Homological Algebra
Lecture 10

Richard Crew

Summer 2021

Sheaves on a topological space

Suppose $\mathcal{C}$ and $\mathcal{A}$ are categories. An $\mathcal{A}$-valued presheaf on $\mathcal{C}$ is simply a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}$. Two particularly important cases are where $\mathcal{A}$ is the category of sets, in which case these are called *presheaves of sets on* $\mathcal{C}$, and where $\mathcal{A}$ is the category of abelian groups, in which case these are called *presheaves of abelian groups on* $\mathcal{C}$, or more simply *abelian presheaves on* $\mathcal{C}$. For any $\mathcal{C}$ the $\mathcal{A}$-valued presheaves form a category in which the morphisms are morphisms of functors.

We now take $\mathcal{C}$ to be the following category. Let X be a topological space. The category X_{Zar} is as follows:

- Objects of X_{Zar} are open subsets of X.
- For any objects U, V (i.e. open subsets), $\mathrm{Hom}(U, V)$ is empty unless $U \subseteq V$, in which case it is a singleton set.
- The rule for composition of morphisms is the only one possible.

Following Grothendieck we call the category X_{Zar} the *Zariski site of* X. For any continuous map $f : X \to Y$ there is a functor $f^* : Y_{\text{Zar}} \to X_{\text{Zar}}$ given by $f^*(U) = f^{-1}(U)$.

A presheaf of sets F on X_{Zar} has the following explicit description:

- to every open set $U \subseteq X$ we associate a set $F(U)$;
- to every inclusion $U \subseteq V$ of open subsets we specify a map $\rho_{UV} : F(V) \to F(U)$.

The maps ρ_{UV} are called the *restriction maps* for the presheaf. They need not be injective. If $s \in F(V)$ and $U \subseteq V$ we will write $s|U$ for $\rho_{UV}(s)$ (as if it were a function).

Presheaves of abelian groups have basically the same description, except that each $F(U)$ is an abelian group and the ρ_{UV} are group homomorphisms.

In what follows "presheaf on X" means "presheaf of sets on X_{Zar} and "abelian presheaf on X" means "presheaf of abelian groups on X_{Zar}." We denote by **PShv**(X) the category of presheaves of sets on X, and by **PAb**(X) the category of abelian presheaves on X.

For historic reasons the sets (or groups) $F(U)$ are called the *sections* of F over U. I will try to explain this laters.

Morphisms of presheaves (of sets, abelian groups...) are just morphisms of the corresponding functors. Thus a morphism of presheaves $\alpha : F \to G$ on X is a collection of maps $\alpha_U : F(U) \to G(U)$ for all open $U \subseteq X$ such that whenever $U \subseteq V$, the diagram

$$\begin{array}{ccc} F(V) & \xrightarrow{\rho_{UV}} & F(U) \\ \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_U} \\ G(V) & \xrightarrow{\rho_{UV}} & G(U) \end{array}$$

is commutative. Similarly for morphisms of abelian sheaves, except now the α_U must be homomorphisms.

If $x \in X$ is a point we denote by I_x the *neighborhood filter* of x, i.e. the set of open neighborhoods of x partially ordered by the inverse of inclusion, i.e. $U \leq V$ if and only if $V \subseteq U$. This is a filtered partially order: given any two U and V in I_x there is a $W \in I_x$ such that $W \subseteq U$ and $W \subseteq V$, i.e. $U \leq W$ and $V \leq W$.

We can identify I_x with a full subcategory of X_{Zar}, so a presheaf $F : X_{\mathrm{Zar}}^{\mathrm{op}} \to \mathbf{Set}$ yields by composition a functor $I_x^{\mathrm{op}} \to \mathbf{Set}$. The *stalk* of F at x is the colimit

$$F_x = \varinjlim_{U \in I_x^{\mathrm{op}}} F(U).$$

Since I_x is filtered, elements of F_x have the following description: they are represented by sections $s \in F(U)$ for some $U \in I_x$, and $t \in F(V)$ represents the same element of F_x if and only if $s|W = t|W$ for some $W \subseteq U \cap V$. If $s \in F(U)$ and $x \in U$ we denote by s_x the image of s in F_x.

