Homological Algebra
Lecture 11

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Let X be a topological space. In the last lecture all results were stated for the category $\mathbf{Shv}(X)$ of sheaves of sets on X, but in fact they all hold, often with no change of statement or proof for the category $\mathbf{Ab}(X)$ of abelian sheaves on X. We now have all the material we need to shows that $\mathbf{Ab}(X)$ is an abelian category. We have shown:

- limits and colimits in $\mathbf{PAb}(X)$ are representable and can be computed object-wise in the sense that

$$(\varprojlim_i F_i)(U) \simeq \varprojlim_i F_i(U)$$

$$(\varinjlim_i F_i)(U) \simeq \varinjlim_i F_i(U)$$

  for all open $U \subseteq X$
- limits in $\mathbf{Ab}(X)$ are representable and coincide with the limits taken in the category of presheaves. In particular the above formula for sections of a limit applies to sheaves.
- A morphism $f : F \to G$ is a monomorphism (resp. epimorphism, isomorphism) if and only if for all $x \in X$ the map on stalks $f_x : F_x \to G_x$ is a monomorphism (resp. epimorphism, isomorphism).

Finally we have shown the the forgetful functor

$$\mathbf{Ab}(X) \to \mathbf{PAb}(X) \qquad G \mapsto G^-$$

has a left adjoint

$$\mathbf{PAb}(X) \to \mathbf{Ab}(X) \qquad F \mapsto F^+.$$

The adjunction is realized by a morphism of presheaves $i_F : F \to (F^+)^-$. Given a morphism $f : F^+ \to G$ of sheaves, $f^- \circ i_F$ is a morphism $F \to G^-$ and $f \mapsto f^- \circ i_F$ is a bijection

$$\mathrm{Hom}_{\mathbf{Ab}(X)}(F^+, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{PAb}(X)}(F, G^-).$$

We can now show that colimits are representable in $\mathbf{Ab}(X)$.

Suppose we are given a functor $i \mapsto F_i$ from some indexing category to $\mathbf{Ab}(X)$. If G is any object of $\mathbf{Ab}(X)$ there are bijections

$$\begin{aligned}\varprojlim_i \mathrm{Hom}_{\mathbf{Ab}(X)}(F_i, G) &\xrightarrow{\sim} \varprojlim_i \mathrm{Hom}_{\mathbf{PAb}(X)}(F_i^-, G^-)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{PAb}(X)}(\varinjlim_i F_i^-, G^-)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Ab}(X)}((\varinjlim_i F_i^-)^+, G)\end{aligned}$$

where the first bijection is valid because $\mathbf{Ab}(X)$ is by definition a full subcategory of $\mathbf{PAb}(X)$, the second is the universal property of a colimit of presheaves and the third is the adjunction. These bijections are functorial in G, so we conclude that the colimit of $i \mapsto F_i$ is representable, namely by the sheafification of the presheaf colimit $\varinjlim_i F_i^-$. The same argument shows that colimits are representable in the sheaf category $\mathbf{Shv}(X)$.

Theorem

For any topological space X the category $\mathbf{Ab}(X)$ of abelian sheaves on X is abelian.

Proof: The argument that $\mathbf{Ab}(X)$ is additive is the same as the one given for $\mathbf{PShv}(X)$ (and applies equally well to $\mathbf{PAb}(X)$). To show that $\mathbf{Ab}(X)$ is preabelian we have to show that kernels and cokernels are representable in $\mathbf{Ab}(X)$. In fact these are special cases respectively of limits and colimits, so they are indeed representable. Finally we have shown that a morphism in $\mathbf{Ab}(X)$ is an isomorphism if it is a monomorphism and an epimorphism, so a theorem proven earlier shows that $\mathbf{Ab}(X)$ is abelian. ∎

The cokernel of a morphism $f : G \to F$ in $\mathbf{Ab}(X)$ has the following description. The presheaf cokernel is just $U \mapsto F(U)/G(U)$, and the cokernel of f in $\mathbf{Ab}(X)$ is the sheafification of this presheaf. This means that a section of $\mathrm{Coker}(f)$ is specified by giving sections $s_i \in F(U_i)$ on some open cover $\{U_i\}$ of U such that for all i and j,

$$(s_i|U_i \cap U_j) - (s_j|U_i \cap U_j) \in G(U_i \cap U_j).$$

Furthermore two systems $(s_i \in F(U_i))$ and $(t_j \in F(V_j))$ yield the same section of $\mathrm{Coker}(f)$ if there is a common refinement $\{W_\ell\}$ of $\{U_i\}$ and $\{V_j\}$ such that $s_i|W_\ell - t_j|W_\ell \in G(W_\ell)$ for all i, j, ℓ such that $W_\ell \subseteq U_i \cap V_j$.

