Introduction the Derived Categories

The spectral sequence of a composite functor is a powerful tool, but it nonetheless does not provide a way of calculating the derived functors of the composite from the derived functors of the two factors in the composition. The problem ultimately is that there is more information in a complex $K^\cdot$ than in the series of cohomology objects $H^n(K^\cdot)$.

For example if M and N are objects of an abelian category $\mathcal{A}$ and $N \to I^\cdot$ is an injective resolution, there is somehow more information in the complex $\mathrm{Hom}(M, I^\cdot)$ than in the cohomology $H^n(\mathrm{Hom}(M, I^\cdot)) = \mathrm{Ext}^n(M, N)$. We would like to somehow work with the complex $\mathrm{Hom}(M, I^\cdot)$ itself rather than its cohomology. The problem is that $\mathrm{Hom}(M, I^\cdot)$ is anything but uniquely defined, since there are many possible choices for $N \to I^\cdot$.

Grothendieck's answer was to try put the entire complex $\mathrm{Hom}(M, I^\cdot)$ in some new category where all the different choices yield isomorphic objects. This leads to the notion of *derived categories.*

Before we see how this is done let's consider the case of a two-term complex $K^\cdot$, say

$$\cdots \to 0 \to K^{-1} \xrightarrow{d} K^0 \to 0 \to 0 \to \cdots$$

and set $M = H^0(K^\cdot) = \mathrm{Coker}(d)$, $N = H^{-1}(K^\cdot) = \mathrm{Ker}(d)$ (all the other H^n are zero). This leads to a 2-extension of M by N, namely

$$0 \to N \to K^{-1} \to K^0 \to M \to 0$$

which by our earlier results yields an element of

$$\mathrm{Ext}^2(M, N) = \mathrm{Ext}^2(H^0(K^\cdot), H^{-1}(K^\cdot)).$$

So this is already extra information. What does it tell us about $K^\cdot$? Let's just consider the case when $\mathcal{A} = \mathbf{Ab}$.

One way of formulating an answer is to construct a category $\mathcal{K}$ for which the

- objects are elements of K^0, and
- a morphism $x \to y$ for $x, y \in K^0$ is a $u \in K^{-1}$ such that

$$du = y - x.$$

If morphisms $f : x \to y$ and $g : y \to z$ are given by $u, v \in K^{-1}$ such that $du = y - x$, $dv = z - y$ then $d(u + v) = z - x$ is a morphism $x \to z$, and we take this condition to define $g \circ f$. The identity $x \to x$ is $0 \in K^{-1}$ since $d0 = x - x$. It is not hard to check that this definition of objects, morphisms and composition makes $\mathcal{K}$ into a category. This cute device is due to Pierre Deligne.

Suppose now $K^{'\cdot}$ is another 2-term complex, and set $H^0(K^{'\cdot}) \simeq M'$, $H^{-1}(K^{'\cdot}) \simeq N'$. The above relations show that a morphism of complexes $f : K^{\cdot} \to K^{'\cdot}$ induces a functor $\mathcal{K} \to \mathcal{K}'$ where $\mathcal{K}'$ is the category associated to $\mathcal{K}$.

If $f : K^{\cdot} \to K'^{\cdot}$ induces isomorphisms $H^{-1}(K^{\cdot}) \to H^{-1}(K'^{\cdot})$ and $H^0(K^{\cdot}) \to H^0(K'^{\cdot})$, the functor $\mathcal{K} \to \mathcal{K}'$ is an equivalence of categories. To be sure, there could be equivalences $\mathcal{K} \to \mathcal{K}'$ that do not arise in this way. In any case I will now show that if f induces an isomorphism on cohomology, both complexes yield the same element of $\operatorname{Ext}^2(M, N)$. Suppose for example that $\mathcal{A}$ has enough projectives and $P_{\cdot} \to M$ is a projective resolution. The class in $\operatorname{Ext}^2(M, N)$ of $K^{\cdot}$ viewed as a 2-extension of M by N is represented by the class of a morphism $\alpha : P_2 \to N$. Consider now the diagram

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & K'^{-1} & \longrightarrow & K'^{0} & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

where the vertical arrows between the middle and bottom rows represent $f : K^{\cdot} \to K^{'\cdot}$. The diagram makes it clear that the class of $K^{'\cdot}$ in $\text{Ext}^2(M, N)$ is represented by the same α.