Since colimits are functorial in their arguments a morphism of presheaves $F \to G$ induces a morhphism $F_x \to G_x$ of stalks. In other words $F \mapsto F_x$ defines a functor $\mathbf{PShv}(X) \to \mathbf{Set}$. When F is an abelian presheaf, F_x is naturally an abelian group and in this way we get a functor $\mathbf{PAb}(X) \to \mathbf{Ab}$.

Proposition

Let X be a topological space. Arbitrary limits and colimits in $\mathbf{PShv}(X)$ are representable, and for any functor

$$I \to \mathbf{PShv}(X) \qquad i \mapsto F_i$$

the limit is

$$(\varprojlim_i F_i)(U) \simeq \varprojlim_i F_i(U)$$

and the colimit is

$$(\varinjlim_i F_i)(U) \simeq \varinjlim_i F_i(U).$$

In other words limits and colimits can be computed "value by value."

Proof (sketch): In fact the formulas define presheaves on X; the universal properties of the limit and colimit then show that they are indeed the limit and colimit of the functor. ∎

Corollary

The category **PAb**(X) *of abelian presheaves on* X *is abelian.*

Proof (really sketchy): We proceed in steps:

(1) **PAb**(X) is preadditive: suppose $f : F \to G$ and $g : F \to G$ are morphisms, so that for each $U \subseteq X$ we are given homomorphisms $f_U : F(U) \to G(U)$ and $g_U : F(U) \to G(U)$. We define $(f+g) : F \to G$ by $(f+g)_U = f_U + g_U$. This gives $\mathrm{Hom}(F, G)$ the structure of an abelian group. The zero object is the zero preheaf $F(U) = 0$.

(2) **PAb**(X) is additive: for abelian preasheaves F and G, the proposition shows that $F \oplus G$ is representable and that $(F \oplus G)(U) = F(U) \oplus G(U)$.

(3) **PAb**(X) is preabelian: again by the proposition, kernels and cokernels exist since this are finite limits and colimits. If $\alpha : F \to G$ is a morphism, the kernel of α is $U \mapsto \mathrm{Ker}(\alpha_U)$ and the cokernel of α is $U \mapsto \mathrm{Coker}(\alpha_U)$.

(4) **PAb**(X) is abelian: use the explicit formulas for the kernel and cokernel. ∎

We now give some examples (in fact important motivating examples).

- Fix a topological space X, and for any open $U \subseteq X$ let $\mathcal{O}_X(U)$ be the set of $\mathbb{R}$-valued continuous functions on X. The maps ρ_{UV} are the usual restriction of functions. It is easily checked that $U \mapsto \mathcal{O}_X(U)$ is a presheaf of abelian groups on X (in fact a presheaf of rings, i.e. a functor $X_{\text{Zar}}^{\text{op}} \to \textbf{Ring}$).
- Now let X be an open subset of $\mathbb{R}^n$, and for open $U \subseteq X$ let $\mathcal{O}_X^\infty(U)$ be the set of infinitely differentiable real-valued functions on U. This is again a presheaf of rings if we take the ρ_{UV} to be the usual restriction maps.
- Finally let X be an open subset of $\mathbb{C}^n$ and for open $U \subseteq X$ let $\mathcal{O}_X^{hol}(U)$ be the set of holomorphic $\mathbb{C}$-valued functions on U. This too is a presheaf of rings if we take the ρ_{UV} to be the usual restriction maps.