Here is a more concrete example. Suppose X is an open subset of $\mathbb{C}$, and let $\mathcal{O}_X$ be the presheaf on X such that for all open $U \subseteq X$, $\mathcal{O}_X(U)$ is the abelian group (under addition) of continuous $\mathbb{C}$-valued functions on U. The restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for $V \subseteq U$ are the usual restriction of functions. Next, define $\mathcal{O}_X^\times$ to be the subsheaf of $\mathcal{O}_X$ such that $\mathcal{O}_X^\times(U)$ is the set of nowhere vanishing sections of $\mathcal{O}_X^\times$. This is an abelian sheaf if we take the group law to be *multiplication* of sections. There is a morphism of abelian sheaves $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ given by

$$f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^\times(U).$$

The kernel of exp is easy to describe: its sections over $U \subseteq X$ consist of locally constant $\mathbb{C}$-valued functions on U (i.e. constant on every connected component) taking their values in $2\pi i\mathbb{Z}$.

Furthermore the morphism $\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ is an *epimorphism*, i.e. the cokernel is zero. In fact if $x \in X$ and $x \in U \subseteq X$ is a *simply connected* neighborhood, then for any $g \in \mathcal{O}_X^\times(U)$ there is a section $f \in \mathcal{O}_X(U)$ such that $\exp(f) = g$. This amounts to the assertion that a continuous nonvanishing function on a simply connected open set U has a logarithm (not unique, but unique up to an element of $2\pi i\mathbb{Z}$).

To see what goes wrong if U is not simply connected, take $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$. If z is the affine parameter on X then $z \in \mathcal{O}_X^\times(U)$, but there is no continuous function $\ell(z)$ on U such that $\exp(\ell(z)) = z$. If $U \subseteq X$ is a simply connected open subset not containing 0 we can define a single-valued continuous function $\arg(z)$ on U and then set $\ell(z) = \log|z| + i\arg(z)$ (as usual) but we cannot do this on $U = \mathbb{C} \setminus \{0\}$.

Summary: we have shown there is an exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

of abelian sheaves on $\mathbb{C}$. Here $2\pi i\mathbb{Z}$ is the "constant sheaf with value $2\pi i\mathbb{Z}$," i.e. its sections on a *connected* open set $U \subseteq \mathbb{C}$ is the group $2\pi i\mathbb{Z}$.

Suppose now X is a topological space. For any $U \subseteq X$ and abelian sheaf F on X the assignment $U \mapsto F(U)$ defines a functor $\mathbf{Ab}(X) \to \mathbf{Ab}$. Tradition demands that this be denoted

$$\Gamma(U, F) = F(U)$$

and as before $\Gamma(U, F)$ is called the group of *sections of F over U*. The reason is that in the early days of sheaf theory, a sheaf F was construed as a kind of topological space together with a local homeomorphism $\pi : F \to X$; the sections $F(U)$ were literally sections of the restricted map $\pi^{-1}(U) \to U$. Presheaves however were defined in the same way as now, and it was Grothendieck who first defined sheaves as a particular kind of presheaf.

We have seen that if

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathbf{Ab}(X)$, the sequence

$$0 \to \Gamma(X, A) \to \Gamma(X, B) \to \Gamma(X, A)$$

is exact (and this would be true with X replaced by any open $U \subseteq X$) but we cannot in general complete this to a short exact sequence: $\Gamma(X, B) \to \Gamma(X, A)$ is not usually surjective. In other words the functor $F \mapsto \Gamma(X, F)$ of "global sections" is left exact but not necessarily exact.

We would like to define the right derived functors of $\Gamma(X, \_)$ but for this we need to show that $\mathbf{Ab}(X)$ has enough injectives. NB: in general $\mathbf{Ab}(X)$ does *not* have enough projectives.

