Homological Algebra
Lecture 12

Richard Crew

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Filtered complexes

Earlier we defined the notion of a filtered module – essentially a module M over a ring A, say, and a sequence of submodules $F^pM \subseteq M$ such that $F^{p+1} \subseteq F^p$. Typically we let p run through the set $\mathbb{N}$ or $\mathbb{Z}$. In the former case it is typical to assume $M = F^0M$.

More generally if $\mathcal{A}$ is an abelian category a *filtered object* of A is a an object M of $\mathcal{A}$ and a sequence of monomorphisms

$$\cdots \to F^{p+1}M \to F^pM \to \cdots \to M.$$

The pth *graded object* is

$$\mathrm{gr}^p F^{\cdot} M = \mathrm{Coker}(F^{p+1}M \to F^pM)$$

(the word "the" is a little misleading). We denote filtered objects by (M, F^pM).

If (M, F^pM) and (M', F^pM') are filtered objects of $\mathcal{A}$ a *morphism* $(M, F^pM) \to (M', F^pM')$ is a morphism $M \to M'$ such that $F^pM \to M \to M'$ factors (necessarily uniquely) through F^pM'. It is evident that this makes the filtered objects of $\mathcal{A}$ into a category. It is a preabelian category but not abelian, as we saw when $\mathcal{A}$ is a module category. In particular morphisms have kernels, cokernels, images and coimages. A morphism f is *strict* if the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Again if $\mathcal{A}$ is an abelian category, a *filtered complex* in $\mathcal{A}$ is just a filtered object of the abelian category of cochain complexes (sometimes, chain complexes). Two standard examples are particularly important. Suppose $K^\cdot$ is a complex in $\mathcal{A}$.

- The complex $\sigma_{\geq p}K^\cdot$ defined by

$$\sigma_{\geq p}K^n = \begin{cases} 0 & n < p \\ K^n & n \geq p. \end{cases}$$

  is a subobject of $K^\cdot$. The filtration $\sigma_{\geq p}K^\cdot$ is called the *dumb filtration*. Here $\mathrm{gr}^pK^\cdot$ has K^p in degree p and 0 elsewhere.

- The *canonical filtration* $\tau_{\leq p} K^{\cdot}$ is defined by

$$\tau_{\leq p} K^n = \begin{cases} K^p & n < -p \\ \mathrm{Ker}(d^{-p}) & n = -p \\ 0 & n > -p. \end{cases}$$

In this case $\mathrm{gr}^p K^{\cdot}$ is the bottom row of

$$\begin{array}{ccccccccc}
K^{-p-2} & \longrightarrow & \mathrm{Ker}(d^{-p-1}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{-p-2} & \longrightarrow & K^{-p-1} & \longrightarrow & \mathrm{Ker}(d^{-p}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{-p-1}/\mathrm{Ker}(d^{-p-1}) & \longrightarrow & \mathrm{Ker}(d^{-p}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Note that $H^{-p}(\tau_{\leq p} K^{\cdot}) \simeq H^{-p}(K^{\cdot})$ and $H^n(\tau_{\leq p} K^{\cdot}) = 0$ for $n \neq -p$.

In the last lecture we saw that the problem of comparing the two definitions of Ext gave rise to a double complex and its associated single complex. In fact the single complex associated to a double complex has *two* natural filtrations. Let (K^{ij}, d_1^i, d_2^j) be a double complex, and recall that the associated single complex has as its degree n term the direct sum

$$\mathrm{Tot}(K^{\cdot,\cdot})^n = \bigoplus_{i+j=n} K^{i,j}$$

and the differential is the sum of the morphisms

$$d = d_1^i + (-1)^i d_2^j : K^{i,j} \to K^{i+1,j} \oplus K^{i,j+1}.$$

The two filtrations of $\mathrm{Tot}(K^{\cdot,\cdot})^{\cdot}$ are

$$F_I^p \,\mathrm{Tot}(K^{\cdot,\cdot})^{\cdot} = \bigoplus_{i+j=n,\ j\geq p} K^{i,j}, \qquad F_{II}^p \,\mathrm{Tot}(K^{\cdot,\cdot})^{\cdot} = \bigoplus_{i+j=n,\ i\geq p} K^{i,j}.$$

with the induced differentials.

