Homological Algebra
Lecture 13

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The Spectral Sequence of a Composite Functor

Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left exact functors between abelian categories. If $\mathcal{A}$ and $\mathcal{B}$ have enough injectives the right derived functors $R^n F$, $R^n G$ and $R^n(G \circ F)$ all exist, and we can ask how they are related. The answer is that with a suitable additional hypothesis on F and G there is a spectral sequence

$$E_2^{pq} = R^p G(R^q F(M)) \Rightarrow R^{p+q}(G \circ F)(M)$$

for any object M of $\mathcal{A}$. I'll start with the additional hypothesis.

An object M of $\mathcal{A}$ is *F-acyclic* if $R^n F(M) = 0$ for all $n > 0$. For example any injective object is acyclic for any left exact functor $F : \mathcal{A} \to \mathcal{B}$. For any particular functor F there can be F-acyclic objects that are not injective.

Suppose

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in $\mathcal{A}$. From the long exact sequence of derived functors we see that

- if M' and M are F-acyclic, so is M'';
- if M' and M'' are F-acyclic, so is M.

We can now state the main result:

Theorem (Grothendieck)

Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left exact functors between abelian categories, and that for any injective object I of $\mathcal{A}$, $F(I)$ is G-acyclic. There is a spectral sequence

$$E_2^{pq} = R^p G(R^q F(M)) \Rightarrow R^{p+q}(G \circ F)(M)$$

functorial in M.

There is a similar result for the left derived functors of right exact functors F and G, which I will leave to your imagination.

The main technical tool for the proof is the notion of a *Cartan-Eilenberg resolution*, which basically does for complexes what an injective resolution does for single objects. Suppose $K^\cdot$ is a complex in the abelian category $\mathcal{A}$. A Cartan-Eilenberg resolution of $K^\cdot$ is a double complex $L^{\cdot,\cdot}$ with each L^{ij} injective and a morphism of complexes $\epsilon : K^\cdot \to L^{\cdot,0}$ satisfying the following conditions:

- The complexes

$$
\begin{aligned}
0 \to & K^i \xrightarrow{\epsilon} L^{i0} \to L^{i1} \to \cdots \\
0 \to & B^i(K^\cdot) \xrightarrow{\epsilon} B^i_I(L^{\cdot 0}) \to B^i_I(L^{\cdot 1}) \to \cdots \\
0 \to & Z^i(K^\cdot) \xrightarrow{\epsilon} Z^i_I(L^{\cdot 0}) \to Z^i_I(L^{\cdot 1}) \to \cdots \\
0 \to & H^i(K^\cdot) \xrightarrow{\epsilon} H^i_I(L^{\cdot 0}) \to H^i_I(L^{\cdot 1}) \to \cdots
\end{aligned}
$$

  are exact;

- The sequences

$$0 \to B^i_K(L^{\cdot,j}) \to Z^i_I(L^{\cdot,j}) \to H^i_I(L^{\cdot,j}) \to 0$$
$$0 \to Z^i_I(L^{\cdot,j}) \to L^{i,j} \to B^{i+1}_I(L^{\cdot,j}) \to 0$$

  are split exact.

One can show that if $\mathcal{A}$ has enough injective and if $K^\cdot$ is *bounded below*, i.e. $K^n = 0$ for $n \ll 0$ then $K^\cdot$ has a Cartan-Eilenberg resolution. The argument is an extension of the ones used earlier in showing that any short exact sequence of objects of $\mathcal{A}$ has, so to speak a short exact sequence of injective resolutions. For details see Gelfand-Manin pp. 210–213.

Since the L^{ij} are all injective the second condition shows that the objects $Z^i_I(L^{\cdot,j})$, $B^i_I(L^{\cdot,j})$ and $H^i_I(L^{\cdot,j})$ are injective as well. In particular the resolutions in the first condition are injective resolutions.

As an example of the use of such resolutions, we have the following:

Proposition

Suppose $F : \mathcal{A} \to \mathcal{B}$ is an additive left exact functor, M is an object of $\mathcal{A}$ and

$$0 \to M \to M^0 \to M^1 \to M^2 \to \cdots$$

is a resolution of M by F-acyclic objects. Then

$$R^n F(M) \simeq H^n(F(M^{\cdot}))$$

for all n.

