Homological Algebra
Lecture 2

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Summary so far: a *preabelian category* is one for which the Hom sets have an abelian group structure compatible with the categorical structure. An *additive category* is a preadditive category with zero objects and finite coproducts. A *preabelian category* is an additive category in which all kernels and cokernels are representable.

We saw finally that all finite limits and colimits are representable in a preabelian category. They do not however necessarily behave as expected. We first consider some ways in which they do.

We say that a commutative diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is *cartesian* if it is a pullback diagram, i.e. if Y' and the morphisms $Y' \to Y$ and $Y' \to X'$ represent the fibered product of $Y \to X$ and $X' \to X$.

It is *cocartesian* if it makes X a pushout of $Y' \to X'$ and $Y' \to Y$. Consider now a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X \end{array} \tag{1}$$

in the preabelian category $\mathcal{A}$. If $i' : K' \to Y'$ is a kernel of f', $f'i = 0$ and thus $fg'i' = gf'i' = 0$. If $i : K \to Y$ is a kernel of f this says that fg' factors uniquely through i, whence a commutative diagram

$$\begin{array}{ccccc} K' & \xrightarrow{i'} & Y' & \xrightarrow{f'} & X' \\ {\scriptstyle h}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[i]{} & Y & \xrightarrow[f]{} & X \end{array}$$

In this situation we call the morphism the *canonical morphism* associated to (1). Applying this to $\mathcal{A}^{\mathrm{op}}$ we get a similar diagram for cokernels.

Proposition

If the diagram (1) is cartesian (resp. cocartesian), the canonical morphism $\mathrm{Ker}(f') \to \mathrm{Ker}(f)$ *(resp.* $\mathrm{Coker}(f') \to \mathrm{Coker}(f)$*) is an isomorphism.*

Proof: This is clearly true if $\mathcal{A}$ is the category of abelian groups. To reduce to this case we pick kernels $K \to Y$ and $K' \to Y'$ of f and f'. For any T in $\mathcal{A}$ the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(T, K') & \longrightarrow & \mathrm{Hom}(T, Y') & \longrightarrow & \mathrm{Hom}(T, X') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(T, K) & \longrightarrow & \mathrm{Hom}(T, Y) & \longrightarrow & \mathrm{Hom}(T, X) \end{array}$$

is commutative and the rows are exact. Since the original diagram was cartesian so is the right hand square, so $\mathrm{Hom}(T, K') \to \mathrm{Hom}(T, K)$ is an isomorphism for all T. By Yoneda $K' \to K$ is an isomorphism. ■

Let's reconsider the diagram

$$\begin{CD} K' @>{i'}>> Y' @>{f'}>> X' \\ @V{h}VV @V{g'}VV @VV{g}V \\ K @>>{i}> Y @>>{f}> X \end{CD}$$

where $K' \to Y'$ is a kernel of f' and $K \to Y$ is a kernel of f. We do not assume the right hand square is cartesian; but we can apply the preceding construction to get a larger diagram

$$\begin{CD} \mathrm{Ker}(h) @>>> \mathrm{Ker}(g') @>>> \mathrm{Ker}(g) \\ @VVV @VVV @VVV \\ K' @>{i'}>> Y' @>{f'}>> X' \\ @V{h}VV @V{g'}VV @VV{g}V \\ K @>>{i}> Y @>>{f}> X \end{CD} \tag{2}$$

in which all squares are commutative.

Proposition

In the diagram (2), $\mathrm{Ker}(h) \to \mathrm{Ker}(g')$ *is a kernel of* $\mathrm{Ker}(g') \to \mathrm{Ker}(g)$.