If you like, these two filtrations the total complexes associated to the two possible "dumb filtrations" of the double complex. Note that

$$\mathrm{gr}_I^p \operatorname{Tot}(K^{\cdot,\cdot})^{\cdot} \simeq K^{\cdot,p}[-p]$$

where we have modified the shift operation $[-p]$ by replacing the differential d_2 by $(-1)^p d_2$. There is a similar description of $\mathrm{gr}_{II} \operatorname{Tot}(K^{\cdot,\cdot})^{\cdot}$ but this time there is no sign change.

Returning to the subject of filtered complexes in general, we now ask the following question: suppose $F^p K^{\cdot}$ is a filtered complex; how are the $H^n(\mathrm{gr}^p K^{\cdot})$ related to $H^n(K^{\cdot})$? It's obviously too much to expect that $H^n(K^{\cdot})$ can be computed from the $H^n(\mathrm{gr}^p K^{\cdot})$, but it's reasonable to expect that they give some information. The answer, roughly is that $H^n(K^{\cdot})$ itself has a natural filtration whose graded pieces in favorable situations can be computed from the $H^n(\mathrm{gr}^p K^{\cdot})$. The mechanism behind this is the theory of *spectral sequences.*

A spectral sequence in an abelian category $\mathcal{A}$ is the following massive package of data:

- For all p, $q \in \mathbb{Z}$ and $r \geq 1$, an object $E_r^{p,q}$ of $\mathcal{A}$.
- For all such p, q and r, a morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. If we view the $E_r^{\cdot,\cdot}$ as a $\mathbb{Z} \times \mathbb{Z}$-graded object of $\mathcal{A}$, d_r is a morphism of graded objects of degree $[r, -r+1]$. We require that $d_r^2 = 0$, or more explicitly that $d_r^{p+r,q-r+1} d_r^{p,q} = 0$, and also that for every p, q there is an $r_0(p,q)$ such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ when $r \geq r_0(p,q)$.
- We define $H^{p,q}(E_r) = \operatorname{Ker}(d_r^{p,q}) / \operatorname{Im}(d_r^{p-r,q+r-1})$. The next block of data are isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q}$. From the last bullet point we see that for any p, q the objects $E_r^{p,q}$ are isomorphic for $r \gg 0$. We denote by $E_\infty^{p,q}$ their common value.
- Finally, for each $n \in \mathbb{Z}$ a filtered object $F^p E^n \subseteq E^n$ of $\mathcal{A}$ and isomorphisms $\beta^{p,q} : \operatorname{gr}^p E^{p+q} \simeq E_\infty^{p,q}$ for all p and q. The grading must satisfy $E^n = \bigcup_p F^p E^n$ and $\bigcap_p F^p E^n = 0$ for all n.

Sometimes we take $r \geq 2$ or $r \geq 0$. The graded object $E^{\cdot}$ is the *ending* or *abutment* of the spectral sequence. We also say that the spectral sequence $(E_r^{p,q}, d_r^{p,q})$ *converges* to (E^n), and use the notation

$$E_1^{p,q} \Rightarrow E^{p+q}$$

(with no explicit mention of $\alpha_r^{p,q}$, $\beta_r^{p,q}$, the filtration of E^n or the differentials $d_r^{p,q}$).

Some important special cases: a *first quadrant spectral sequence* is one for which $E_1^{p,q} = 0$ if $p < 0$ or $q < 0$; i.e. $E_1^{\cdot,\cdot}$ is "supported in the first quadrant". In this case the condition $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ is automatic for $r \geq p + q + 1 = r_0(p,q)$. Similarly for *third quadrant spectral sequences*. These are the most common cases, but by no means the only ones.