Summary: if $f : K^{\cdot} \to K^{'\cdot}$ induces an isomorphism on cohomology, the induced functor $\mathcal{K} \to \mathcal{K}'$ is an equivalence of categories and $K^{\cdot}$ and $K^{'\cdot}$ have the same class in $\text{Ext}^2(M, N)$. We are now lead to ask if there is some converse to this, specifically:

1. If $K^{\cdot}$ and $K^{'\cdot}$ have the same class in $\text{Ext}^2(M, N)$, are the corresponding categories equivalent?
2. If so, is there an equivalence that this induced by a morphism $f : K^{\cdot} \to K^{'\cdot}$?

The short answer is: yes for (1) and no for (2). Let's concentrate on (1); I'll leave it as an optional exercise to come up with a counterexample for (2).

Let's go back the diagram two slides back. Since the composite

$$P_3 \xrightarrow{d} P_2 \xrightarrow{\alpha} N$$

is zero, the top two rows can be contracted to a morphism

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P_2/dP_3 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and one easily checks that this factors through a morphism of complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N & \longrightarrow & E_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where

$$E_1 = N \amalg_{P_2/dP_3} P_1$$

and $N \to E_1$ is the canonical monomorphism. The morphism $E_1 \to K^{-1}$ is induced by the universal property and the morphism $E_1 \to P_0$ is induced by the morphisms $N \xrightarrow{0} P_0$, $P_2/dP_3 \xrightarrow{d} P_0$.

Finally one can easily check that the top row of this last diagram is exact, so we have here is morphism of 2-extensions of M by N. In particular if we let $P(\alpha)^{\cdot}$ be the 2-term complex

$$\cdots \to 0 \to E_1 \to P_0 \to 0 \to \cdots$$

then middle vertical arrows of the last diagram are a morphism $f : P(\alpha)^{\cdot} \to K^{\cdot}$ such that

$$H^n(f) : H^n(P(\alpha)^{\cdot}) \to H^n(K^{\cdot})$$

are isomorphisms for all n (note that it is only for $n = -1, 0$ that the H^n are nonzero).

Now the construction of $P(\alpha)^{\cdot}$ only used the resolution $P_{\cdot} \to M$ and the morphism $\alpha : P_2 \to N$. In particular the exact same argument works with $K^{'\cdot}$ in place of $K^{\cdot}$, yielding a morphism $f' : P(\alpha) \to K^{'\cdot}$ inducing isomorphisms on all H^n and in particular on H^{-1} and H^0.

If $\mathcal{P}(\alpha)$ is the category constructed from the 2-term complex $P(\alpha)^{\cdot}$ the morphisms $f : P(\alpha)^{\cdot} \to K^{\cdot}$, $f' : P(\alpha) \to K^{'\cdot}$ give rise to equivalences $\mathcal{P}(\alpha) \xrightarrow{\sim} \mathcal{K}$ and $\mathcal{P}(\alpha) \xrightarrow{\sim} \mathcal{K}'$. This implies that $\mathcal{K}$ and $\mathcal{K}'$ are equivalent. This is the affirmative answer to question (1).