These three examples all share a common important property, arising from the fact the in each case the group. $\mathcal{O}(U)$, $\mathcal{O}^\infty(U)$ $\mathcal{O}^{hol}(U)$ assigned to U is defined by local properties. This means that the groups of sections satisfy certain "patching" properties. Suppose $\{U_i\}_{i\in I}$ is an open covering of U and let F be one of $\mathcal{O}(U)$, $\mathcal{O}^\infty(U)$ or $\mathcal{O}^{hol}(U)$.

1. The morphism $F(U) \to \prod_{i\in I} F(U_i)$ induced by the restrictions $\rho_{U_i,U}$ is injective.
2. An element $(x_i) \in \prod_i F(U_i)$ lies in the image of $F(U) \to \prod_{i\in I} F(U_i)$ if and only if
$$x_i|U_i \cap U_j = x_j|U_i \cap U_j$$
for all i and $j \in I$.

Definition

A presheaf F on X is a *sheaf* if it satisfies the properties (1) and (2) above for all open $U \subseteq X$ and open coverings $\{U_i\}_{i\in I}$ of U.

The presheaves $\mathcal{O}_X$, $\mathcal{O}_X^\infty$ and $\mathcal{O}_X^{hol}$ defined above are all sheaves, since they are all defined by local conditions. For example you can tell if a real-valued function from an open subset of $\mathbb{R}^n$ is differentiable at a point x if you know its values in any open neighborhood of x, no matter how small. This shows that $\mathcal{O}_X^\infty$ satisfies (1), and (2) is clear: if I am given functions f_i on the U_i that agree on the overlaps $U_i \cap U_j$ then each f_i is the restriction to U_i of some function on U.

Here is a presheaf that is not a sheaf: take $X = \mathbb{R}$ and for open $U \subseteq X$ let $\mathcal{O}^b(U)$ be the group of real-valued *bounded* continuous functions on U. With the usual restriction maps $\mathcal{O}^b$ is a presheaf on $\mathbb{R}$. It is not a sheaf: let $U = \mathbb{R}$ and for $i \in \mathbb{N}$ let U_i be the open disk with center 0 and radius i. Clearly $\{U_i\}_{i\in\mathbb{N}}$ is an open covering of $\mathbb{R}$. Suppose now f is a continuous real-valued function on $\mathbb{R}$; since f is continuous its restriction f_i to each U_i is bounded, so $f_i \in \mathcal{O}^b(U_i)$. For all i and j, f_i and f_j have the same restriction to $U_i \cap U_j$. If there is a section of $\mathcal{O}^b(\mathbb{R})$ whose restriction to U_i is f_i, that section must necessarily be f, but a continuous function on $\mathbb{R}$ is not necessarily bounded. Thus condition (2) fails.

The problem with the last example is that boundedness is a global condition, not a local one. A function is not necessarily bounded if it is bounded on every element of some open covering.

The following lemma gives one way of saying that sections of sheaves are defined locally:

Lemma

Suppose F is a sheaf on X, $U \subseteq X$ and $s, t \in F(U)$. Then $s = t$ if and only if $s_x = t_x$ for all $x \in U$.

Proof: The "if" part is evident. Suppose conversely that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there is an open neighborhood $x \in U_x \subseteq U$ such that $s|U_x = t|U_x$. The set $\{U_x\}_{x \in U}$ is an open cover of U, so by condition (1) we must have $s = t$. ■

Remark: this lemma only used condition (1), so it holds for a more general class of presheaves. We say that a presheaf on X is *separated* if it satisfies condition (1). Many of the results to be proven later will be stated for sheaves but in fact are true for separated presheaves.

We can rephrase conditions (1) and (2) in a more categorical manner. If F is a presheaf on X and $\{U_i\}_{i\in I}$ is an open covering of an open subset $U \subseteq X$ there are two morphisms from $\prod_i F(U_i) \to \prod_{ij} F(U_i \cap U_j)$. The first, say f_1 is induced by the universal property of the product from the composition

$$\prod_i F(U_i) \xrightarrow{p_i} F(U_i) \to F(U_i \cap U_j)$$

where p_i is the projection and the second map is the restriction. The second, f_2 is the composition

$$\prod_i F(U_i) \xrightarrow{p_j} F(U_j) \to F(U_i \cap U_j).$$

The conjunction of (1) and (2) says that the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \prod_{ij} F(U_i \cap U_j)$$

is an equalizer diagram.