Proof: We must show that

$$0 \to \mathrm{Hom}(T, \mathrm{Ker}(h)) \to \mathrm{Hom}(T, \mathrm{Ker}(g')) \to \mathrm{Hom}(T, \mathrm{Ker}(g))$$

is an exact sequence in **Ab** for any T. But if we apply $\mathrm{Hom}(T, \ )$ to the diagram (2) we get a similar diagram of Hom groups in **Ab**, and the top row is exact by the snake lemma. ■

Dually we may consider a diagram of the form

$$\begin{array}{ccccccc} X' & \longrightarrow & Y' & \longrightarrow & C' & \longrightarrow & 0 \\ \downarrow f & & \downarrow f' & & \downarrow h & & \\ X & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in which $Y' \to C'$ and $Y \to C$ are the cokernels of $X' \to Y'$ and $X \to Y$.

Applying the first proposition in the case of cokernels yields a series of morphisms

$$\mathrm{Coker}(f) \to \mathrm{Coker}(f') \to \mathrm{Coker}(h) \to 0$$

and the same argument as before shows that this is an exact sequence in **Ab**.

We could say that the last proposition and the remark following it say that the snake lemma is "half true" in a preabelian category. What prevents us from proving the entire snake lemma? Basically, the circumstance that a morphism in a preabelian category can be a monomorphism and an epimorphism without being an isomorphism. This also implies that there is no obvious notion of exact sequence. To explain what is missing we make the following constructions. The next few slides will be rather confusing.

Suppose $\mathcal{A}$ is a preabelian category and $f : X \to Y$ is a morphism in $\mathcal{A}$. Since kernels and cokernels exist in $\mathcal{A}$ we get a diagram

$$\mathrm{Ker}(f) \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{p} \mathrm{Coker}(f)$$

in $\mathcal{A}$. By construction $pf = 0$, so there is a unique $q : X \to \mathrm{Ker}(p)$ such that $f = jq$, where $j : \mathrm{Ker}(p) \to Y$ is a kernel of p:

$$\mathrm{Ker}(f) \xrightarrow{i} X \xrightarrow{q} \mathrm{Ker}(p) \xrightarrow{j} Y \xrightarrow{p} \mathrm{Coker}(f)$$

Here j is a monomorphism, i.e. $j : \mathrm{Ker}(p) \to Y$ is a subobject of Y and is unique up to unique isomorphism. It is called the *image* of f and we denote it by $j : \mathrm{Im}\, f \to Y$:

$$\mathrm{Ker}(f) \xrightarrow{i} X \xrightarrow{q} \mathrm{Im}(f) \xrightarrow{j} Y \xrightarrow{p} \mathrm{Coker}(f)$$

On the other hand, $fi = 0$ implies that f *also* factors through the cokernel of $i : \mathrm{Ker}(f) \to X$. We denote (any choice of a) cokernel by $k : X \to \mathrm{Coim}(f)$. So now we have a picture

$$\mathrm{Ker}(f) \xrightarrow{i} X \xrightarrow{k} \mathrm{Coim}(f) \xrightarrow{f''} Y \xrightarrow{p} \mathrm{Coker}(f)$$

in which $f''k = f$. The cokernel $k : X \to \mathrm{Coim}(f)$ is called the *coimage* of f.

The next step is to observe that we can *combine* these two pictures. The cokernel $k : X \to \mathrm{Coim}(f)$ is an epimorphism, so $pf = pf''k = 0$ implies $pf'' = 0$ (since $pf''k = 0k$). It follows that $f'' : \mathrm{Coim}(f) \to Y$ factors through the kernel $j : \mathrm{Im}(f) \to Y$ of the cokernel $p : Y \to \mathrm{Coker}(f)$. The picture is now

$$\mathrm{Ker}(f) \xrightarrow{i} X \xrightarrow{q} \mathrm{Coim}(f) \xrightarrow{\bar{f}} \mathrm{Im}(f) \xrightarrow{j} Y \xrightarrow{p} \mathrm{Coker}(f).$$

The factorization $f = j\bar{f}q$ is called the *canonical factorization* of f. In fact the morphism $\bar{f}$ is unique, for we know that q is an epimorphism (being a cokernel) and j is a monomorphism (being a kernel). Thus if $f = j\tilde{f}q$,