Proof: Let $L^{\cdot,\cdot}$ be a Cartan-Eilenberg resolution of $M^{\cdot}$. Since $L^{i,\cdot}$ is a resolution of M^i,

$$H^q(L^{p,\cdot}) = \begin{cases} M^p & q = 0 \\ 0 & q > 0 \end{cases}$$

and the E_2 terms of the first spectral sequence of the double complex are

$$ {}^{I}E_2^{pq} = \begin{cases} M & p = q = 0 \\ 0 & \text{otherwise.} \end{cases} $$

since $M \to M^{\cdot}$ is a resolution. It follows that $H^0(\mathrm{Tot}(L^{\cdot,\cdot})) = M$ and $H^n(\mathrm{Tot}(L^{\cdot,\cdot})) = 0$ if $n > 0$, i.e. $\mathrm{Tot}(L^{\cdot,\cdot})$ is an injective resolution of M. Thus we can use $\mathrm{Tot}(L^{\cdot,\cdot})$ to compute the $R^n F(M)$:

$$ R^n F(M) \simeq H^n(F(\mathrm{Tot}(L^{\cdot,\cdot}))) \simeq H^n(\mathrm{Tot}(F(L^{\cdot,\cdot}))) $$

where the second isomorphism holds because F is additive, being left exact.

Since each M^i is F-acyclic,

$$ H^q(F(L^{\cdot,p}) = \begin{cases} F(M^p) & q = 0 \\ 0 & q > 0 \end{cases} $$

and thus the E_2 terms of the first spectral sequence of the double complex $\mathrm{Tot}(F(L^{\cdot,\cdot}))$ are

$$ {}^{I}E_2^{pq} = \begin{cases} H^p(F(M^\cdot)) & q = 0 \\ 0 & q > 0 \end{cases} $$

and we conclude that the spectral sequence degenerates at E_2, and

$$ R^n F(M) \simeq H^n(F(\mathrm{Tot}(L^{\cdot,\cdot}))) \simeq H^n(F(M^\cdot)) $$

as asserted. ■

The proof of Grothendieck's theorem is quite similar. We start with an injective resolution $M \to I^\cdot$ of M, and find a Cartan-Eilenberg resolution $L^{\cdot,\cdot}$ of $F(I^\cdot)$. Finally, we consider both spectral sequences for the double complex $G(L^{\cdot,\cdot})$. Since each $F(I^p)$ is G-acyclic,

$$ H^q(G(L^{p,\cdot})) = \begin{cases} GF(I^p) & q = 0 \\ 0 & q > 0 \end{cases} $$

and therefore the E_2 term is

$$ {}^{I}E_2^{pq} = \begin{cases} H^p(GF(I^\cdot)) & q = 0 \\ 0 & q > 0 \end{cases}. $$

Thus the first spectral sequence degenerates at E_2, and since $I^\cdot$ is an injective resolution of M,

$$H^n(G(L^{\cdot,\cdot})) \simeq R^n(G \circ F)(M).$$

Consider now the second spectral sequence. Since $L^{\cdot,\cdot}$ is a Cartan-Eilenberg resolution of $F(I^\cdot)$, $H^q_{II}(L^{p,\cdot})$ is an injective resolution of $H^q(F(I^\cdot)) \simeq R^qF(M)$. Therefore

$$R^pG(R^qF(M)) \simeq H^p_I H^q_{II}(G(L^{\cdot,\cdot})) \simeq {}^{II}E_2^{pq}.$$

The last two isomorphisms show that the second spectral sequence of the double complex $G(L^{\cdot,\cdot})$ is

$$E_2^{pq} = R^pG(R^qF(M)) \Rightarrow R^{p+q}(G \circ F)(M).$$

■

I will now describe the more famous examples of this spectral sequence. First, suppose $f : Y \to X$ is a continuous map of topological spaces and E is a sheaf on Y. We have seen that

$$\Gamma(Y, E) \simeq \Gamma(X, f_*(E))$$

so we are led to consider the functors

$$\begin{aligned} \mathbf{Ab}(Y) \to \mathbf{Ab}(X) &\quad E \mapsto f_*(E) \\ \mathbf{Ab}(X) \to \mathbf{Ab} &\quad E \mapsto \Gamma(X, E). \end{aligned}$$

To apply the theorem we need to check that if I is an injective abelian sheaf on Y then $f_*(I)$ is a Γ-acyclic sheaf on X. This is an instructive case since $f_*(I)$ is not generally injective. But we can show that $f_*(I)$ in fact has a slightly stronger property. We need a bit of general sheaf theory.