If in fact $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for all $r \geq r_0$ we say that the spectral sequence *degenerates at* E_{r_0}.

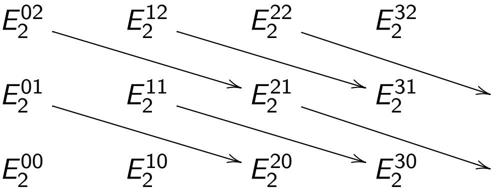
Let's look at a first-quadrant spectral sequence and see what it has to say about the first three terms E^0, E^1, E^2 of the abutment. The array of E_1 terms is

$$\begin{array}{ccccccccc} E_1^{02} & \longrightarrow & E_1^{12} & \longrightarrow & E_1^{22} & \longrightarrow & E_1^{32} & \longrightarrow \\ \\ E_1^{01} & \longrightarrow & E_1^{11} & \longrightarrow & E_1^{21} & \longrightarrow & E_1^{31} & \longrightarrow \\ \\ E_1^{00} & \longrightarrow & E_1^{10} & \longrightarrow & E_1^{20} & \longrightarrow & E_1^{30} & \longrightarrow \end{array}$$

where I have only indicated the possibly nonzero differentials. The rows are complexes, and the E_2 terms are just the cohomology of these complexes. Note that for $r > 1$ the differentials entering and exiting E_r^{00} are zero, so we must have $E_\infty^{00} = E_2^{00} = \mathrm{Ker}(d_1^{00})$. This is the only E_∞ term that can appear as a graded piece of E^0, so

$$E^0 = E_2^{00} = \mathrm{Ker}(d_1^{00}).$$

The array of E_2 terms and their differentials is then



where again I have only indicated nonzero differentials (except for the ones on the edges of the diagram). From this we see that $E_3^{01} \simeq \mathrm{Ker}(d_2^{01})$ and $E_3^{10} = E_\infty^{10}$. Furthermore $E_2^{10} = E_\infty^{10}$ since there is no nonzero differential entering or exiting E_r^{10} for $r \geq 0$. Since $\mathrm{gr}^0 E^1 \simeq E_\infty^{01}$ and $\mathrm{gr}^1 E^1 \simeq E_\infty^{10}$ we have a short exact sequence

$$0 \to E_2^{10} \to E^1 \to E_2^{01} \xrightarrow{d_2^{01}} E_2^{20}$$

called the *exact sequence of low-degree terms* of the spectral sequence. We also find

$$E_3^{20} \simeq \mathrm{Coker}(d_2^{01}), \quad E_3^{11} \simeq \mathrm{Ker}(d_2^{11}), \quad E_3^{02} \simeq \mathrm{Ker}(d_2^{02}).$$

The only possible nonzero differential in the E_3 terms contributing to E is the "edge morphism" $d_3^{02}: E_3^{02} \to E_3^{30}$, and then

$$E_r^{02} = E_4^{02} = \mathrm{Ker}(d_3^{02}) \qquad r \geq 4.$$

This yields

$$\mathrm{gr}^0 E^2 \simeq E_4^{02} = \mathrm{Ker}(d_3^{02}), \quad \mathrm{gr}^1 E^2 \simeq E_3^{11}, \quad \mathrm{gr}^2 E^2 \simeq E_3^{20}.$$

To compute $\mathrm{gr}^{\cdot} E^3$ we would have to find the E_5^{pq} for $p + q = 5$, and so on up the ladder.

In general a spectral sequence is an infinite collection of exact sequences arranged so as to make it possible to get information out of them in a comprehensible way. The next theorem is one of the more important examples:

Theorem

Suppose $F^p K^\cdot \subseteq K^\cdot$ is a filtered complex in an abelian category such that for every n, $F^p K^n = K^n$ for $p \ll 0$ and $F^p K^n = 0$ for $p \gg 0$. There is a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p K^\cdot) \Rightarrow H^{p+q}(K^\cdot)$$

for which the filtration on $H^{p+q}(K^\cdot)$ is the one induced by the morphisms $F^p K^\cdot \to K^\cdot$.