More precisely, since $\mathcal{P}(\alpha) \to \mathcal{K}$ is an equivalence we can find a quasi-inverse functor $\mathcal{K} \to \mathcal{P}(\alpha)$, and then the desired equivalence $\mathcal{K} \to \mathcal{K}'$ is the composition of this quasi-inverse with $\mathcal{P}(\alpha) \to \mathcal{K}'$. What we *don't* know is whether $\mathcal{K} \to \mathcal{P}(\alpha)$ is derived from a morphism $K^{\cdot} \to P(\alpha)^{\cdot}$ of complexes. It's embarrassingly easy to find examples of morphisms of complexes $P^{\cdot} \to K^{\cdot}$ such that $H^n(P^{\cdot}) \to H^n(K^{\cdot})$ is an isomorphism for all n but there is no morphism $K^{\cdot} \to P^{\cdot}$ yielding the inverses on H^n, for example

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

For both complexes $H^{-1} = 0$ and $H^0 = \mathbb{Z}/2\mathbb{Z}$ and the map in cohomology is an isomorphism, but there is no morphism $K^\cdot \to P^\cdot$ such that $H^0(K^\cdot) \to H^0(P^\cdot)$ is an isomorphism.

Summary: Suppose $K^\cdot$ and $K'^\cdot$ are 2-term complexes of abelian groups such that $H^{-1}(K^\cdot) \simeq H^{-1}(K'^\cdot)$ and $H^0(K^\cdot) \simeq H^0(K'^\cdot)$.

- If there is a morphism of complexes $f : K^\cdot \to K'^\cdot$ inducing these isomorphisms then $K^\cdot$ and $K'^\cdot$ have the same class in $\mathrm{Ext}^2(H^0(K^\cdot), H^{-1}(K^\cdot))$ and the corresponding categories are equivalent.
- Conversely if they have the same class then the corresponding categories are equivalent but the equivalence is not necessarily induced by a morphism $f : K^\cdot \to K'^\cdot$.

In general, a morphism of complexes $f : K^\cdot \to K'^\cdot$ is a *quasi-isomorphism* (or in Bourbaki, a *homologism*) if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. The previous discussion suggests the following question: if $\mathcal{A}$ is an abelian category, is there a "natural" functor $f : C^\cdot(\mathcal{A}) \to \mathcal{C}$ for some $\mathcal{C}$ which turns quasi-isomorphisms into isomorphisms?

We have realized a small piece of this program, restricted to the subcategory of complexes of abelian groups supported in degrees -1 and 0, and we can take as $\mathcal{C}$ the category of small categories (objects are categories and morphisms are isomorphism classes of functors). The category $\mathcal{C}$ here is in some sense optimal, if we regard the class of a complex $K^\cdot$ in $\mathrm{Ext}^2(H^0(K^\cdot), H^{-1}(K^\cdot))$ as representing all the useful information that can be gotten out of $K^\cdot$ in addition to $H^0(K^\cdot)$ and $H^{-1}(K^\cdot)$. Exactly how one generalizes this construction is not entirely clear even when $\mathcal{A}$ is the category of abelian groups.

Instead we pose the question on a more abstract level. Suppose $\mathcal{C}$ is a category and W is a class (not necessarily a set) of morphisms in $\mathcal{C}$. We would like to have a category $\mathcal{C}(W^{-1})$ and a functor $i_W : \mathcal{C} \to \mathcal{C}(W^{-1})$ with the property that any $f \in W$ becomes an isomorphism in $\mathcal{C}(W^{-1})$. Furthermore $\mathcal{C}(W^{-1})$ should be universal for this property: if $F : \mathcal{C} \to \mathcal{D}$ is any functor such that $f \in W$ implies that $F(f)$ is an isomorphism, then F should factor through a functor $\mathcal{C}(W^{-1}) \to \mathcal{D}$, uniquely up to isomorphism.

This would be a categorical version of localization in a ring, and amazingly it is always possible, although in general it can be very difficult to understand what $\mathcal{C}(W^{-1})$ really is. With a few hypotheses on W there is a relatively simple construction that is quite useful.