Theorem

For any topological space X the abelian category $\mathbf{Ab}(X)$ has enough injectives.

Proof: One can use Grothendieck's criterion by showing that $\mathbf{Ab}(X)$ has a generator an satisfies axiom **AB5**, but it is easier to do this explicitly. Suppose F is an abelian sheaf; since $\mathbf{Ab}$ has enough injectives we may choose for all $x \in X$ a monomorphism $F_x \to I_x$ for some injective (i.e. divisible) abelian group I_x. We now set

$$\mathcal{I} = \prod_{x \in X} i_x(I_x)$$

where as before $i_x(I_x)$ is the skyscraper sheaf on X at x defined by the abelian group I_x. I will show that $\mathcal{I}$ is an injective abelian sheaf and produce a monomorphism $F \to \mathcal{I}$. We first observe that for any abelian sheaf G there are isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Ab}(X)}(G,\mathcal{I}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Ab}(X)}(G, \prod_{x\in X} i_x(I_x)) \\ &\xrightarrow{\sim} \prod_{x\in X} \mathrm{Hom}_{\mathbf{Ab}(X)}(G, i_x(I_x)) \\ &\xrightarrow{\sim} \prod_{x\in X} \mathrm{Hom}_{\mathbf{Ab}}(G_x, I_x)\end{aligned}$$

where the second isomorphism is the universal property of the product and the third follows from the isomorphisms $\mathrm{Hom}_{\mathbf{Ab}(X)}(G, i_x(A)) \simeq \mathrm{Hom}_{\mathbf{Ab}}(G_x, A)$ which hold for any G and any abelian group A.

In particular the given monomorphisms $F_x \to I_x$ correspond under the above to a morphism $F \to \mathcal{I}$. The stalk of this at a point $x \in X$ is $F_x \to \mathcal{I}_x$, and one checks easily that $\mathcal{I}_x \simeq I_x$. Then $F_x \to \mathcal{I}_x$ is injective for all x, so $F \to \mathcal{I}$ is a monomorphism.

It remains to show that $\mathcal{I}$ is injective, i.e. for any epimorphism $G \to H$ in $\mathbf{Ab}(X)$ the map $\mathrm{Hom}_{\mathbf{Ab}(X)}(G, \mathcal{I}) \to \mathrm{Hom}_{\mathbf{Ab}(X)}(H, \mathcal{I})$ is surjective. By the above computation this map is the same as

$$\prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(G_x, I_x) \to \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(H_x, I_x)$$

which is surjective: since $G \to H$ is an epimorphism the maps $G_x \to H_x$ are surjective, whence $\mathrm{Hom}_{\mathbf{Ab}}(G_x, I_x) \to \mathrm{Hom}_{\mathbf{Ab}}(H_x, I_x)$ is surjective since I_x is injective, and finally a product of surjective maps is surjective (serious use of the axiom of choice on this last step). ■

We can now define sheaf cohomology as the right derived functors of the global section functor:

$$H^n(X, F) = R^n\Gamma(X, F).$$

Thus if

$$0 \to F \to G \to H \to 0$$

is a short exact sequence of abelian sheaves there is a long exact sequence

$$\cdots \to H^i(X,F) \to H^i(X,G) \to H^i(X,H) \to H^{i+1}(X,F) \to \cdots$$

for all i.

If X is a "reasonable" topological space (a paracompact manifold, say) and $\mathbb{Z}$ denotes the "constant sheaf $\mathbb{Z}$ on X (its sections over a a connected open $U \subseteq X$ is $\mathbb{Z}$) then $H^i(X,\mathbb{Z})$ is isomorphic to the ith singular cohomology of X. Recall now our earlier short exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

of sheaves on $X = \mathbb{C} \setminus \{0\}$. One can show that $H^i(X,\mathcal{O}_X) = 0$ for all $i > 0$. The long exact sequence associated to this breaks up into an exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathcal{O}_X^\times) \to H^1(X,2\pi i\mathbb{Z}) \to 0$$

and isomorphisms

$$H^i(X, \mathcal{O}_X^\times) \xrightarrow{\sim} H^{i+1}(X, 2\pi i\mathbb{Z})$$

for all $i > 0$. In particular we recover our earlier observation that $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X^\times)$ is not surjective, and in fact now we have and idea of by how much it fails to be surjective. Since $X = \mathbb{C} \setminus \{0\}$ has the homotopy type of the circle S^1, $H^1(X, 2\pi i\mathbb{Z}) \xrightarrow{\sim} 2\pi i\mathbb{Z}$ is an infinite cyclic group. We also find that $H^i(X, \mathcal{O}_X^\times) = 0$ for all $i > 0$.

