any two kernels of $f : X \to Y$ are unique up to unique isomorphism: if $i : K \to x$ and $i' : K' \to X$ are both kernels of $f : X \to Y$, there is a unique isomorphism $j : K \xrightarrow{\sim} K'$ such that

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "j"'] & X \\ K' \arrow[ur, "i'"'] & \end{tikzcd}$$

commutes.

It is immediate from the definition that a morphism $f : X \to Y$ in $\mathcal{C}$ is a monomorphism if and only if the unique morphism $0 \to X$ is a kernel of f.

The dual notion is that of a *cokernel* of $f : X \to Y$, which is a morphism $p : Y \to C$ such that

$$0 \to \mathrm{Hom}(C, T) \to \mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T)$$

is exact for all T in $\mathcal{C}$. In particular the injectivity of $\mathrm{Hom}(C, T) \to \mathrm{Hom}(Y, T)$ shows that $p : Y \to C$ is an *epimorphism*, or in other words that $Y \to C$ is a *quotient* of Y. As before cokernels are unique up to unique isomorphism when they exist. A morphism $f : X \to Y$ in $\mathcal{C}$ is an epimorphism if only if $Y \to 0$ is a cokernel of f.

Definition

An additive category $\mathcal{A}$ is *preabelian* if the kernel and cokernel of every morphism in $\mathcal{A}$ is representable.

Evidently a kernel (resp. cokernel) of a morphism $X \to Y$ in $\mathcal{A}$ is the cokernel (resp. kernel) of the corresponding morphism $Y \to X$ in $\mathcal{A}^{\mathrm{op}}$. Therefore the notion of a preadditive category is self-dual.

The condition defining a kernel $i : K \to X$ of a morphism $f : X \to Y$ can be phrased as follows: if $h : T \to X$ is such that $fh = 0$ then h factors uniquely through a morphism $\bar{h} : T \to K$.

Suppose now $\mathcal{A}$ is preabelian and f and g are morphisms $X \to Y$ in $\mathcal{A}$. If $h : T \to X$ is any morphism with target X,

$$fh = gh \iff (f - g)h = 0$$

and from this we deduce that a kernel of $f - g$ is an equalizer of the morphisms f and g. Therefore in any preadditive category, equalizers are representable. The dual argument shows that coequalizers are also representable; in fact a cokernel of $f - g$ is a coequalizer of f and g.

If $f : X \to Y$ is a morphism in a preabelian category we denote an unspecified kernel (resp. cokernel) by $\mathrm{Ker}(f) \to X$ (resp. $Y \to \mathrm{Coker}(f)$). It must be remembered that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are not unique, just unique up to unique isomorphism.

Proposition

Finite limits and colimits are representable in any preabelian category.

Proof: A category has finite limits (resp. colimits) if and only if finite products and equalizers) (resp. finite coproducts and coequalizers) are representable. ■

In particular we get the important fact that fibered products, or *pullbacks* and cofibered coproducts, or *pushouts* are always representable in a preabelian category.

If $f : X \to Z$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$ we denote by $X \times_Z Y$ an unspecified fibered product of f and g. Similarly if $f : Z \to X$ and $g : Z \to Y$ are are morphisms, $X \amalg^Z Y$ is an unspecified cofibered coproduct.