Returning to the subject of derived functors we see that the utility of such a construction is clear: given, say a left exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories and supposing as always that $\mathcal{A}$ has enough injectives, and finally M is an object of $\mathcal{A}$, we find an injective resolution $M \to I^{\cdot}$ and form the complex $F(I^{\cdot})$; this of course is dependent on the choice of $M \to I^{\cdot}$ but the $H^n(F(I^{\cdot}))$ are not. Suppose now W is the class of quasi-isomorphisms in $C^{\cdot}(\mathcal{B})$; then the image $i_W(F(I^{\cdot}))$ of $F(I^{\cdot})$ in $C^{\cdot}(\mathcal{B})(W^{-1})$ will be independent of the choice of $M \to I^{\cdot}$ up to isomorphism.

If we could use this construction to define a functor $\mathcal{A} \to C^{\cdot}(\mathcal{B})(W^{-1})$, the latter would deserve to be called "the" derived functor of F, since the usual derived functors can be obtained from it. But there is a further problem.

The problem is that even though $i_W(F(I^\cdot))$ is unique up to isomorphism it is not unique up to unique isomorphism. If $M \to J^\cdot$ is a second choice of injective resolution there is a morphism of complexes $I^\cdot \to J^\cdot$ unique up to homotopy, and inducing a quasi-isomorphism $F(I^\cdot) \to F(J^\cdot)$. But the choice of $I^\cdot \to J^\cdot$ is not unique.

We are thus lead to a 2-step process:

- Construct a category $K^\cdot(\mathcal{A})$ and a morphism $h : C^\cdot(\mathcal{A}) \to K^\cdot(\mathcal{A})$ such that $f \sim g$ in $C^\cdot(\mathcal{A})$ implies $h(f) = h(g)$ in $K^\cdot(\mathcal{A})$. We also require that for $K^\cdot$ in $C^\cdot(\mathcal{A})$, the $H^n(K^\cdot)$ depend only on $h(K^\cdot)$, functorially in $K^\cdot$.
- It then makes sense to speak of a morphism in $K^\cdot(\mathcal{A})$ being a quasi-isomorphism. Let W be the class of quasi-isomorphisms in $K^\cdot(\mathcal{A})$ and let $i_W : K^\cdot(\mathcal{A}) \to K^\cdot(\mathcal{A})(W^{-1})$ be the localization functor. With F, M and $M \to I^\cdot$ as before, $i_W(h(F(I^\cdot)))$ is independent of the choice of $M \to I^\cdot$ up to unique isomorphism.

In a moment I will describe a fairly simple construction of a category $K^{\cdot}(\mathcal{A})$. With this choice, the category $K^{\cdot}(\mathcal{A})(W^{-1})$ is called the *derived category* of $\mathcal{A}$ and is denoted by $D(\mathcal{A})$. I will denote by $i_{\mathcal{A}}$ the composite functor $i_W \circ h$. Since the object $i_{\mathcal{A}}(F(I^{\cdot}))$ of $D(\mathcal{A})$ is independent of the choice of $M \to I^{\cdot}$ up to canonical isomorphism, the construction $M \mapsto i_{\mathcal{A}}(F(I^{\cdot}))$ defines a functor $RF : \mathcal{A} \to D(\mathcal{A})$, the *right derived functor* of F. For any object M of $\mathcal{A}$ the usual derived functors of F can be calculated as $H^n(RF(M))$.

Suppose now $G : \mathcal{B} \to \mathcal{C}$ is another left exact functor and that $\mathcal{B}$ has enough injectives. If F takes injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$ one can show that there is an isomorphism

$$R(G \circ F) \simeq RG \circ RF.$$

This takes the place of the spectral sequence of derived functors.

The construction of $K^{\cdot}$ is fairly simple. Recall first that the relation of homotopy on morphisms is an equivalence relation, and it is compatible with composition: if $f \sim f'$ and $g \sim g'$ then $gf \sim g'f'$ (assuming that the morphisms are composable. We can therefore define $K^{\cdot}(\mathcal{A})$ to be the category in which

- objects are the objects of $C^{\cdot}(\mathcal{A})$, and
- for any two objects $K^{\cdot}$, $L^{\cdot}$ of $C^{\cdot}(\mathcal{A})$, $\mathrm{Hom}_{K(\mathcal{A})}(K^{\cdot}, L^{\cdot})$ is the quotient of $\mathrm{Hom}_{C(\mathcal{A})}(K^{\cdot}, L^{\cdot})$ by the relation of homotopy.