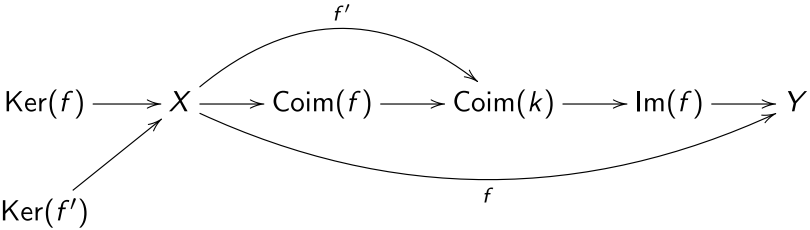
$$j\bar{f}q = j\tilde{f}q \implies \bar{f}q = \tilde{f}q \implies \bar{f} = \tilde{f}.$$

Since kernels (resp. cokernels) in $\mathcal{A}$ are cokernels (resp. kernels) in $\mathcal{A}^{\mathrm{op}}$, coimages (resp. images) in $\mathcal{A}$ are images (resp. coimages) in $\mathcal{A}^{\mathrm{op}}$. It follows that $\bar{f}$ is "self-dual" in the sense that it is the same in $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$.

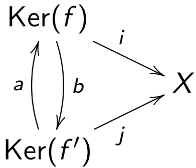
Lemma

The canonical morphism $\bar{f} : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ *is a monomorphism and an epimorphism.*

Proof: By duality it suffices to check that it is a monomorphism. Let $k : \mathrm{Ker}(\bar{f}) \to \mathrm{Coim}(f)$ be the kernel of $\bar{f}$, which thus factors $\mathrm{Coim}(f) \to \mathrm{Coim}(k) \to \mathrm{Im}(f)$. If f' is the composite $X \to \mathrm{Coim}(f) \to \mathrm{Coim}(k)$ there is a diagram



Since f factors through f' and $\mathrm{Ker}(f') \to X \xrightarrow{f'} \mathrm{Coim}(k)$ is zero, $\mathrm{Ker}(f') \to X$ factors uniquely through $\mathrm{Ker}(f)$. On the other hand $\mathrm{Ker}(f) \to X \xrightarrow{f'} \mathrm{Coim}(k)$ is zero, so $\mathrm{Ker}(f) \to X$ factors uniquely through $\mathrm{Ker}(f') \to X$. The picture is now



and

$$j = ia, \qquad i = jb$$

whence

$$j = jba, \qquad i = iab.$$

Since i and j are monomorphisms,

$$ba = 1_{\mathrm{Ker}(f')} \quad \text{and} \quad ab = 1_{\mathrm{Ker}(f)}.$$

Since a and b are inverse isomorphisms of $\mathrm{Ker}(f)$ with $\mathrm{Ker}(f')$, $\mathrm{Coim}(f) \to \mathrm{Coim}(k)$ is an isomorphism, i.e. $\mathrm{Ker}(k) = 0$ and $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is a monomorphism. ∎

Definition

A preabelian category $\mathcal{A}$ is *abelian* if for every morphism $f : X \to Y$ in $\mathcal{A}$, the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

If $\mathcal{A}$ is abelian we will not distinguish $\mathrm{Coim}(f)$ from $\mathrm{Im}(f)$. The canonical factorization of a morphism $f : X \to Y$ is now

$$X \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow Y.$$

and it is the unique way of writing f as the composite of an epimorphism with a monomorphism.

For example the category $\mathbf{Mod}_R$ of (left) modules over a ring R is abelian: kernels, cokernels and images are what they usually are, and the assertion that $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism is essentially the first isomorphism theorem. At the moment, examples of preabelian categories that are *not* abelian categories might be more useful.