There is a generalization of the global sections functor that is important in algebraic and analytic geometry. Suppose $f : Y \to X$ is a continuous map of topological spaces and F is a sheaf on Y. We get a presheaf on X by sending

$$U \mapsto F(f^{-1}(U))$$

for any open $U \subseteq X$ (this makes sense since $f^{-1}(U) \subseteq Y$ is open). Since inverse images are compatible in an obvious sense with intersections this construction in face defines a sheaf on X, which we denote by f_*F. It is called the *direct image* of F by the map $f : Y \to X$.

The same argument we used before shows that f_* is left exact. Since $\mathbf{Ab}(Y)$ has enough injectives, f_* has right derived functors which we simply denote by $R^n f_*$; they are called the *higher direct images* of f. Thus for any short exact sequence

$$0 \to F \to G \to H \to 0$$

we get a long exact sequence

$$\cdots \to R^n f_*(F) \to R^n f_*(G) \to R^n f_*(H) \to R^{n+1} f_*(F) \to \cdots$$

of higher direct images.

There are other left exact functors whose derived functors are important. We have already seen the Ext functors which are defined in any abelian category with enough injective. Particular to the case of sheaf categories are the so-called *local* Ext functors, which are defined as follows.

Suppose X is a topological space and $U \subseteq X$ is an open set. Any open subset of U in the induced topology is automatically an open subset of X, so a sheaf F on X has a natural "restriction" $F|U$ to a sheaf on U: for any open $V \subseteq U$ we define $(F|U)(V) = F(V)$ (this is a very particular case of the *inverse image* functor which we will not make use of). If F and G are abelian sheaves on X we can define a presheaf on X by

$$U \mapsto \mathrm{Hom}_{\mathbf{Ab}(U)}(F|U, G|U).$$

This assignment is clearly compatible with the restrictions, so this does indeed define a presheaf. Slightly less obvious is the fact that this presheaf is in fact a sheaf. Suppose the $\{U_i\}$ is an open covering of an open set $U \subseteq X$ and $f_i : F|U_i \to G|U_i$ are morphisms of sheaves such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i and j. We must show that the f_i are the restrictions of a unique $f : F \to G$. To this end we let $V \subseteq X$ be any open subset and observe that $\{V \cap U_i\}$ is an open covering of V. We then have a commutative diagram of solid arrows

$$
\begin{array}{ccccccc}
0 \longrightarrow & F(V) & \longrightarrow & \prod_i F(V \cap U_i) & \longrightarrow & \prod_{ij} F((V \cap U_i) \cap (V \cap U_j)) \\
& \vdots & & \Big\downarrow {\scriptstyle \prod_i f_i|V\cap U_i} & & \Big\downarrow {\scriptstyle \text{etc.}} \\
0 \longrightarrow & G(V) & \longrightarrow & \prod_i G(V \cap U_i) & \longrightarrow & \prod_{ij} G((V \cap U_i) \cap (V \cap U_j))
\end{array}
$$

with exact rows (expressing that F and G are sheaves. A quick diagram chase shows that the dotted arrow can be filled in uniquely so as to make a commutative diagram. It is easily checked (by means of a 3-dimensional diagram) that the morphisms $F(V) \to G(V)$ just constructed are compatible with restrictions, so they define a morphism $f : F \to G$ of sheaves.

We denote by $Hom_X(F, G)$ the sheaf defined by

$$Hom_X(F, G)(U) = \mathrm{Hom}_{\mathbf{Ab}(U)}(F|U, G|U).$$

It is called simply the "sheaf Hom" of F and G.