By the previous remarks, composition of morphisms in $C^{\cdot}(\mathcal{A})$ induces compositions in $K^{\cdot}(\mathcal{A})$, and it is evident that this makes $K^{\cdot}(\mathcal{A})$ into a category. It is called the category of *cochain complexes in $\mathcal{A}$ up to homotopy.*

Another construction of the groups $\mathrm{Hom}_{K(\mathcal{A})}(K^\cdot, L^\cdot)$ is useful. Consider the double complex $\mathrm{Hom}_{\mathcal{A}}(K^\cdot, L^\cdot)$ whose (i,j)-term is $\mathrm{Hom}_{\mathcal{A}}(K^{-i}, L^j)$ and whose differentials are induced by the differentials d_K, d_L of $K^\cdot$ and $L^\cdot$. We then define

$$\mathrm{Hom}^\cdot_{\mathcal{A}}(K^\cdot, L^\cdot) = \mathrm{Tot}(\mathrm{Hom}_{\mathcal{A}}(K^\cdot, L^\cdot))$$

and give it the usual differential

$$d = d_K^i + (-1)^i d_L.$$

A brief consideration of the construction shows that

- A 0-chain of $\mathrm{Hom}^\cdot_{\mathcal{A}}(K^\cdot, L^\cdot)$ is a morphism $K^\cdot \to L^\cdot$ of graded objects;
- A 0-cocycle of $\mathrm{Hom}^\cdot_{\mathcal{A}}(K^\cdot, L^\cdot)$ is a morphism of complexes, and
- If $f : K^\cdot \to L^\cdot$ is a 0-cocycle and $f = dh$ then h is a null homotopy of f.

From this we deduce that

$$H^0(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(K^{\cdot}, L^{\cdot})) \simeq \mathrm{Hom}_{K(\mathcal{A})}(K^{\cdot}, L^{\cdot}).$$

It is not too hard to show that for all m and n there are maps

$$\mathrm{Hom}^m_{\mathcal{A}}(L^{\cdot}, M^{\cdot}) \otimes \mathrm{Hom}^n_{\mathcal{A}}(K^{\cdot}, L^{\cdot}) \to \mathrm{Hom}^{m+n}_{\mathcal{A}}(K^{\cdot}, M^{\cdot})$$

which for $m = n = 0$ reduces to composition of morphisms up to homotopy. This shows that $K^{\cdot}(\mathcal{A})$ has a natural "enrichment in the category of chain complexes of abelian groups," i.e. we can define Hom groups that belong to $C^{\cdot}(\mathbf{Ab})$.

We now give a construction of the localization functor $i_W : \mathcal{C} \to \mathcal{C}(W^{-1})$ for an arbitrary category $\mathcal{C}$ with the following hypotheses on W, which define what it means for W to *admit a calculus of fractions:*

- W is closed under composition and contains all identities.
- (Ore condition) Given any diagram of solid arrows

$$\begin{array}{ccc} T & \overset{v}{\dashrightarrow} & Y \\ {\scriptstyle t}\,\vdots & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Z \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} T & \xrightarrow{u} & Y \\ {\scriptstyle s}\downarrow & & \vdots\,{\scriptstyle t} \\ X & \overset{v}{\dashrightarrow} & Z \end{array}$$

  with $s \in W$, the rest of the diagram can be filled in with some T and $t \in W$.

- If the composites

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Z \qquad \text{resp.} \qquad Z \xrightarrow{s} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

  are equal and $s \in W$, there is a $T \in W$ and $t : T \to X$ in W (resp. $t : Y \to T$) such that the composites

$$T \xrightarrow{t} X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad \text{resp.} \qquad X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{t} T$$

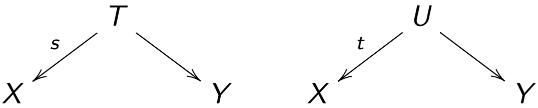
  are equal.