A *topological abelian group* is an abelian group X with a topology such that the maps

$$m : X \times X \to X \qquad m(x, y) = xy$$
$$i : X \to X \qquad i(x) = x^{-1}$$

are continuous. Note that i must actuall be a homeomorphism. A morphism of topological abelian groups is a continuous homomorphism. It is easily checked that this category is additive. In fact it is preabelian: the kernel of a morphism $f : X \to Y$ is the ordinary group-theoretic kernel $\mathrm{Ker}(f)$ with the topology induced as a subset of X, and the cokernel $\mathrm{Coker}(f)$ is the usual cokernel with the topology induced as a quotient of Y. The canonical factorization

$$X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$$

then has the following description: $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ are the same group (the usual image) but $\mathrm{Coim}(f)$ has the topology induced as a *quotient* of X, while $\mathrm{Im}(f)$ has the topology induced as a *subset* of Y. These topologies are usually distinct.

Suppose R is a ring. A *(left) filtered R-module* is a left R-module M and a set of submodules $F^k M \subseteq M$ for all $k \in \mathbb{Z}$ such that $F^{k+1}M \subseteq F^k M$ for all k. A morphism $(M, F^{\cdot} M) \to (N, F^{\cdot} N)$ of filtered R-modules is an R-module homomorphism $f : M \to N$ such that $f(F^k M) \subseteq F^k N$ for all k. The category of filtered R-modules and their morphisms is additive. It is preabelian: the kernel of $f : M \to N$ is the usual kernel $K = \mathrm{Ker}(f)$ with the filtration $F^k K = K \cap F^k M$. The cokernel is the usual cokernel $\mathrm{Coker}(f)$ with the filtration such that

$$F^k \mathrm{Coker}(f) = \mathrm{Im}(F^k N \to \mathrm{Coker}(f)).$$

From this we see that $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ are the same R-module but the filtrations of $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ are given by

$$F^k \,\mathrm{Coim}(f) = (F^k M + \mathrm{Ker}(f))/\mathrm{Ker}(f), \quad F^k \,\mathrm{Im}(f) = \mathrm{Im}(f) \cap F^k N$$

and they are usually distinct.

Proposition

Suppose $\mathcal{A}$ is a preabelian category. The following are equivalent:

1. *$\mathcal{A}$ is abelian;*
2. *every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel;*
3. *a morphism that is a monomorphism and an epimorphism is an isomorphism.*

Proof: Suppose first that $\mathcal{A}$ is abelian. It will suffice treat the case of a monomorphism since the other is dual to the first. Suppose $f : X \to Y$ is a monomorphism. Then the kernel of f is $0 \to X$ and the coimage is then $1_X : X \to X$. The canonical factorization of f is therefore

$$0 \to X \xrightarrow{1} X \to \mathrm{Im}(f) \to Y \to \mathrm{Coker}(f).$$

By hypothesis $X \to \mathrm{Im}(f)$ is an isomorphism, and as $\mathrm{Im}(f) \to Y$ is by definition the kernel of the cokernel of f, $f : X \to Y$ is the kernel of its cokernel.

Suppose (2) holds and $f : X \to Y$ is both a monomorphism and an epimorphism. Then $\mathrm{Ker}(f) = 0$ and the cokernel of the kernel of f is the identity 1_X. Since f is an epimorphism it is the cokernel of its kernel, so $1_X : X \to X$ and $f : X \to Y$ are both cokernels of $0 = \mathrm{Ker}(f) \to X$. It follows that f is an isomorphism.

Suppose finally that (3) holds and $f : X \to Y$ is a morphism in $\mathcal{A}$. By the last lemma we know that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is a monomorphism and an epimorphism. By hypothesis it is then an isomorphism, so $\mathcal{A}$ is abelian. ∎

The assertion that an epimorphism is the cokernel of its kernel is a version of the first isomorphism theorem, and like it is used construct morphisms. Suppose for example we want to construct a morphism $X \to T$ and are given an epimorphism $f : Y \to X$. If $i : K \to Y$ is the kernel of f, f is the cokernel of i and thus

$$0 \to \mathrm{Hom}(X, T) \to \mathrm{Hom}(Y, T) \to \mathrm{Hom}(K, T)$$

is exact. This says that there is a bijection between morphism $X \to T$ and morphisms $h : Y \to T$ such that $hi = 0$.