Since the (global) Hom is left exact in its second argument it follows easily that the sheaf Hom is left exact in its second argument. It therefore has derived functors which are denoted by $Ext^n_X(F, G)$. As usual they are computed by choosing an injective resolution $G \to I^\cdot$, and then

$$Ext^n_X(F, G) = H^n(Hom_X(F, I^\cdot)).$$

We can now raise the following question. Some of the left exact functors we have been discussing are compositions of other left exact functors of our acquaintance. For example if $f : Y \to X$ is a continuous map of topological spaces, our definition of the direct image f_* shows that

$$\Gamma(X, f_*F) \simeq \Gamma(Y, F)$$

for any abelian sheaf F on Y. Similarly if F and G are abelian sheaves on X, the definition of the sheaf Hom showed that

$$\Gamma(X, Hom_X(F, G)) \simeq \mathrm{Hom}_{\mathbf{Ab}(X)}(F, G).$$

We then ask: in these cases can the derived functors of the target (the $H^i(Y, F)$ in the first case, and the $\mathrm{Ext}^i(F, G)$ in the second) be somehow computed in terms of the derived functors appearing on the left? In other words, how are the $H^i(Y, F)$ related to the $H^i(X, R^j f_* F)$? How are the $\mathrm{Ext}^i(F, G)$ related to the $H^i(X, Ext^j_X(F, G))$?

More generally, if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are functors between abelian categories, and if $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, how are the derived functors $R^n(G \circ F)$ of the composite related to the objects $R^i F \circ R^j G$? Is $R^n(G \circ F)$ isomorphic to $R^n F \circ R^n G$? Evidently this is true for $n = 0$; it's almost always false for $n > 0$.

The answer is that there is a relation and it's not a simple one. You can't really compute $R^n(G \circ F)$ from the $R^i F \circ R^j G$, but in favorable cases you can at least get a handle on them. The method for doing this is the theory of spectral sequences. This is also one way of showing that the Ext groups in a module category (or in any abelian category with enough projectives and injectives) can be computed using either projective or injective resolutions, so I will use this to motivate the theory.

Suppose $\mathcal{A}$ is an abelian category and A, B are objects of $\mathcal{A}$. If $P_\cdot \to A$ is a projective resolution and $B \to I^\cdot$ is an injective resolution we have seen that we can compute the Ext groups $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ either as $H^n(\mathrm{Hom}(A, I^\cdot))$ or as $H^n(\mathrm{Hom}(P_\cdot, B))$. A natural idea for showing that these are isomorphic is to compare them with a third object that uses both resolutions at the same time. This leads us to contemplate the diagram

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d_2'' & & \uparrow d_2'' & & \uparrow d_2'' & & \\
\mathrm{Hom}(P_0, I^2) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^2) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^2) & \xrightarrow{d_2'} & \cdots \\
\uparrow d_1'' & & \uparrow d_1'' & & \uparrow d_1'' & & \\
\mathrm{Hom}(P_0, I^1) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^1) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^1) & \xrightarrow{d_2'} & \cdots \\
\uparrow d_0'' & & \uparrow d_0'' & & \uparrow d_0'' & & \\
\mathrm{Hom}(P_0, I^0) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^0) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^0) & \xrightarrow{d_2'} & \cdots
\end{array}
$$

and in particular, to wonder how it might compute the Ext groups.

To do this the obvious idea is to turn it into some kind of complex, possibly by taking the "antidiagonal sums" $E^n = \bigoplus_{i+j} \mathrm{Hom}(P_i, I^j)$ as the nth degree component. The question is then how to find a differential $d : E^n \to E^{n+1}$. To do this it suffices to find for all $i + j = n$ a morphism $\mathrm{Hom}(P_i, I^j) \to E^{n+1}$. Now we have morphisms

$$d'_i : \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_{i+1}, I^j) \to E^{n+1}$$
$$d''_j : \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_i, I^{j+1}) \to E^{n+1}$$

which it might make sense to add to get $\mathrm{Hom}(P_i, I^j) \to E^{n+1}$, and then take the direct sum of all of these to get $d : E^n \to E^{n+1}$. The problem is that we don't get $d^2 = 0$. What happens is that the component $\mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_{i+1}, I^{j+1})$ induced by d^2 is $d'_i d''_j + d''_j d'_i \neq 0$ (the other two are $d'_i d'_{i+1} : \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_{i+2}, I^j)$ and $d''_j d''_{j+1} : \mathrm{Hom}(P_i, I^j) \to \mathrm{Hom}(P_i, I^{j+2})$, which are 0).