A sheaf E on X is *flasque* if for every open $U \subseteq X$ the restriction $E(X) \to E(U)$ is surjective. From this it clearly follows that for all open subsets $U \subseteq V \subseteq X$ the restriction $E(V) \to E(U)$ is surjective.

Lemma

Suppose

$$0 \to E \to F \to G \to 0$$

is a short exact sequence of abelian sheaves on X. If E is flasque then

$$0 \to E(U) \to F(U) \to G(U) \to 0$$

is exact for every open $U \subseteq X$.

Proof: Suppose $s \in G(U)$ and let $\mathcal{S}$ be the set of pairs (V, t) with $V \subseteq U$ open and $t \in F(V)$ mapping to $s|V$. Define a partial order on $\mathcal{S}$ by requiring that $(V, t) \leq (V', t')$ if $V \subseteq V'$ and $t'|V = t$. Since $F \to G$ is an epimorphism, $\mathcal{S}$ is nonempty. If C is a chain in $\mathcal{S}$, say $C = \{(V_c, t_c) \,|\, c \in C\}$ then $W = \bigcup_{c \in C} V_c$ is open in U, and since F is a sheaf there is a $t \in F(W)$ such that $t|V_c = t_c$ for all $c \in C$. By Zorn there is a maximal element (V, t) of $\mathcal{S}$ and it suffices to show that $V = U$. If not, there is an $x \in U \setminus V$, an open neighborhood W of x and $t' \in F(W)$ such that $t' \mapsto s|W$.

Since both $t|V \cap W$ and $t'|V \cap W$ both map to $s|V \cap W$, $(t - t')|V \cap W$ maps to 0 and is therefore the image of some $u \in E(V \cap W)$. Since E is flasque there is a $u' \in E(W)$ such that $u'|V \cap W = u$. Then $t|V \cap W = (t' + u')|V \cap W$, and since F is a sheaf there is a section $t'' \in E(V \cup W)$ such that $t''|V = t$ and $t''|W = t' + u'$. Since u' is a section of E, $t''|V$ and $t''|W$ map to $s|V$ and $s|W$ respectively. Therefore $(V \cup W, t'') \in \mathcal{S}$ and $V \cup W \neq V$, contradicting the maximality of (V, t). ■

Corollary

Suppose

$$0 \to E \to F \to G \to 0$$

is a short exact sequence of abelian sheaves on X. If E and F are flasque, so is G. If E and G are flasque, so is F.

Proof: Snake lemma. ■

Suppose $U \subseteq X$ is open, $j : U \to X$ is the inclusion and E is a sheaf on U. The "extension by zero" of E to X is

$$j_! E(V) = \begin{cases} E(V) & V \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that this is indeed a sheaf on X and that the functor the functor $E \mapsto j_! E$ is the left adjoint to the restriction functor $F \mapsto F|U$, i.e.

$$\mathrm{Hom}_{\mathbf{Ab}(X)}(j_! E, F) \simeq \mathrm{Hom}_{\mathbf{Ab}(U)}(E, F|U)$$

for all abelian sheaves E on U and F on X. If we take $E = F|U$ we get a functorial morphism $j_!(F|U) \to F$ for any abelian sheaf F, and by checking stalks it is clear that $j_!(F|U) \to F$ is a monomorphism.

If A is an abelian group we denote by $\underline{A}$ the "constant sheaf with value A on X, i.e. sheafification of the presheaf sending $U \mapsto A$ for all $U \subseteq X$. It is easily checked that $\underline{A}(U) \simeq A^{\pi_0(U)}$ for all open $U \subseteq X$.

Lemma

An injective abelian sheaf on X is flasque.

Proof: Suppose I is injective, $U \subseteq X$ is open and let $j : U \to X$ be the inclusion. For any sheaf E on X there is are functorial isomorphisms

$$\mathrm{Hom}(j_!\underline{\mathbb{Z}}, E) \simeq \mathrm{Hom}(\underline{\mathbb{Z}}, E|U) \simeq E(U)$$

and this is in particular true for I. Suppose $s \in I(U)$ and identify it with a morphism $j_!\underline{\mathbb{Z}} \to I$. Since I is injective and $j_!\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$ is a monomorphism there is a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & j_!\underline{\mathbb{Z}} & \longrightarrow & \underline{\mathbb{Z}} \\ & & \downarrow & \swarrow & \\ & & I & & \end{array}$$

By the above adjunction the morphism $\underline{\mathbb{Z}} \to I$ corresponds to an element of $I(X)$ which restricts to $s \in I(U)$. ■

Proposition

Suppose $f : Y \to X$ is a continuous map of topological spaces. A flasque sheaf on Y is f_*-acyclic.