We can now define $\mathcal{C}(W^{-1})$. It has the same objects as $\mathcal{C}$. A morphism $X \to Y$ in $\mathcal{C}(W^{-1})$ is a diagram of the form

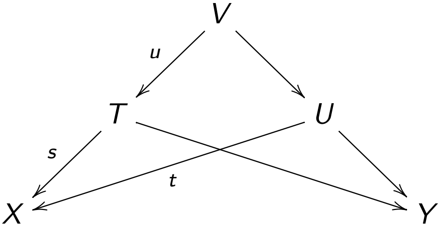
$$X \xleftarrow{s} T \longrightarrow Y$$

with $s \in W$, up to the following equivalence:

The diagrams



represent the same morphism if and only if they fit into a diagram



with $u \in W$. One must check that this defines an equivalence relation.

Composition of morphisms is defined as follows. Suppose $X \to Y$ and $Y \to Z$ are morphisms in $\mathcal{C}(W^{-1})$ represented as diagrams

$$\begin{array}{ccccccc} & T & & & & U & \\ {}^{s}\swarrow & & \searrow^{f} & & {}^{t}\swarrow & & \searrow^{g} \\ X & & Y & & Y & & Z \end{array}$$

The Ore condition says that there is a $u : V \to T$ in W and a $h : V \to U$ such that

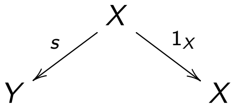
$$\begin{array}{ccccccc} & & & V & & & \\ & & {}^{u}\swarrow & & \searrow^{h} & & \\ & T & & & & U & \\ {}^{s}\swarrow & & \searrow^{f} & & {}^{t}\swarrow & & \searrow^{g} \\ X & & & Y & & & Z \end{array}$$

is commutative. The composite $X \to Z$ is then represented by the pair $su : V \to X$ and $gh : V \to Z$. Note that $su \in W$.

Of course one needs to check that composition so defined is independent of the choice of representing diagrams, which verification makes extensive use of the other two conditions on W. And then check that composition is associative...

A more serious problem is that $\mathrm{Hom}(X, Y)$ in $\mathcal{C}(W^{-1})$ might turn out to be a proper class instead of a set. There are various ways of avoiding this problem, none of them entirely satisfactory.

It's clear that if $s : X \to Y$ is in W then s is invertible in $\mathcal{C}(W^{-1})$, an inverse being represented by



.

I'll let you do the computation showing that this works. Finally, the universal property of $\mathcal{C}(W^{-1})$ is clear: $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(f)$ is an isomorphism for all $f \in W$, then F factors through a functor $\mathcal{C}(W^{-1}) \to \mathcal{D}$.

Finally the Ore condition shows that there is a morphism $X \to Y$ in $\mathcal{C}(W^{-1})$ if and only if there is a diagram

$$X \longrightarrow T \xleftarrow{\ s\ } Y$$

with $s \in W$. This leads to an alternate "dual" definition of composition, and I leave it as an exercise to show that one in fact gets the same composition law on morphisms.

We are just about done. The last step is to show that for any abelian category $\mathcal{A}$ the class of quasi-isomorphisms in $K^\cdot(\mathcal{A})$ admits a calculus of fractions. This leads to the following problem: the category $C^\cdot(\mathcal{A})$ is abelian, and while the homotoply category $K^\cdot(\mathcal{A})$ is additive it is not generally abelian, so few of the usual tools and arguments we use in abelian categories are available. However $K^\cdot(\mathcal{A})$ does have another sort of structure that is very useful, that of a *triangulated category.*

Before I explain what this means I want to introduce the following very useful construction. All of our homological algebra arguments make use of the long exact sequence associated to a short exact sequence of complexes. Miraculously, there is a long exact sequence associated to certain composable pairs of morphisms in $C^\cdot(\mathcal{A})$ even when they don't give a short exact sequence.