If F is an abelian sheaf, we can express this by saying that the sequence

$$0 \to F(U) \to \prod_i F(U_i) \xrightarrow{f_1 - f_2} \prod_{ij} F(U_i \cap U_j)$$

is exact.

We denote by $\mathbf{Shv}(X)$ the full subcategory of $\mathbf{PShv}(X)$ consisting of sheaves of sets. Likewise $\mathbf{Ab}(X)$ is the full subcategory $\mathbf{PAb}(X)$ consisisting of sheaves of abelian groups. There are evident forgetful functors $\mathbf{Shv}(X) \to \mathbf{PShv}(X)$ and $\mathbf{Ab}(X) \to \mathbf{PAb}(X)$. We will usually not use any notation for these forgetful functors.

That the condition defining a sheaf can be expressed in terms of a diagram involving limits has many interesting consequences. The main one, for the moment is that a limit of sheaves, computed in the presheaf category $\mathbf{PShv}(X)$ is actually a sheaf. Similarly for limits of abelian sheaves.

Theorem

Let X be a topological space. Arbitrary limits are representable in $\mathbf{Shv}(X)$*, and for any open* $U \subseteq X$*,*

$$(\varprojlim_i F_i)(U) = \varprojlim_i F(U_i).$$

In other words limits can be computed "value-by-value," like presheaves. We will see later that colimits $\mathbf{Shv}(X)$ are also representable, but the formula for them is more complicated.

Proof: The theorem says that a limit of sheaves in the presheaf category $\mathbf{PShv}(X)$ is in fact a sheaf. Suppose $\ell \mapsto F_\ell$ is a functor from some indexing category to $\mathbf{Shv}(X)$. Since each F_ℓ is a sheaf,

$$F_\ell(U) \longrightarrow \prod_i F_\ell(U_i) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \prod_{ij} F_\ell(U_i \cap U_j)$$

is an equalizer diagram for all ℓ.

We now recall the categorical principal that "inverse limits commute," and in particular that limits of equalizers is an equalizer of limits. Thus the limit of the preceding diagram

$$\varprojlim_\ell F_\ell(U) \longrightarrow \varprojlim_\ell \left(\prod_i F_\ell(U_i)\right) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \varprojlim_\ell \left(\prod_{ij} F_\ell(U_i \cap U_j)\right)$$

is another equalizer diagram. By the same token, the inverse limits commute with products, so

$$\varprojlim_\ell F_\ell(U) \longrightarrow \prod_i \varprojlim_\ell F_\ell(U_i) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \prod_{ij} \varprojlim_\ell F_\ell(U_i \cap U_j)$$

is yet another equalizer diagram. Appealing to our very first proposition, we write this as

$$(\varprojlim_\ell F_\ell)(U) \longrightarrow \prod_i (\varprojlim_\ell F_\ell)(U_i) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \prod_{ij} (\varprojlim_\ell F_\ell)(U_i \cap U_j)$$

which says that the presheaf limit $\varprojlim_\ell F_\ell$ is a sheaf.

It remains to check that the presheaf limit $\varprojlim_\ell F_\ell$ is also a limit in **Shv**(X), but this is immediate since **Shv**(X) is a full subcategory of **PShv**(X). ■

The same argument shows that arbitrary limits are representable in **Ab**(X), and coincide with their limits in **PAb**(X).

Corollary

A morphism $F \to G$ of sheaves is a monomorphism if and only if $F(U) \to G(U)$ is injective for all open $U \subseteq X$.