The proof is a prolonged agony of computation and I will only sketch the construction of the $E_r^{p,q}$ and $d_r^{p,q}$, and pretend that I am working in a module category. We start with the subobject

$$Z_r^{p,q} = d^{-1}(F^{p+r} K^{p+q+1}) \cap F^p K^{p+q}$$

which in a module category would be the set of $x \in F^p K^{p+q}$ such that dx lies in the $(p+r)$-th filtered piece. In an arbitrary abelian category the right hand side should be expressed as a fibered product.

In particular an element of $Z_1^{p,q}$ would represent an element of $H^{p+q}(\mathrm{gr}^p K^\cdot)$. Now $Z_r^{p,q}$ clearly contains two subobjects: one is

$$Z_{r-1}^{p+1,q-1} = d^{-1}(F^{p+r}K^{p+q+1}) \cap F^{p+1}K^{p+q}$$

and as

$$Z_{r-1}^{p-r+1,q+r-2} = d^{-1}(F^p K^{p+q}) \cap F^{p-r+1}K^{p+q-1}$$

another is

$$dZ_{r-1}^{p-r+1,q+r-2} = d(F^{p+r-1}K^{p+q-1}) \cap F^p K^{p+q}.$$

since $d^2 = 0$. We then set

$$B_r^{p,q} = Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}, \qquad E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}.$$

I claim that d induces morphisms

$$d_r^{p,q} : Z_r^{p,q} \to Z_r^{p+r,q-r+1}, \qquad d_r^{p,q} : B_r^{p,q} \to B_r^{p+r,q-r+1}.$$

In the first case $Z_r^{p+r,q-r+1} = d^{-1}(F^{p+2r}K^{p+q+1}) \cap F^{p+r}K^{p+q+1}$, and $dZ_r^{p,q} \subseteq F^{p+r}K^{p+q+1}$ by construction; finally since $d^2 = 0$, $dZ_r^{p,q} \subseteq d^{-1}(F^{p+2r}K^{p+q+1})$. In the case of $B_r^{p,q}$,

$$dB_r^{p,q} = dZ_{r-1}^{p+1,q-1}$$

since $d^2 = 0$, and since

$$B_r^{p+r,q-r+1} = Z_{r-1}^{p+r+1,q-r-2} + dZ_{r-1}^{p+1,q-1},$$

we see that $dB_r^{p,q} \subseteq B_r^{p+r,q-r+1}$ as required. We conclude that d induces a morphism

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

for all p, q and r.

I refer you the the text of Gelfand-Manin for the rest of the argument. I will illustrate the above computations by finding the E_1 and E_2 terms of the filtered complexes arising from a double complex $K^{\cdot,\cdot}$. As before d_I and d_{II} are the differentials of degrees $[1,0]$ and $[0,1]$, and the differential of the total complex $\mathrm{Tot}(K^{\cdot,\cdot})$ is induced by $d_I + (-1)^p d_{II}$ on $K^{p,q}$. We recall that the degree n component of $C^{\cdot} := \mathrm{Tot}(K^{\cdot,\cdot})$ is

$$C^n = \mathrm{Tot}(K^{\cdot,\cdot})^n = \bigoplus_{i+j=n} K^{i,j}$$

and that the first filtration of the complex is

$$F_I^p C^n = \bigoplus_{i+j=n,\ j \geq p} K^{i,j}.$$

Our first task is to compute $Z_1^{p,q}$, where $p+q=n$. In a module category it's the subobject of $F^p C^{p+q}$ consisting of $x \in F^p C^{p+q}$ such that $dx \in F^{p+1} C^{p+q}$. Now $d = d_I + (-1)^p d_{II}$ and $d_{II}(x) \in F^{p+1} C^{p+q}$ automatically. We conclude that

$$Z_1^{p,q} = \mathrm{Ker}(d_{II}|K^{p,q}) \oplus \bigoplus_{i+j=p+q,\ j\geq p+1} K^{i,j}.$$