The solution is clear when we recall that $d_i' d_j'' = d_j'' d_i'$ in the diagram, so that $d_i' d_j'' - d_j'' d_i' = 0$. We must therefore *alternate* signs in the definition of d. The puzzling thing is that there are many natural ways to do this. One popular choice is

$$d = d_i' + (-1)^n d_j'' : \mathrm{Hom}(P_i, I^j) \to E^{n+1}$$

for $i + j = n$ but this is not the only one seen in the literature. This amounts to using the naive definition on the complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d_2'' \uparrow & & -d_2'' \uparrow & & d_2'' \uparrow & & \\
\mathrm{Hom}(P_0, I^2) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^2) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^2) & \xrightarrow{d_2'} & \cdots \\
-d_1'' \uparrow & & d_1'' \uparrow & & -d_1'' \uparrow & & \\
\mathrm{Hom}(P_0, I^1) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^1) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^1) & \xrightarrow{d_2'} & \cdots \\
d_0'' \uparrow & & -d_0'' \uparrow & & d_0'' \uparrow & & \\
\mathrm{Hom}(P_0, I^0) & \xrightarrow{d_0'} & \mathrm{Hom}(P_1, I^0) & \xrightarrow{d_1'} & \mathrm{Hom}(P_2, I^0) & \xrightarrow{d_2'} & \cdots
\end{array}$$

and with this description it's easy to see that $d^2 = 0$. Alternatively we could just compute

$$\begin{aligned} d^2 &= (d_i' + (-1)^n d_j'')(d_i' + (-1)^{n+1} d_j'') \\ &= d_i' d_i' + (-1)^n d_j'' d_i' + d_i'(-1)^{n+1} d_j'' - d_j'' d_j'' \\ &= 0 + (-1)^n (d_j'' d_i' - d_i' d_j'') + 0 = 0. \end{aligned}$$

This kind of business arises quite frequently so we make some general definitions. A *double complex* or *bicomplex* in $\mathcal{A}$ is a $\mathbb{Z}^2$-graded object $E^{\cdot,\cdot}$ equipped with two commuting morphisms

$$d' : E \to E[1,0], \qquad d'' : E \to E[0,1]$$

such that $d'd' = d''d'' = 0$. Here $E[i,j]$ is the $\mathbb{Z}^2$-graded object whose component in degree (m,n) is $E^{m+i,n+j}$. The *associated simple complex* $\mathrm{Tot}(E)$ is the cochain complex whose degree n component and differential are

$$\mathrm{Tot}(E)^n = \bigoplus_{i+j=n} E^{i,j}, \qquad d = d' + (-1)^n d'' : E^n \to E^{n+1}.$$

If $E^{\cdot\cdot}$ is a double complex with differentials d', d'' we get two series of cochain complexes by fixing the first or second index:

$$d' : E^{\cdot j} \to E^{\cdot j}[1], \qquad d'' : E^{i\cdot} \to E^{i\cdot}[1]$$

This in turn give rise to cohomology objects which we denote by

$${}'H^n(E^{\cdot j}), \qquad {}''H^n(E^{j\cdot}).$$

Then the differentials d', d'' induce morphisms

$$d'' : {}'H^n(E^{\cdot j}) \to {}'H^n(E^{\cdot j+1}), \qquad d' : {}''H^n(E^{j\cdot}) \to H^n(E^{j+1\cdot})$$

which since $(d'')^2 = (d')^2 = 0$ make the ${}'H^n(E^{\cdot j})$ and ${}''H^n(E^{j\cdot})$ into complexes. We thus arrive at $\mathbb{Z}^2$-graded objects

$${}''H^p({}'H^q(E^{\cdot\cdot})), \qquad {}'H^p({}''H^q(E^{\cdot\cdot}))$$

by taking cohomology.

The goal is to get as much information as we can about $H^n(\mathrm{Tot}(E^{\cdot\cdot}))$ using the $''H^p('H^q(E^{\cdot\cdot}))$ and $'H^p(''H^q(E^{\cdot\cdot}))$. But this is best conceptualized by generalizing the situation even further, as we shall see next time.