Proof: In any category with finite limits, $F \to G$ is a monomorphism if and only if the relative diagonal $F \to F \times_G F$ is an isomorphism.

In $\mathbf{Shv}(X)$, the relative diagonal is an isomorphism if and only if $F(U) \to (F \times_G F)(U)$ is a bijection for all U. By the theorem, this is equivalent to $F(U) \to F(U) \times_{G(U)} F(U)$ being bijective for all U, i.e. $F(U) \to G(U)$ being injective. ∎

To deal with colimits we will need some preliminaries about stalks. In what follows we fix a final object f in the category of sets, the singleton set $f = \{\emptyset\}$. If S is a set and $x \in X$ is a point there is a presheaf of sets $i_{x,S}$ on X defined by

$$i_{x,S}(U) = \begin{cases} S & x \in U \\ f & x \notin U. \end{cases}$$

It is easily checked that $i_{x,S}$ is in fact a sheaf, and we call $i_{x,S}$ the *skyscraper sheaf at x with value S*. It is easily seen to be functorial in S: a map $S \to T$ of sets yields a morphism $i_{x,S} \to i_{x,T}$ of sheaves.

Now if F is any sheaf of sets on X and $x \in X$ there is a canonical morphism of sheaves $\alpha_x : F \to i_{x,F_x}$ where as before F_x is the stalk of F at x. We must define maps $\alpha_{xU} : F(U) \to i_{x,F_x}(U)$ for all open $U \subseteq X$ compatible with the restriction maps. If $x \in U$, $i_{x,F_x}(U) = F_x$ and α_{xU} sends a section $s \in F(U)$ to its image in the stalk F_x. If $x \notin U$, $i_{x,F_x}(U) = f = \{\emptyset\}$ and we send s to $\emptyset \in f$. It is clear that this is compatible with the restriction maps, so this is a morphism of sheaves.

Theorem

Suppose $f : F \to G$ is a morphism in $\mathbf{Shv}(X)$, and for $x \in X$ let $f_x : F_x \to G_x$ be the induced morphism on stalks. Then f is a monomorphism (resp. epimorphism) if and only if f_x is a monomorphism (resp. epimorphism) for all $x \in X$.

Proof: We first consider the case of monomorphisms. By an earlier corollary it suffices to show that $f_U : F(U) \to G(U)$ is injective for all open $U \subseteq X$ if and only if f_x is injective for all $x \in X$.

Since the partially ordered set I_x of neighborhoods of x is filtered, the injectivity of $f_U : F(U) \to G(U)$ for all $U \in I_x$ implies that $\varinjlim_{U \in I_x} F(U) \to \varinjlim_{U \in I_x} G(U)$ is injective as well, i.e. $f_x : F_x \to G_x$ is injective.

Suppose conversely that $f_x : F_x \to G_x$ is injective for all $x \in X$ and suppose s, $t \in F(U)$ are such that $f_U(s) = f_U(t)$. From this it follows that for all $x \in U$, $f_x(s) = f_x(t)$ and since f_x is injective, s and t have the same image in F_x for all $x \in U$. This means that for all x there is a open neighborhood $U_x \subseteq U$ of x such that $s|U_x = t|U_x$ for all $x \in U$. Since the $\{U_x\}_{x \in U}$ is an open cover of U, condition (1) in the definition of a sheaf shows that $s = t$.

Suppose next that $f_x : F_x \to G_x$ is surjective for all $x \in X$ and that g, $h : G \to H$ are two morphisms in $\mathbf{Shv}(X)$ such that $gf = hf$; we want to show that $g = h$. The equality $gf = hf$ implies $(gh)_x = (hf)_x$, i.e. $g_x f_x = h_x f_x$ and since f_x is surjective, $g_x = h_x$ for all $x \in X$. To show that $g = h$ it suffices to show that for all open $U \subseteq X$, $g_U = h_U$ as maps $G(U) \to H(U)$.