We must next compute

$$B_1^{p,q} = Z_0^{p+1,q-1} + dZ_0^{p,q-1}.$$

Note that $Z_0^{p,q} = F^p C^{p+q}$ by definition, so

$$B_1^{p,q} = F^{p+1} C^{p+q} + dF^p C^{p+q-1}.$$

Since $Z_0^{p+1,q-1} \subseteq F^{p+1} C^{p+q}$, we find that

$$B_1^{p,q} = \mathrm{Im}(d_{II}|K^{p,q-1}) \oplus \bigoplus_{i+j=p+q,\ j\geq p+1} K^{i,j}.$$

We conclude that

$$E_1^{p,q} = \operatorname{Ker}(d_{II}|K^{p,q})/\operatorname{Im}(d_{II}|K^{p,q-1}) = H^q(K^{p,\cdot}).$$

We know that $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by the action of $d = d_I + (-1)^p d_{II}$ on C^{p+q}. Since d_{II} annihilates $Z_1^{p,q}$ and $B_1^{p,q}$ we find that

$$d_1^{p,q} : H^q(K^{p,\cdot}) \to H^q(K^{p+1,\cdot})$$

is induced by the morphism of complexes $d_I : K^{p,\cdot} \to K^{p+1,\cdot}$. We conclude that

$$E_2^{p,q} = H_I^p H_{II}^q(K^{\cdot,\cdot})$$

where the subscript I refers to the first index and the subscript II to the second.

In what follows we will denote this E_2 by ${}^I E_2^{p,q}$ to distinguish it from the E_2 term of the spectral sequence arising from the second filtration, which we recall is

$$F_{II}^p C^n = \bigoplus_{i+j=n,\ i \geq p} K^{i,j}.$$

An exactly parallel argument shows that the E_2 term for the spectral sequence arising from this filtration is

$$^{II}E_2^{p,q} = H_{II}^p H_I^q(K^{\cdot,\cdot})$$

which we should compare with the previous formula

$$^{I}E_2^{p,q} = H_I^p H_{II}^q(K^{\cdot,\cdot})$$

It is customary to take the spectral sequences as starting with the E_2, so we write these as

$$^{I}E_2^{p,q} = H_I^p H_{II}^q(K^{\cdot,\cdot}) \Rightarrow H^{p+q}(\mathrm{Tot}(K^{\cdot,\cdot}))$$
$$^{II}E_2^{p,q} = H_{II}^p H_I^q(K^{\cdot,\cdot}) \Rightarrow H^{p+q}(\mathrm{Tot}(K^{\cdot,\cdot}))$$

They are called the *first* and *second* spectral sequences associated to the double complex.

We will now use this vast machine to show that when the abelian category $\mathcal{A}$ has both enough injectives and enough projectives, the Ext groups can be computed using either projective resolutions of the first argument or injective resolutions of the second. Suppose M and N are objects of $\mathcal{A}$, $P_{\cdot} \to M$ is a projective resolution and $N \to I^{\cdot}$ is an injective resolution. We consider the double complex $K^{\cdot,\cdot}$ for which (i,j)-component is

$$K^{i,j} = \mathrm{Hom}_{\mathcal{A}}(P_i, I^j)$$

and the differentials are

$$d_I^j : \mathrm{Hom}_{\mathcal{A}}(P_i, I^j) \to \mathrm{Hom}_{\mathcal{A}}(P_{i+1}, I^j)$$
$$d_{II}^j : \mathrm{Hom}_{\mathcal{A}}(P_i, I^j) \to \mathrm{Hom}_{\mathcal{A}}(P_i, I^{j+1}).$$