Let's now recall the diagram (1)

$$
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
{\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
Y & \xrightarrow[f]{} & X
\end{array}
$$

and we assume this takes place in an abelian category $\mathcal{A}$.

Proposition

Suppose the diagram (1) is cartesian (resp. cocartesian). If g is an epimorphism (resp. monomorphism) then so is g', and (1) is cocartesian (resp. cartesian).

Proof: it suffices to treat the case where the diagram is cartesian. The composite morphism

$$X' \xrightarrow{i_1} X' \times Y \xrightarrow{gp_1 - fp_2} X$$

is the same as the morphism $g : X' \to X$, so it is an epimorphism. Then

$$X' \times Y \xrightarrow{gp_1 - fp_2} X$$

is an epimorphism as well. On the other hand in the diagram

$$Y' \xrightarrow{(f', g')} X' \times Y \xrightarrow{gp_1 - fp_2} X$$

the morphism $Y' \xrightarrow{(f', g')} X' \times Y$ is the kernel of $X' \times Y \xrightarrow{gp_1 - fp_2} X$; this just expresses the fact that (1) is cartesian. Since $\mathcal{A}$ is abelian and $X' \times Y \xrightarrow{gp_1 - fp_2} X$ is an epimorphism, it is the cokernel of $Y' \xrightarrow{(f', g')} X'$.

Applying the functor Hom(, T) to this sequence yields an exact sequence

$$0 \to \mathrm{Hom}(X, T) \to \mathrm{Hom}(X', T) \times \mathrm{Hom}(Y, T) \to \mathrm{Hom}(Y', T) \qquad (3)$$

whose exactness says that (1) is cocartesian. The diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(Y, T) & \longrightarrow & \mathrm{Hom}(Y \times X', T) & \longrightarrow & \mathrm{Hom}(X', T) \\
 & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & \mathrm{Hom}(Y', T) & \longrightarrow & \mathrm{Hom}(Y', T) & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows, and the kernel of the middle vertical arrow is $\mathrm{Hom}(X, T)$ by the exactness of (3). By the second proposition today the sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Hom}(X, T) \to \mathrm{Hom}(X', T)$$

is exact. Since $g : X' \to X$ is an epimorphism by hypothesis, $\mathrm{Hom}(X, T) \to \mathrm{Hom}(X', T)$ is injective and thus $\mathrm{Ker}(\alpha) = 0$. Since T was arbitrary this says that $g' : Y' \to Y$ is an epimorphism. ∎

Corollary

Let $f : Y \to X$ be a morphism in an abelian category. If f is an epimorphism it is the coequalizer of the projections p_1, $p_2 : Y \times_X Y \to X$. If f is a monomorphism it is the equalizer of the injections i_1, $i_2 : Y \to X \amalg^Y X$.

Proof: If f is an epimorphism, apply the proposition to the cartesian square

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p_1} & Y \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & X \end{array} .$$

Dual argument for monomorphisms. ■

In **Ab**, if f is an epimorphism then $Y \times_X Y$ is an equivalence relation on Y – it's the relation "congruent modulo the kernel of f" – and then X is the quotient of Y by this relation.

Some important buzzwords are connected with the proposition and its corollary. We say that an epimorphism $f : Y \to X$ is

- *squarable* (French *carrable*) if $Y \times_X T$ exists for any $T \to X$,
- *universal* if it is squarable and $p_2 : Y \times_X T \to T$ is an epimorphism for all T, and
- *effective* if f is the coequalizer of the projections p_1, $p_2 : Y \times_X Y \to Y$.

In a abelian category any morphism is squarable. The proposition and its corollary say that any epimorphism is a universal effective epimorphism.

Similar definitions can be made for universal and effective monomorphisms.