Proof: If E is a flasque sheaf on Y we will show that $R^n f_* E = 0$ for $n > 0$ by induction on n. When $n = 1$ we choose a monomorphism $E \to I$ with I injective and define F by the exactness of

$$0 \to E \to I \to F \to 0.$$

Since I is injective the long exact sequence of derived functors of f_* starts out

$$0 \to f_* E \to f_* I \to f_* F \to R^1 f_* E \to R^1 f_* I = 0$$

so that $R^1 f_* E$ is a cokernel of $f_* I \to f_* F$. I will show that $f_* I \to f_* F$ is an epimorphism, which will imply that $R^1 f_* E = 0$. In fact if $U \subseteq X$ is any open, $f_* I(U) \to f_* F(U)$ is $I(f^{-1}(U)) \to F(f^{-1}(U))$ which is surjective since E is flasque. It follows that $f_* I \to f_* F$ is an epimorphism.

Suppose we have shown that $R^k f_* E = 0$ for all positive $k < n$ and all flasque sheaves E. Let E be any flasque sheaf and choose I and F as before. Since I is injective it is flasque, and then F is flasque by an earlier observation. Part of the long exact sequence of the $R^{\cdot} f_*$ is

$$0 = R^{n-1} f_* F \to R^n f_* E \to R^n f_* I = 0$$

and it follows that $R^n f_* E = 0$ as required.

This gives us our spectral sequence, usually known as the *Leray-Serre spectral sequence:*

Theorem

For any continuous map $f : Y \to X$ and any abelian sheaf F on Y there is a spectral sequence

$$E_2^{pq} = H^p(X, R^q f_* F) \Rightarrow H^{p+q}(Y, F).$$

The short exact sequence of low-degree terms is

$$0 \to H^1(X, f_* F) \to H^1(Y, F) \to H^0(X, R^1 f_* F) \to H^2(X, f_* F).$$

Again with X a topological space and sheaves F, G on X we recall the local Hom functor $Hom_X(F, G)$, defined as the sheafification of

$$U \mapsto \mathrm{Hom}_U(F|U, G|U)$$

and the isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}(X)}(F, G) \simeq \Gamma(X, Hom_X(F, G)).$$

Lemma

If G and I are abelian sheaves on X and I is injective then $Hom_X(G, I)$ is flasque.

Proof: If $j : U \to X$ is the inclusion of an open there is are functorial isomorphisms

$$Hom(G, I)(U) \simeq \mathrm{Hom}(G|U, I|U) \simeq \mathrm{Hom}(j_!(G|U), I).$$

Since $j_!(G|U) \to G$ is a monomorphism and I is injective, a morphism $j_!(G|U) \to I$ extends to a morphism $G \to I$, i.e to a global section of $Hom(G, I)$. ∎

We can therefore apply Grothendieck's theorem to the functors $\Gamma(X, \_)$ and $Hom_X(F, \_)$. Recall that the derived functors of the left exact functor $Hom_X(F, \_)$ are denoted by $Ext^n_X(F, \_)$. The resulting spectral sequence is known as the *local-to-global spectral sequence:*

Theorem

For any two sheaves F, G on a topological space there is a spectral sequence

$$E_2^{pq} = H^p(X, Ext^q_X(F, G)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{Ab}(X)}(F, G).$$

The exact sequence of low-degree terms is

$$0 \to H^1(X, Hom_X(F, G)) \to \mathrm{Ext}^1_{\mathbf{Ab}(X)}(F, G) \\ \to H^0(X, Ext^1_X(F, G)) \to H^2(X, Hom_X(F, G)).$$

We remarked earlier that a Cartan-Eilenberg resolution was an extension of the notion of injective resolution to complexes. This allows us to extend the construction of derived functors to complexes of objects of an abelian category with enough injectives. I will merely sketch how this works.