We will drop the subscript $\mathcal{A}$ from now on. As before the associated total complex $C^{\cdot}$ has

$$C^n = \mathrm{Tot}(K^{\cdot,\cdot})^n = \bigoplus_{i+j=n} \mathrm{Hom}(P_i, I^j).$$

Since $\mathrm{Hom}(P_p, \_)$ is an exact functor,

$$H^q_{II}(\mathrm{Hom}(P_p, I^\cdot)) = \begin{cases} \mathrm{Hom}(P_p, N) & q = 0 \\ 0 & q > 0 \end{cases}$$

so the E_2 term of the first spectral sequence is

$${}^I E_2^{p,q} = \begin{cases} H^p(\mathrm{Hom}(P_\cdot, N)) & q = 0 \\ 0 & q > 0 \end{cases}.$$

The spectral sequence degenerates at E_2 since only nonzero entries in the array of E_2 terms are in the bottom row. We conclude that

$$H^p(C^\cdot) \simeq H^p(\mathrm{Hom}(P_\cdot, N))$$

This is the computation of $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ using the projective resolution $P_\cdot \to M$.

Let's now compute $H^n(C^\cdot)$ using the second spectral sequence. The procedure is entirely parallel. Since $\mathrm{Hom}(\_, I^q)$ is an exact functor,

$$H_I^p(\mathrm{Hom}(P_\cdot, I^q) = \begin{cases} \mathrm{Hom}(M, I^q) & p = 0 \\ 0 & p > 0 \end{cases}.$$

and the E_2 term is

$${}^{II}E_2^{p,q} = \begin{cases} H^q(\mathrm{Hom}(M, I^\cdot)) & p = 0 \\ 0 & p > 0 \end{cases}.$$

Once again the spectral sequence degenerates at E_2 since the only nonzero E_2 terms are in the first column. We conclude that

$$H^q(C^\cdot) \simeq H^q(\mathrm{Hom}(M, I^\cdot))$$

which is $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ computed by means of an injective resolution of N.

Summary: if $\mathcal{A}$ has enough projectives and enough injectives we can compute the $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ using a projective resolution of M, and injective resolution of N, or by using the total complex of a double complex that uses both resolutions. In terms of the latter, and element of $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$ is represented by an $(n+1)$-tuple

$$(u_{pq})_{p+q=n} \qquad u_{pq} \in \mathrm{Hom}(P_p, I^q)$$

if $P_\cdot \to M$ and $N \to I^\cdot$ are resolutions of the appropriate sort. The u_{pq} must satisfy

$$d_I u_{pq} + (-1)^{p+1} d_{II} u_{p+1,q-1} = 0$$

for $0 \le p < n$. Two $(n+1)$-tuples (u_{pq}), (u'_{pq}) yield the same element of $H^n(C^\cdot)$ if there is an n-tuple $(v_{pq})_{p+q=n-1}$ such that $u' - u = dv$. Concretely this means that

$$u_{pq} = d_I v_{p-1,q} + (-1)^p d_{II} v_{p,q-1} \qquad 0 \le p \le n,\ \ p+q=n$$

if we set $v_{-1,n} = v_{n,-1} = 0$.

The explicit isomorphism between $H^n(\mathrm{Hom}(P_., N)$ and $H^n(\mathrm{Hom}(M, I^\cdot)$ is the composition

$$H^n(P_., N) \simeq {}^I E_\infty^{n0} \xrightarrow{\sim} H^n(C^\cdot) \xrightarrow{\sim} {}^{II} E_\infty^{0n} \simeq H^n(\mathrm{Hom}(M, I^\cdot)).$$