Pick $s \in G(U)$; since $g_x = h_x$ there is for every $x \in U$ an open neighborhood $x \in U_x \subseteq U$ such that $g(s)|U_x = h(s)|U_x$, and the argument of the last paragraph shows that $g(s)|U = h(s)|U$. Therefore $g = h$, showing that f is an epimorphism.

Suppose finally that f_x is *not* surjective for some $x \in X$; I will show that f is not an epimorphism. Since f_x is not surjective, $G_x \setminus \operatorname{Im}(f_x)$ is nonempty. Let S be a two-element set, say $S = \{0, 1\}$ and let $H = i_{x,S}$ be the skyscraper sheaf with value S. There is a map $G_x \to S$ which sends $s \in G_x$ to 0 if $s \in \operatorname{Im}(f_x)$ and 1 otherwise. This yields a morphism $g_x : i_{x,G_x} \to i_{x,S}$ and we define $g : G \to i_{x,S}$ to be the composition

$$G \xrightarrow{\alpha_x} i_{x,G_x} \xrightarrow{g_x} i_{x,S}.$$

Similarly the constant map $G_x \to S$ sending every element of G_x to $0 \in S$ defines a morphism of sheaves $h_x : i_{x,G_x} \to i_{x,S}$, and we define $h : G \to i_{x,S}$ to be the composition

$$G \xrightarrow{\alpha_x} i_{x,G_x} \xrightarrow{h_x} i_{x,S}.$$

I will show that $gf = hf$ and $g \neq h$, which will show that f is not an epimorphism.

Suppose U is a neighborhood of x and $s \in G(U)$ is a section such that the image of s in G_x is not contained in $\mathrm{Im}(f_x)$. Then $g_x(s) = 1$ and $h_x(s) = 0$, which shows that $g \neq h$. On the other hand if $t \in F(U)$ then $(gf)_x(t) = g_x f_x(t) = 0$ and $(hf)_x(t) = h_x f_x(t) = 0$ by construction. From this it follows that $gf = hf$, as asserted. ∎

Theorem

If $f : F \to G$ is a morphism in $\mathbf{Shv}(X)$ that is both a monomorphism and an epimorphism, it is an isomorphism.

Proof: The last theorem shows that if f is both a monomorphism and an isomorphism then $f_x : F_x \to G_x$ is a bijection for all $x \in X$. Furthermore $f_U : F(U) \to G(U)$ is injective for all open $U \subseteq X$ since f is a monomorphism, so we must show it is surjective as well. Pick $t \in G(U)$ and for $x \in X$ let $x \in U_x \subseteq U$ be an open neighborhood for which there is a section $s(x) \in F(U_x)$ such that $f_x(s(x)_x) = t_x$. Since $t|U_x$ and $f_{U_x}(s(x))$ both have the same image $t_x \in G_x$, we arrange to have $f_{U_x}(s(x)) = t|U_x$ by shrinking U_x if necessary.

Suppose now x and y are any two points of U; then $f_{U_x}(s(x)) = t|U_x$ and $f_{U_y}(s(y)) = t|U_y$ imply that

$$\begin{aligned} f_{U_x \cap U_y}(s(x)|U_x \cap U_y) &= f_{U_x}(s(x))|U_x \cap U_y \\ &= f_{U_x}(s(y))|U_x \cap U_y \\ &= f_{U_x \cap U_y}(s(y)|U_x \cap U_y) \end{aligned}$$

and since $f_{U_x \cap U_y}$ is injective, $s(x)|U_x \cap U_y = s(y)|U_x \cap U_y$ for all x and y. Since the U_x for all $x \in U$ form an open cover of U, there is a section $s \in F(U)$ such that $s|U_x = s(x)$ (this is the first time we have used sheaf property (2)). We show, finally that $f_U(s) = t$: for this it suffices by sheaf property (1) that $f_U(s)|U_x = t|U_x$ for all x, but this is clear since

$$f_U(s)|U_x = f_U(s|U_x) = f_U(s(x)) = t|U_x.$$

■

In what follows I will denote the forgetful functor $\mathbf{Shv}(X) \to \mathbf{PShv}(X)$ by $F \mapsto F^-$ (this is not standard notation). The next step is to construct a left adjoint to this functor. If F is a presheaf of sets on X we denote by F^+ the following presheaf: for open $U \subseteq X$ $F^+(U)$ is the subset of $\prod_{x \in U} F_x$ such that