Suppose $u : K^\cdot \to L^\cdot$ is a morphism in $C^\cdot(\mathcal{A})$. The *cone* of u is the following complex: as a graded object, it is

$$C(u)^\cdot = K^\cdot[1] \oplus L^\cdot$$

and from now on we take the differential of $K^\cdot[1]$ to be induced by the *negative* of that of $K^\cdot$: $d_{K[1]} = -d_K$. The differential on $C(u)^\cdot$ is

$$d^n(k, \ell) = (-d_K^{n+1}(k), u_{n+1}(k) + d_L^n(\ell))$$

where we have $k \in K^{n+1}$ and $\ell \in L^n$.

It is not hard to check directly that $d^2 = 0$ and there are morphisms of complexes

$$L^\cdot \to C(u)^\cdot \qquad \ell \mapsto (0, \ell)$$
$$C(u)^\cdot \to K^\cdot[1] \qquad (k, \ell) \mapsto k$$

fitting into a diagram

$$K^\cdot \xrightarrow{u} L^\cdot \to C(u)^\cdot \to K^\cdot[1]$$

which we call an *exact triangle*. The miracle is that for any exact triangle there is a long exact sequence of cohomology

$$\to H^n(K^\cdot) \to H^n(L^\cdot) \to H^n(C(u)^\cdot) \to H^{n+1}(K^\cdot) \to$$

for all n, *without any condition on u*. This is not hard to check directly but following argument is slicker. The trick is to form the double complex

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K^2 & \xrightarrow{u} & L^2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_K} & & \uparrow{\scriptstyle d_L} & & \\
0 & \longrightarrow & K^1 & \xrightarrow{u} & L^1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle d_K} & & \uparrow{\scriptstyle d_L} & & \\
0 & \longrightarrow & K^0 & \xrightarrow{u} & L^0 & \longrightarrow & 0
\end{array}
$$

where K^n is in bidegree $(-1, n)$ and L^n is in degree $(0, n)$. With our standard sign conventions the total complex of this double complex is none other than $C(u)^{\cdot}$. The E_1 terms for the second filtration are

$$E_1^{-1,q} = H^q(K^{\cdot}), \qquad E_1^{0q} = H^q(L^{\cdot})$$

and all the others are 0. It follows that the E_2 therms are

$$E_2^{-1,q} = \mathrm{Ker}(H^q(K^{\cdot}) \xrightarrow{H^q(u)} H^q(L^{\cdot}))$$
$$E_2^{0,q} = \mathrm{Coker}(H^q(K^{\cdot}) \xrightarrow{H^q(u)} H^q(L^{\cdot}))$$

and the spectral sequence degenerates at E_2. From this we see that the sequence

$$\to H^{q-1}(C(u)^\cdot) \to H^q(K^\cdot) \to H^q(L^\cdot) \to H^q(C(u)^\cdot) \to H^{q+1}(K^\cdot) \to$$

is exact.

The situation in the homotopy category is less satisfactory but we can still say the following:

- The translation functor $K^\cdot \to K^\cdot[1]$ in $C^\cdot(\mathcal{A})$ extends to $K^\cdot(\mathcal{A})$.
- If u and v are homotopic maps $K^\cdot \to L^\cdot$ there is a map $C(u)^\cdot \to C(v)^\cdot$ with a homotopy inverse.

We can then say that a diagram

$$K^\cdot \xrightarrow{u} L^\cdot \xrightarrow{v} M^\cdot \xrightarrow{w} K^\cdot[1]$$

in $K^\cdot(\mathcal{A})$ is a *triangle* if there is a morphism of complexes $u : K^\cdot \to L^\cdot$ such that $M^\cdot \simeq C(u)^\cdot$, and v and w are induced by the canonical maps in $C^\cdot(\mathcal{A})$.