Suppose $K^\cdot$ is an cochain complex in the abelian category $\mathcal{A}$ (assumed to have enough injectives) and suppose that $K^\cdot$ is *bounded below*, i.e. $K^n = 0$ for $n \ll 0$. An *injective resolution* of $K^\cdot$ is a morphism of complexes $f : K^\cdot \to I^\cdot$ which I^n injective for all n and such that the induced morphism $H^n(f) : H^n(K^\cdot) \to H^n(I^\cdot)$ is an isomorphism for all n. Such a morphism is called a *quasi-isomorphism* of complexes (or Bourbaki, a *homologism*).

Suppose for example that M is an object of $\mathcal{A}$ and $K^\cdot$ is the complex with M in degree 0 and zero elsewhere, i.e. $K^0 = M$ and $K^n = 0$ for $n \neq 0$. Any injective resolution $M \to I^\cdot$ gives rise to an injective resolution of $K^\cdot$, as one sees from the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

since both complexes have $H^0(K^\cdot) \simeq M$ and the morphism on $H^0(K^\cdot) \to H^0(I^\cdot)$ is an isomorphism.

Suppose that $K^\cdot$ is a cochain complex that is bounded below. A "simple" extension of the arguments used earlier show that if $K^\cdot \to I^\cdot$ and $K^\cdot \to J^\cdot$ are injective resolutions there is a morphism $I^\cdot \to J^\cdot$ such that

$$\begin{array}{ccc}
K^\cdot & \longrightarrow & I^\cdot \\
 & \searrow & \downarrow \\
 & & J^\cdot
\end{array}$$

commutative, and any two such morphisms are homotopic.

The question remains as to whether injective resolutions of a complex $K^\cdot$ exist (as always, $K^\cdot$ is bounded below). The answer is that they can be constructed as follows:

Lemma

Suppose $\mathcal{A}$ has enough injectives and $K^\cdot$ is a cochain complex in $\mathcal{A}$ that is bounded below. If $L^{\cdot,\cdot}$ is a Cartan-Eilenberg resolution of $K^\cdot$ then $\mathrm{Tot}(L^{\cdot,\cdot})$ is a resolution of $K^\cdot$.

Proof: The morphism $i : K^\cdot \to \mathrm{Tot}(L^{\cdot,\cdot})$ is induced by the morphisms $K^n \to L^{n,0}$ for all n. We have to show that $H^n(i) : H^n(K^\cdot) \to H^n \mathrm{Tot}(L^{\cdot,\cdot})$ is an isomorphism for all n. Since $L^{\cdot,\cdot}$ is a Cartan-Eilenberg resolution of $K^\cdot$,

$$H^q_{II}(L^{p,\cdot}) \simeq \begin{cases} K^p & q = 0 \\ 0 & q > 0 \end{cases}$$

with the isomorphism induced by $K^p \to L^{p,0}$.

The E_2 term of the first spectral sequence of $L^{\cdot,\cdot}$ is

$$E_2^{pq} = \begin{cases} H^p(K^\cdot) & q = 0 \\ 0 & q > 0 \end{cases}$$

and the spectral sequence degenerates at E_2. Therefore $H^p(K^\cdot) \xrightarrow{\sim} H^p(\mathrm{Tot}(L^{\cdot,\cdot})$ and the isomorphism is induced by $K^p \to L^{p,0}$. ■

Suppose now $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor and $K^\cdot$ is bounded below. The arguments above show that for any injective resolution $K^\cdot \to I^\cdot$, the objects $H^n(F(I^\cdot))$ are independent of the choice of resolution $K^\cdot \to I^\cdot$. They can be shown to be functorial in $K^\cdot$ by something like the usual methods. We therefore arrive at the following notion.

Let $C^{\cdot}_{+}(\mathcal{A})$ be the full subcategory of $C^{\cdot}(\mathcal{A})$ consisting of cochain complexes in $\mathcal{A}$ that are bounded below. Furthermore let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. The *hyperderived functors* of F are the functors

$$R^n F : C^{\cdot}_{+}(\mathcal{A}) \to \mathcal{B}$$

defined by

$$R^n F(K^{\cdot}) = H^n(F(I^{\cdot}))$$

for any injective resolution $K^{\cdot} \to I^{\cdot}$. When $K^{\cdot}$ consists of a single object M degree zero, the example earlier shows that $R^n F(K^{\cdot}) \simeq R^n F(M)$.