- $(s_x)_{x \in U} \in F^+(U)$ if and only if for all $x \in U$ there is a neighborhood $x \in U_x \subseteq U$ and a section $s(x) \in F(U_x)$ such that $s(x)_y = s_y$ for all $y \in U_x$.

In other words, a section of F^+ over U is a tuple of elements of the stalks F_x for all $x \in U$ that "locally come from the section of F."

It is "evident" that F^+ is actually a sheaf, so we will view it as an object of $\mathbf{Shv}(X)$. Furthermore there is a morphism of presheaves $i_F : F \to (F^+)^-$ which to $s \in F(U)$ assigns the tuple $(s_x)_{x \in U}$ such that s_x is the stalk of s at x. Observe finally that for any $x \in X$, i_F induces an isomorphism $F_x \xrightarrow{\sim} (F^+)_x$. We call the pair (F^+, i_F) the *sheafification* of F.

Theorem

For any sheaf G on X the morphism $i_F : F \to (F^+)^-$ induces a bijection

$$\mathrm{Hom}_{\mathbf{Shv}(X)}(F^+, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{PShv}(X)}(F, G^-)$$

functorially in G.

The theorem implies that $F \mapsto F^+$ defines a functor of F, so that $F \mapsto F^+$ is the left adjoint of $G \mapsto G^-$.

Proof: The map itself comes about as follows: given $f : F^+ \to G$, apply the forgetful functor to obtain $f^- : (F^+)^- \to G^-$, and compose this with $i_F : F \to (F^+)^-$. Suppose conversely that we are given a morphism of presheaves $g : F \to G^-$. For each $x \in X$ this induces maps $g_x : F_x \to (G^-)_x = G_x$ and thus for all $U \subseteq X$ a map $\prod_{x \in U} F_x \to \prod_{x \in U} G_x$. Suppose now $U \subseteq X$ and $s = (s_x)_{x \in U} \subseteq \prod_{x \in U}$ lies in $F^+(U)$. For every $x \in U$ there is a neighborhood $x \in U_x \subseteq U$ and $s(x) \in F(U_x)$ such that $s(x)_y = s_y$ for all $y \in U_x$.

Then $g_{U_x}(s(x)) \in G(U_x)$ for all x. Suppose now $y \in U$ is another point and $s(y) \in F(U_y)$ is such that $s(y)_z = s_z$ for all $z \in U_y$. If $z \in U_x \cap U_y$ then

$$g_{U_x}(s(x))_z = g_z(s(x)_z) = g_z(s_z) = g_z(s(y)_z) = g_{U_y}(s(y))_z$$

and since G is a sheaf, an earlier lemma shows that

$$g_{U_x}(s(x))|U_x \cap U_y = g_{U_y}(s(y))|U_x \cap U_y.$$

Again since G is a sheaf there is a unique section $g(s) \in G(U)$ such that $g_{U_x}(s(x)) = g(s)|U_x$ for all $x \in U$. The construction shows that the the formation of $g(s)$ is compatible with the restriction maps, so $s \mapsto g(s)$ defines a morphism of sheaves $g : F^+ \to G$. This defines a map $\mathrm{Hom}_{\mathbf{PShv}(X)}(F, G^-) \to \mathrm{Hom}_{\mathbf{Shv}(X)}(F^+, G)$ and I will leave it to you to check that it is inverse to the previous one. ∎