The previous description of $H^n(C^\cdot)$ leads to the following method for computing this isomorphism. Consider a class in $H^n(\mathrm{Hom}(P_., N)$ represented by an $f \in \mathrm{Hom}(P_n, N)$ in the kernel of $\mathrm{Hom}(P_n, N) \to \mathrm{Hom}(P_{n+1}, N)$. The image u_{n0} of f in $\mathrm{Hom}(P_n, I^0)$ satisfies $d_I u_{n0} = d_{II} u_{n0} = 0$, so $(0, \cdots, 0, u_{n0})$ is an $(n+1)$-tuple representing the image of f under $H^n(P_., N) \to H^n(C^\cdot)$. Since $\mathrm{Hom}(\_, I^0)$ is exact there is a $v_{n-1,0} \in \mathrm{Hom}(P_{n-1}, I^0)$ such that $d_I v_{n-1,0} = u_{n0}$. If we set $u_{n-1,0} = (-1)^{n-1} d_{II} v_{n-1,0}$ then

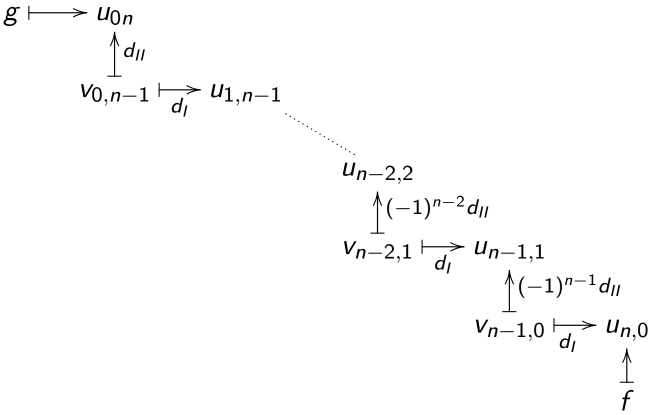
$$(0, \cdots, 0, u_{n0}) \sim (0, \cdots, 0, u_{n-1,1}, 0)$$

i.e. both sides have the same image in $H^n(C^\cdot)$.

Clearly $d_{II} u_{n-1,1} = 0$, and since $\mathrm{Hom}(\_, I^1)$ is exact there is a $v_{n-2,1} \in \mathrm{Hom}(P_{n-2}, I^2)$ such that $u_{n-1,1} = d_I v_{n-2,1}$. If we set $u_{n-2,2} = (-1)^{n-2} d_{II} v_{n-2,1}$ then

$$(0, \cdots, 0, u_{n-1,1}, 0) \sim (0, \cdots, 0, u_{n-2,2}, 0, 0).$$

Continuing in this fashion we eventually wind up with an $(n+1)$-tuple $(u_{0n}, 0, \cdots, 0)$ that represents the same element of $H^n(C^\cdot)$ as $(0, \cdots, 0, u_{0n})$. Now $u_{0n} \in \mathrm{Hom}(P_0, I^n)$ and $d_I u_{0n} = 0$, so there is a $g \in \mathrm{Hom}(M, I^n)$ mapping to u_{0n} under $\mathrm{Hom}(M, I^n) \to \mathrm{Hom}(P_0, I^n)$. Finally the image of the class of f in $H^n(\mathrm{Hom}(P_., N))$ under the isomorphism $H^n(\mathrm{Hom}(P_., N) \xrightarrow{\sim} H^n(\mathrm{Hom}(M, I^\cdot))$ is the class of $g \in \mathrm{Hom}(M, I^n)$. The entire process can be seen from the following diagram:



Note that the argument is completely reversible: starting with a $g \in \mathrm{Hom}(M, I^n)$ the same considerations lead to an $f \in \mathrm{Hom}(P_n, N)$.

The diagram yields the barest hint of the nightmare of signs that is always just below the surface whenever you are doing explicit computations in this subject. The particular signs we have here are a consequence of the particular choice of differential for the total complex of the double complex $\mathrm{Hom}(P_., I^\cdot)$, namely $d = d_I + (-1)^p d_{II}$. Different choices are possible, and indeed quite frequent in the literature, and would lead to different isomorphisms of $H^n(\mathrm{Hom}(P_., N)$ with $H^n(\mathrm{Hom}(M, I^\cdot)$.

In particular, you never really know if the commutative square you are looking at is commutative or anticommutative, unless you work out exactly what the maps are.