There are spectral sequences relating $R^n F(K^{\cdot})$ to values of the "ordinary" derived functors. To see how these come about we choose a Cartan-Eilenberg resolution $L^{\cdot,\cdot}$ of $K^{\cdot}$ and consider the two spectral sequences of the double complex $F(L^{\cdot,\cdot})$. The total complex is

$$\mathrm{Tot}(F(L^{\cdot,\cdot})) \simeq F(\mathrm{Tot}(L^{\cdot,\cdot}))$$

so the abutment of these spectral sequences is $R^n F(K^{\cdot})$.

Recall that the E_1 term of the first spectral sequence is

$$^{I}E_1^{pq} = H^q(F(L^{p,\cdot})) \simeq R^qF(K^p)$$

so the first spectral sequence is

$$^{I}E_1^{pq} = R^qF(K^p) \Rightarrow R^{p+q}F(K^{\cdot}).$$

The E_2 term of the second spectral sequence is

$$^{II}E_2^{pq} = H_{II}^pH_I^qF(L^{\cdot,\cdot})$$

and since the sequences

$$\begin{aligned} 0 \to & B_K^i(L^{\cdot,j}) \to Z_I^i(L^{\cdot,j}) \to H_I^i(L^{\cdot,j}) \to 0 \\ 0 \to & Z_I^i(L^{\cdot,j}) \to L^{i,j} \to B_I^i(L^{\cdot,j}) \to 0 \end{aligned}$$

are split exact

$$H_I^q(F(L^{\cdot,\cdot})) \simeq F(H_I^q(L^{\cdot,\cdot})).$$

Since

$$0 \to H^q(K^\cdot) \to H_I^q(L^{\cdot,0}) \to H_I^q(L^{\cdot,0}) \to \cdots$$

is an injective resolution,

$${}^{II}E_2^{pq} \simeq R^pF(H^q(K^\cdot))$$

and the spectral sequence is

$${}^{II}E_2^{pq} = R^pF(H^q(K^\cdot)) \Rightarrow R^{p+q}F(K^\cdot).$$

Several cases of this in differential geometry, analytic geometry and algebraic geometry are too famous not to be mentioned. The basic case is the following: X is a smooth manifold and for $p \geq 0$, E^p is the sheaf of smooth $\mathbb{C}$-valued differential p-forms on X; in particular the sheaf E^0 is the sheaf of smooth $\mathbb{C}$-valued functions on X. The exterior derivative $d : E^p \to E^{p+1}$ makes $E^\cdot$ into a complex and the Poincaré lemma says that

$$0 \to \mathbb{C} \to E^0 \xrightarrow{d} E^1 \xrightarrow{d} E^2 \to \cdots$$

is a resolution of the constant sheaf $\mathbb{C}$. One can also show that each E^p is Γ-acyclic, so

$$H^p(X, \mathbb{C}) \simeq H^p(\Gamma(X, E^{\cdot}))$$

by our earlier result on acyclic resolutions. This is known as *de Rham's theorem.*

Suppose next that X is a complex manifold and Ω_X^p is the sheaf of holomorphic differential p-forms on X. The holomorphic exterior derivative again makes $\Omega_X^{\cdot}$ into a complex, called the (holomorphic) de Rham complex and this is again a resolution of $\mathbb{C}$. This case is more interesting since the $\Omega_X^{\cdot}$ are not Γ-acyclic in general. Nonetheless an injective resolution of $\Omega_X^{\cdot}$ yields an injective resolution of the constant sheaf $\mathbb{C}$, so the hypercohomology of $\Omega_X^{\cdot}$ computes $H^n(X, \mathbb{C})$.

In this situation the first spectral sequence of hypercohomology is

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

and is usually called the Hodge spectral sequence in spite of the fact that it was actually discovered by Frölicher. The interesting thing here is the relation between the topological invariants (the abutment) and the analytic ones (the E_2 terms). This is especially important in the case when X is a projective variety:

Theorem (Hodge)

Suppose X is a smooth projective variety over $\mathbb{C}$ and let X^{an} be the corresponding complex manifold. The Hodge spectral sequence for X^{an} degenerates at E_1.

In particular we get an "analytic" formula for the nth Betti number:

$$B_n = \dim_{\mathbb{C}} H^n(X^{an}, \mathbb{C}) = \sum_{p+q=n} \dim_{\mathbb{C}} H^q(X, \Omega_X^p).$$